Divisibility on Right Circulant Matrices with Sum of the Terms of Two Geometric Sequences

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***Abstract*—We provide divisibility results on the determinants of right circulant matrices with sum of the terms of two geometric sequence. We also give some open problems.**

***Index Terms*—Divisibility, right circulant matrix, geometric sequence, Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal- Lucas numbers.**

***MSC 2010 Codes* – 15A15, 15A18, 15A60**

I. Introduction

BUENO [1] introduced the right circulant matrices of the form

$$C_n(\vec{h}) = \begin{pmatrix} h_0 & h_1 & h_2 & \dots & h_{n-2} & h_{n-1} \\ h_{n-1} & h_0 & h_1 & \dots & h_{n-3} & h_{n-2} \\ h_{n-2} & h_{n-1} & h_0 & \dots & h_{n-4} & h_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_2 & h_3 & h_4 & \dots & h_0 & h_1 \\ h_1 & h_2 & h_3 & \dots & h_{n-1} & h_0 \end{pmatrix} \quad (1)$$

where $h_n = ar^n + bs^n$, the sum of the n^{th} elements of $\{ar^k\}$ and $\{br^k\}$. Using this sequence, obtained the eigenvalues and the determinant of right circulant matrices with Fibonacci numbers (F_n), Lucas numbers (L_n), Pell numbers (P_n), Pell-Lucas numbers (Q_n), Jacobsthal numbers (J_n) and Jacobsthal-Lucas numbers (K_n).

Our goal is to obtain some divisibility on these numbers using Bueno's results on the determinant of $C_n(\vec{h})$.

II. Divisibility Results

Theorem 2.1: If h_n is integer for all n, then

$$1 - (r^n + s^n) + (rs)^n$$

divides

$$(h_0 - h_n)^n - [(as + br) - (ar^n s + brs^n)]^n$$

Proof: If $h_n \in \mathbb{Z}$ for all n, then the $det[C_n(\vec{h})]$ is an integer becuase the determinant of integer matrix is also an integer. Hence

$$\frac{(h_0 - h_n)^n - [(as + br) - (ar^n s + brs^n)]^n}{1 - (r^n + s^n) + (rs)^n}$$

is an integer. From this, it immediately follows that

$$1 - (r^n + s^n) + (rs)^n$$

divides

$$(h_0 - h_n)^n - [(as + br) - (ar^n s + brs^n)]^n$$

The following divisibility results on Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers and Jacobsthal-Lucas numbers are consequences of *Theorem 2.1.*

Corollary 2.2:

$$1 - L_n + (-1)^n$$

divides

$$(-F_n)^n - (F_{n-1} - 1)^n$$

and

$$(2 - L_n)^n - (L_{n-1} + 1)^n$$

Corollary 2.3:

$$1 - L_n + (-1)^n$$

divides

$$M[(-F_n)^n - (F_{n-1} - 1)^n] + N[(2 - L_n)^n - (L_{n-1} + 1)^n]$$

where $M, N \in \mathbb{Z}/\{0\}$.

Corollary 2.4:

$$1 - Q_n + (-1)^n$$

divides

$$(-P_n)^n - (P_{n-1} + 2)^n$$

and

$$(2 - Q_n)^n - (Q_{n-1} + 2)^n$$

Corollary 2.5:

$$1 - Q_n + (-1)^n$$

divides

$$M[(-P_n)^n - (P_{n-1} + 2)^n] + N[(2 - Q_n)^n - (Q_{n-1} + 2)^n]$$

where $M, N \in \mathbb{Z}/\{0\}$.

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Corollary 2.6:

$$1 - K_n + (-2)^n$$

divides

$$(-J_n)^n - (2J_{n-1} - 1)^n$$

and

$$(2 - K_n)^n - (2K_{n-1} + 1)^n$$

Corollary 2.7:

$$1 - K_n + (-2)^n$$

divides

$M[(-J_n)^n - (2J_{n-1} - 1)^n] + N[(2 - K_n)^n - (2K_{n-1} + 1)^n]$

where $M, N \in \mathbb{Z}/\{0\}$.

III. Open Problems

Here are possible ventures for further study

- Periodicty of $det[C_n(\vec{h})]$
- Investigate the matrix given by

$$S_n(\vec{h}) = \begin{pmatrix} h_0 & h_1 & h_2 & ... & h_{n-2} & h_{n-1} \\ -h_{n-1} & h_0 & h_1 & ... & h_{n-3} & h_{n-2} \\ -h_{n-2} & -h_{n-1} & h_0 & ... & h_{n-4} & h_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -h_2 & -h_3 & -h_4 & ... & h_0 & h_1 \\ -h_1 & -h_2 & -h_3 & ... & -h_{n-1} & h_0 \end{pmatrix}$$

References

[1] A.C.F. Bueno, ”Right Circulant Matrices with Sum of the Terms of Two Geometric Sequences”, *International Journal of Mathematics and Scientific Computing*, vol. 4, no. 2, pp. 51-52, 2014.