On Some Properties of Lind's Circulant Matrices

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***Abstract*—We investigate some properties of the eigenvalues and the determinant of Lind's circulant matrices regarding limits.**

***Index Terms*—circulant matrix, eigenvalue, determinant, limit**

***MSC 2010 Codes* – 15A15, 15A18, 15B05**

I. INTRODUCTION

LIND [3], introduced the following circulant matrices $S_R(\vec{W}_r) =$

$$= \begin{pmatrix} W_r & W_{r+1} & \cdots & W_{n+r-2} & W_{n+r-1} \\ W_{n+r-1} & W_r & \cdots & W_{n+r-3} & W_{n+r-2} \\ W_{n+r-2} & W_{n+r-1} & \cdots & W_{n+r-4} & W_{n+r-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ W_{r+2} & W_{r+3} & \cdots & W_r & W_{r+1} \\ W_{r+1} & W_{r+2} & \cdots & W_{n+r-1} & W_r \end{pmatrix}$$

and $S_R(\vec{W}_r) =$

$$\begin{pmatrix} W_r & W_{r+1} & \cdots & W_{n+r-2} & W_{n+r-1} \\ -W_{n+r-1} & W_r & \cdots & W_{n+r-3} & W_{n+r-2} \\ -W_{n+r-2} & -W_{n+r-1} & \cdots & W_{n+r-4} & W_{n+r-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -W_{r+2} & -W_{r+3} & \cdots & W_r & W_{r+1} \\ -W_{r+1} & -W_{r+2} & \cdots & -W_{n+r-1} & W_r \end{pmatrix}$$

where $W_n = A\alpha^n + B\beta^n$, the n^{th} term of the sequence $\{W_j\}_{j=0}^{+\infty}$ which satisfy the reucurrence relation $W_n = pW_{n-1} - qW_n$. In W_n, $\alpha = \frac{p+\sqrt{p^2-4q}}{2}$, $\beta = \frac{p-\sqrt{p^2-4q}}{2} \in \mathbb{R}$, $\alpha > \beta$ and $p^2 - 4q \neq 0$. He derived the explicit forms of the eigenvalues and the determinant of these matrices and thus establishing the following results:

1) The eigenvalues of $C_R(\vec{W}_r)$ are given by
$$\lambda(n,r,m) = \frac{W_r - W_{n+r} - q(W_{r-1} - W_{n+r-1})\omega^{-m}}{(1-\alpha\omega^{-m})(1-\beta\omega^{-m})}$$
where m=0, 1, ..., n-1 and $\omega = e^{2\pi i/n}$, the n^{th} root of 1.
2) The eigenvalues of $S_R(\vec{W}_r)$ are given by
$$\mu(n,r,m) = \frac{W_r + W_{n+r} - q(W_{r-1} + W_{n+r-1})\xi^{-m}}{(1-\alpha\xi^{-m})(1-\beta\xi^{-m})}$$
where m=0, 1, ..., n-1 and $\xi = e^{\pi i/n}$, the n^{th} root of -1.
3) The determinant of $C_R(\vec{W}_r)$ is given by
$$D(n,r) = \frac{(W_r - W_{n+r})^n - q^n(W_{r-1} - W_{n+r-1})^n}{1 - V_n + q^n}$$
where $V_n = \alpha^n + \beta^n$.
4) The determinant of $S_R(\vec{W}_r)$ is given by
$$E(n,r) = \frac{(W_r + W_{n+r})^n + q^n(W_{r-1} + W_{n+r-1})^n}{1 + V_n + q^n}$$

Our aim in this study is to investigate the following limits

$$\lim_{r\to+\infty} \frac{\lambda(n,r+j,m)}{\lambda(n,r,m)} \quad (1)$$
$$\lim_{r\to+\infty} \frac{\mu(n,r+j,m)}{\mu(n,r,m)} \quad (2)$$
$$\lim_{r\to+\infty} \frac{D(n,r+j)}{D(n,r)} \quad (3)$$
$$\lim_{r\to+\infty} \frac{E(n,r+j)}{E(n,r)} \quad (4)$$

provided that $\lambda(n,r,m)$, $\mu(n,r,m)$, $D(n,r)$ and $E(n,r)$ are all non-zero.

II. MAIN RESULTS

Theorem 2.1:
$$\lim_{r\to+\infty} \frac{\lambda(n,r+j,m)}{\lambda(n,r,m)} = \lim_{r\to+\infty} \frac{\mu(n,r+j,m)}{\mu(n,r,m)} = \alpha^j$$

Proof:
$$\begin{aligned} &\lim_{r\to+\infty} \frac{\lambda(n,r+j,m)}{\lambda(n,r,m)} \\ &= \lim_{r\to+\infty} \frac{W_{r+j} - W_{n+r+j} - q(W_{r+j-1} - W_{n+r+j-1})\omega^{-m}}{W_r - W_{n+r} - q(W_{r-1} - W_{n+r-1})\omega^{-m}} \\ &= \lim_{r\to+\infty} \frac{\Omega}{1 - W_{n+r}/W_r - q(W_{r-1}/W_r - W_{n+r-1}/W_r)\omega^{-m}} \end{aligned}$$

where
$$\Omega = W_{r+j}/W_r - W_{n+r+j}/W_r - q(W_{r+j-1}/W_r - W_{n+r+j-1}/W_r)\omega^{-m}$$

Claim:
$$\lim_{r\to+\infty} W_{r+j}/W_r = \alpha^j$$

$$\begin{aligned} \lim_{r\to+\infty} W_{r+j}/W_r &= \lim_{r\to+\infty} \frac{A\alpha^{r+j} - B\beta^{r+j}}{A\alpha^r - B\beta^r} \\ &= \lim_{r\to+\infty} \frac{A\alpha^j - \beta^j B\beta^r/\alpha^r}{A - B\beta^r/\alpha^r} \\ &= \frac{A\alpha^j}{A};\ since\ |\beta/\alpha| < 1 \\ &= \alpha^j \end{aligned}$$

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Continuing and using our proven claim we obtain

$$\begin{aligned}
&\lim_{r\to+\infty}\frac{\lambda(n,r+j,m)}{\lambda(n,r,m)}\\
&=\frac{\alpha^j-\alpha^{n+j}-q(\alpha^{j-1}-\alpha^{n+j-1})\omega^{-m}}{1-\alpha^n-q(1/\alpha-\alpha^{n-1})}\\
&=\frac{\alpha^{j+1}-\alpha^{n+j+1}-q(\alpha^j-\alpha^{n+j})}{\alpha-\alpha^{n+1}-q(1-\alpha^n)\omega^{-m}}\\
&=\frac{\alpha^{j+1}(1-\alpha^n)-q\alpha^j(1-\alpha^n)\omega^{-m}}{\alpha(1-\alpha^n)-q(1-\alpha^n)\omega^{-m}}\\
&=\frac{\alpha^{j+1}-q\alpha^j\omega^{-m}}{\alpha-q\omega^{-m}}\\
&=\frac{\alpha^j(\alpha-q\omega^{-m})}{\alpha-q\omega^{-m}}\\
&=\alpha^j
\end{aligned}$$

as desired.

Similarly,

$$\lim_{r\to+\infty}\frac{\mu(n,r+j,m)}{\mu(n,r,m)}=\alpha^j$$

Theorem 2.2:

$$\lim_{r\to+\infty}\frac{D(n,r+j)}{D(n,r)}=\lim_{r\to+\infty}\frac{E(n,r+j)}{E(n,r)}=\alpha^{jn}$$

Proof:

$$\begin{aligned}
\lim_{r\to+\infty}\frac{D(n,r+j)}{D(n,r)} &= \lim_{r\to+\infty}\frac{\prod_{m=0}^{n-1}\lambda(n,r+j,m)}{\prod_{m=0}^{n-1}\lambda(n,r,m)}\\
&= \prod_{m=0}^{n-1}\left[\lim_{r\to+\infty}\frac{\lambda(n,r+j,m)}{\lambda(n,r,m)}\right]\\
&= \alpha^{jn};\ by\ Theorem\ 2.1
\end{aligned}$$

Similarly,

$$\lim_{r\to+\infty}\frac{E(n,r+j)}{E(n,r)}=\alpha^{jn}$$

III. Conclusion

In summary, we have established that the following limits:

$$\lim_{r\to+\infty}\frac{\lambda(n,r+j,m)}{\lambda(n,r,m)}$$

$$\lim_{r\to+\infty}\frac{\mu(n,r+j,m)}{\mu(n,r,m)}$$

$$\lim_{r\to+\infty}\frac{D(n,r+j)}{D(n,r)}$$

$$\lim_{r\to+\infty}\frac{E(n,r+j)}{E(n,r)}$$

that involve the eigenvalues and determinant of Lind's circulant matrices are all related to $\alpha=\frac{p+\sqrt{p^2-4q}}{2}$, a root of the characteristic equation of the sequence $\{W_j\}_{j=0}^{+\infty}$ whose elements are used to form the said matrices.

References

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