Divisibility on Right Circulant Matrices with Sum of the Terms of Two Geometric Sequences

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Abstract—We provide divisibility results on the determinants of right circulant matrices with sum of the terms of two geometric sequence. We also give some open problems.

Index Terms—Divisibility, right circulant matrix, geometric sequence, Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers. We also give some open problems.

MSC 2010 Codes – 15A15, 15A18, 15A60

I. INTRODUCTION

Bueno introduced the right circulant matrices of the form

\[ C_n(\mathbf{h}) = \begin{pmatrix}
    h_0 & h_1 & h_2 & \cdots & h_{n-2} & h_{n-1} \\
    h_{n-1} & h_0 & h_1 & \cdots & h_{n-3} & h_{n-2} \\
    h_{n-2} & h_{n-1} & h_0 & \cdots & h_{n-4} & h_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    h_2 & h_3 & h_4 & \cdots & h_0 & h_1 \\
    h_1 & h_2 & h_3 & \cdots & h_{n-1} & h_0
\end{pmatrix} \]

where \( h_n = ar^n + bs^n \), the sum of the \( n \)th elements of \( \{ar^k\} \) and \( \{br^k\} \). Using this sequence, obtained the eigenvalues and the determinant of right circulant matrices with Fibonacci numbers \( (F_n) \), Lucas numbers \( (L_n) \), Pell numbers \( (P_n) \), Pell-Lucas numbers \( (Q_n) \), Jacobsthal numbers \( (J_n) \) and Jacobsthal-Lucas numbers \( (K_n) \).

Our goal is to obtain some divisibility on these numbers using Bueno’s results on the determinant of \( C_n(\mathbf{h}) \).

II. DIVISIBILITY RESULTS

Theorem 2.1: If \( h_n \) is integer for all \( n \), then

\[ 1 - (r^n + s^n) + (rs)^n \]

divides

\[ (h_0 - h_n)^n - [(as + br) - (ar^n s + brs^n)]^n \]

Proof: If \( h_n \in \mathbb{Z} \) for all \( n \), then the \( \det(C_n(\mathbf{h})) \) is an integer. From this, it immediately follows that

\[ 1 - (r^n + s^n) + (rs)^n \]

divides

\[ (h_0 - h_n)^n - [(as + br) - (ar^n s + brs^n)]^n \]

The following divisibility results on Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers and Jacobsthal-Lucas numbers are consequences of Theorem 2.1.

Corollary 2.2:

\[ 1 - L_n + (-1)^n \]

divides

\[ (-F_n)^n - (F_{n-1} - 1)^n \]

Corollary 2.3:

\[ 1 - L_n + (-1)^n \]

divides

\[ M[(2 - L_n)^n - (L_{n-1} + 1)^n] + N[(2 - L_n)^n - (L_{n-1} + 1)^n] \]

where \( M, N \in \mathbb{Z}/\{0\} \).

Corollary 2.4:

\[ 1 - Q_n + (-1)^n \]

divides

\[ (-P_n)^n - (P_{n-1} + 2)^n \]

Corollary 2.5:

\[ 1 - Q_n + (-1)^n \]

divides

\[ M[(2 - Q_n)^n - (Q_{n-1} + 2)^n] + N[(2 - Q_n)^n - (Q_{n-1} + 2)^n] \]

where \( M, N \in \mathbb{Z}/\{0\} \).

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Corollary 2.6:

\[ 1 - K_n + (-2)^n \]

divides

\[ (-J_n)^n - (2J_{n-1} - 1)^n \]

and

\[ (2 - K_n)^n - (2K_{n-1} + 1)^n \]

Corollary 2.7:

\[ 1 - K_n + (-2)^n \]

divides

\[ M[(-J_n)^n - (2J_{n-1} - 1)^n] + N[(2 - K_n)^n - (2K_{n-1} + 1)^n] \]

where \( M, N \in \mathbb{Z}/\{0\} \).

III. OPEN PROBLEMS

Here are possible ventures for further study

- Periodicity of \( \det[C_n(\tilde{h})] \)
- Investigate the matrix given by

\[
S_n(\tilde{h}) =
\begin{pmatrix}
h_0 & h_1 & h_2 & \cdots & h_{n-2} & h_{n-1} \\
-h_{n-1} & h_0 & h_1 & \cdots & h_{n-3} & h_{n-2} \\
-h_{n-2} & -h_{n-1} & h_0 & \cdots & h_{n-4} & h_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-h_2 & -h_3 & -h_4 & \cdots & h_0 & h_1 \\
-h_1 & -h_2 & -h_3 & \cdots & -h_{n-1} & h_0
\end{pmatrix}
\]

REFERENCES