On Arithmetic Right Circulant Matrix Sequences

A.C.F. Bueno

***Abstract*—In this study, we introduce the concept of arithmetic right circulant matrix sequence. Furthermore, we also present some properties of this sequence.**

***Index Terms*—arithmetic sequence, right circulant matrix, arithmetic right circulant matrix sequence**

***MSC 2010 Codes* – 15A15, 15A18, 15A60**

I. Introduction

SEQUENCES are the building blocks of right circulant matrices. Because of this, various number sequences are used as entries, and the properties of the resulting right circulant matrices are investigated. Research on these special matrices normally would revolve around the investigation of their determinants, eigenvalues, norms, and inverses.

Bozkurt worked on right circulant matrices with different number sequences, using in particular, the Jacobsthal and Lucas-Jacobsthal in [3] and Pell and Pell-Lucas numbers in [4]. He then generalized these works in [2] by using the general numbers sequence. In his works, Bozkurt provided formulas for the determinant and inverses of the matrices defined in [2],[3],[4] using matrix decompositions.

A recent work by Bahsi and Solak [1] involved the construction of right circulant matrices with arithmetic sequence as entries. They provided explicit formulas for the determinant, eigenvalues, norms, and inverses of these special type of right circulant matrices. They also investigated the Euclidean and spectral norms of the inverses of the resulting matrices.

Other recent works include that of Bueno. He dealt with right circulant matrices with terms of geometric sequences [5], Fibonacci numbers [6] and Jacobsthal numbers [7]. He solved for the eigenvalues, determinant, norms and the inverse of the said matrices.

Other researchers dealt only with the norms of right circulant matrices. To name a few, we have Civciv and Turkmen [8] (with Lucas numbers as entries), Nalli and Sen [11] (with Fibonacci numbers as entries) and Shen and Cen [12] (with (k,h)-Fibonacci numbers and (k,h)-Lucas numbers as entries).

In our study, instead of making number sequences as entries for right circulant matrices, we now make a sequence of right circulant matrices. We will focus on the right circulant matrix sequence patterned from the arithmetic sequence. This sequence will be called the arithmetic right circulant matrix sequence. We will investigate some properties of this sequence.

Aldous Cesar F. Bueno is a special science teacher in the Philippine Science High School - Central Luzon Campus which is located at the Clark Free Port Zone, Angeles City, Pampanga. (e-mail: aldouz_cezar@yahoo.com).

II. Preliminaries

Let $C_R(\vec{A}_1)$ and $C_R(\vec{D})$ be $n\times n$ right circulant matrices, which take the form

$$C_R(\vec{A}_1) = C_R(a_0,a_1,...,a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & ... & a_{n-2} & a_{n-1}\\ a_{n-1} & a_0 & a_1 & ... & a_{n-3} & a_{n-2}\\ a_{n-2} & a_{n-1} & a_0 & ... & a_{n-4} & a_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ a_2 & a_3 & a_4 & ... & a_0 & a_1\\ a_1 & a_2 & a_3 & ... & a_{n-1} & a_0 \end{pmatrix}$$

and

$$C_R(\vec{D}) = C_R(d_0,d_1,...,d_{n-1}) = \begin{pmatrix} d_0 & d_1 & d_2 & ... & d_{n-2} & d_{n-1}\\ d_{n-1} & d_0 & d_1 & ... & d_{n-3} & d_{n-2}\\ d_{n-2} & d_{n-1} & d_0 & ... & d_{n-4} & d_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ d_2 & d_3 & d_4 & ... & d_0 & d_1\\ d_1 & d_2 & d_3 & ... & d_{n-1} & d_0 \end{pmatrix}$$

respectively, with $d_j \neq 0$ for all j. The sequence $\left\{C_R(\vec{A}_k)\right\}_{k=0}^{+\infty}$ whose terms satisfy the recurrence relation:

$$C_R(\vec{A}_r) = C_R(\vec{A}_{r-1}) + C_R(\vec{D})$$

with initial value $C_R(\vec{A}_0)$, is an arithmetic right circulant matrix sequence.

For the rest of the paper, we will use the following notations:

- $C_R(\vec{A}_r)$ for the r^{th} term of the sequence;
- S_r for the sum of the first r terms;
- $\lambda(m,r)$ where $m=0,\ 1,\ \dots,\ n-1$, for the eigenvalues of $C_R(\vec{A}_r)$;
- $\Lambda(m,r)$ where $m=0,\ 1,\ \dots,\ n-1$, for the eigenvalues of S_r;
- Δ_m where $m=0,\ 1,\ \dots,\ n-1$, for the eigenvalues of $C_R(\vec{D})$; and
- $|M|$ for the determinant of a matrix M.

III. Main Results

Theorem 3.1: The r^{th} term of $\left\{C_R(\vec{A}_k)\right\}_{k=0}^{+\infty}$ is given by

$$C_R(\vec{A}_r) = C_R(\vec{A}_1) + (r-1)C_R(\vec{D}) \quad (1)$$

Moreover,

$$C_R(\vec{A}_r) = C_R(\alpha_0,\alpha_1,...,\alpha_{n-1}) \quad (2)$$

where $\alpha_j = a_j + (r-1)d_j$ and $j=0,\ 1,\ \dots,\ n-1$.

Proof: For (1), we use mathematical induction.

Basis step:

If $r = 1$, we have

$$C_R(\vec{A}_1) = C_R(\vec{A}_1) + 0 \cdot C_R(\vec{D})$$

Induction step:

Assume that for all r that

$$C_R(\vec{A}_r) = C_R(\vec{A}_1) + (r-1)C_R(\vec{D})$$

is true. Hence, we need to show that for $r+1$, the statement is also true.

$$\begin{aligned} C_R(\vec{A}_{r+1}) &= C_R(\vec{A}_1) + (r-1)C_R(\vec{D}) + C_R(\vec{D}) \\ &= C_R(\vec{A}_1) + rC_R(\vec{D}) \end{aligned}$$

as desired.

Adding the matrices, $C_R(\vec{A}_1)$ and $(r-1)C_R(\vec{D})$, we easily obtain (2). □

Theorem 3.2: The sum of the first r terms is given by

$$\begin{aligned} S_r &= \frac{r}{2}\left[C_R(\vec{A}_1) + C_R(\vec{A}_r)\right] && (3) \\ &= rC_R(\vec{A}_1) + \frac{r(r-1)}{2}C_R(\vec{D}) && (4) \end{aligned}$$

Moreover,

$$S_r = C_R\left(\Omega_0, \Omega_1, ..., \Omega_{n-1}\right) \tag{5}$$

where $\Omega_j = \frac{r(a_j+\alpha_j)}{2} = \frac{2ra_j + r(r-1)d_j}{2}$ and $\alpha_j = a_j + (r-1)d_j$ and $j = 0,\ 1,\ \ldots,\ n-1$.

Proof: Like the previous theorem, (3) and (4) be easily shown by mathematical induction. Furthermore, we obtain (5) by simply combining the matrices in (3) or (4).

Theorem 3.3:

$$\lambda(m,r) = \lambda(m,1) + (r-1)\Delta_m \tag{6}$$

Proof: The eigenvalues of a right circulant matrix are obtained through the discrete Fourier transform of its first row, *i.e.*

$$\begin{aligned} \lambda(m,r) &= \sum_{k=0}^{n-1} \alpha_k \omega^{-mk};\ where\ \omega = e^{2\pi i/n} \\ &= \sum_{k=0}^{n-1} [a_k + (r-1)d_k]\omega^{-mk} \\ &= \sum_{k=0}^{n-1} a_k\omega^{-mk} + (r-1)\sum_{k=0}^{n-1} d_k\omega^{-mk} \\ &= \lambda(m,1) + (r-1)\Delta_m \end{aligned}$$

Theorem 3.4:

$$\Lambda(m,r) = r\lambda(m,1) + \frac{r(r-1)}{2}\Delta_m \tag{7}$$

Proof:

$$\begin{aligned} \Lambda(m,r) &= \sum_{k=0}^{n-1} \Omega_k\omega^{-mk} \\ &= \sum_{k=0}^{n-1}\left[\frac{2ra_j + r(r-1)d_j}{2}\right]\omega^{-mk} \\ &= r\sum_{k=0}^{n-1} a_k\omega^{-mk} + \frac{r(r-1)}{2}\sum_{k=0}^{n-1} d_k\omega^{-mk} \\ &= r\lambda(m,1) + \frac{r(r-1)}{2}\Delta_m \end{aligned}$$

Theorem 3.5:

$$\left|C_R(\vec{A}_r)\right| = \prod_{m=0}^{n-1}\left(\lambda(m,1) + (r-1)\Delta_m\right) \tag{8}$$

Proof:

Multiplying all the eigenvalues from Thereom 3.3 proves the theorem.

Theorem 3.6:

$$|S_r| = \prod_{m=0}^{n-1}\left(r\lambda(m,r) + \frac{r(r-1)}{2}\Delta_m\right) \tag{9}$$

Proof:

Multiplying all the eigenvalues from Theorem 3.4, we get the desired result.

IV. Conclusion

We have established some properties of the arithmetic right circulant matrix sequence $\left\{C_R(\vec{A}_k)\right\}_{k=0}^{+\infty}$. In particular, we have obtained the explicit forms of the following:

- $C_R(\vec{A}_r)$, the r^{th} term of the sequence;
- S_r, the sum of the first r terms of the sequence; and
- eigenvalues and determinant $C_R(\vec{A}_r)$ and S_r.

Furthermore, for each m, the eigenvalues $\lambda(m,r)$ of the r^{th} term form an arithmetic sequence.

Possible works on this study is to solve for the Euclidean norm and spectral norm of $C_R(\vec{A}_r)$ and S_r. Other matrix sequences can also be created using the recurrence relation of other sequences and thus the properties of the formed sequence can be investigated.

References

[1] M. Bahsi and S. Solak, "On the circulant matrices with arithmetic sequence," *International Journal of Contemporary Mathematical Sciences*, vol. 5, no. 25, pp. 1213-1222, 2010.

[2] D. Bozkurt, "On the determinants and inverses of circulant matrices with a general number sequence," arXiv: 1202.1068v1 [math.NA] 6 Feb 2012.

[3] D. Bozkurt, "On the determinants and inverses of circulant matrices with Jacobsthal and Lucas-Jacobsthal Numbers," arXiv: 1201.6058v1 [math.NA] 29 Jan 2012.

[4] D. Bozkurt, "On the determinants and inverses of circulant matrices with Pell and Pell-Lucas Numbers," arXiv: 1201.6061v1 [math.NA] 29 Jan 2012.

[5] A. Bueno, "Right circulant matrices with geometric progression," *International Journal of Applied Mathematical Research,* vol.1, no. 4, pp. 593-603, 2012.

[6] A. Bueno, "Right circulant matrices with Fibonacci sequence," *International Journal of Mathematics and Scientific Computing,* vol. 2, no. 2, pp. 8-9, 2012.

[7] A. Bueno, "Right circulant matrices with Jacobsthal numbers," *International Journal of Advanced Mathematical Sciences,* vol. 1, no. 2, pp. 87-90, 2013.

[8] H. Civciv and R. Turkmen, "Notes on norms of circulant matrices with Lucas numbers," *International Journal of Information and System Sciences,* vol. 4, no. 1, pp. 142-147, 2008.

[9] R. Gray, "Toeplitz and Circulant matrices: a review," http://ee.stanford.edu/ gray/toeplitz.pdf

[10] H. Karner, J. Schneid and C. Ueberhuber, "Spectral decomposition of real circulant matrices," *Linear Algebra and Its Applications,* vol. 367, pp. 301-311, 2003.

[11] A. Nalli and M. Sen, "On the norms of circulant matrices with generalized Fibonacci numbers," *Selcuk Journal of Applied Mathematics,* vol. 11, no. 1, 107-116, 2010.

[12] S. Shen and J. Cen, "On the norms of circulant matrices with the $(k,h)-$Fibonacci and $(k,h)-$Lucas numbers," *International Journal of Contemporary Mathematics and Sciences,* vol. 6, no. 19, pp. 887-894, 2011.