2- Tensor Product of Graphs

U. P. Acharya and H. S. Mehta

***Abstract*—The tensor product $G_1 \otimes G_2$ and cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 are very well-known product and studied in detail in the literature. The concept of $G_1 \times G_2$ has been generalized by introducing 2- cartesian product $G_1 \times_2 G_2$ in our recent paper in 2014. In this paper, we extend the concept of tensor product $G_1 \otimes G_2$ by defining 2-tensor product $G_1 \otimes_2 G_2$ and obtain it for special graphs as P_n and C_n. We also discuss connectedness for this product.**

***Index Terms*—Tensor product of graphs, 2-tensor product of graphs, component.**

***2010 Mathematics Subject Classification* – 05C76, 15A69.**

I. Introduction

THE tensor product $G_1 \otimes G_2$ is one of the most important graph product, and can be viewed as the category of graphs. This product has been extended by introducing 2 - tensor product $G_1 \otimes_2 G_2$ of two graphs G_1 and G_2 with the help of distance $d(u, v)$ between two vertices u and v. In this paper, we obtain $G_1 \otimes_2 G_2$, for special graphs as P_n and C_n.

II. Preliminaries

Let $G = (V, E)$ be a finite, simple graph (without loops and multiple edges), where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. If edge set is empty then G is a null graph.

A graph G is connected, if there is at least one path between every pair of vertices in G. Otherwise G is disconnected graph. A maximal connected subgraph of G is called a component of G. We say that it is nontrivial component of G, if it is not a single vertex. If G is a connected graph then G has only one component, G itself.

Also $d_G(u, u')$ is the length of the shortest path between u and u' in the connected graph G. The diameter of G is defined as $\max\{d_G(u, u') : u, u' \in V\}$ and is denoted by $D(G)$.

If the graph G is a disjoint union of r similar components H, then we shall denote it by,

$$G = \bigcup_{\overset{\circ}{i=1}}^{r} H^{(i)}$$

Throughout the paper, we fix G to be finite, connected and simple graph. We denote the path graph, cycle graph and complete graph with n vertices by P_n, C_n and K_n respectively.

For the basic terminology, concepts and results of graph theory, we refer to ([3], [4], [6]).

U. P. Acharya is a associate professor in the Department od Applied Sciences and Humanities, A.D. Patel Institute of technology, New Vallabh Vidyanagar-388121, Gujarat, India. (E-mail: urvaship04@yahoo.co.in)

H. S. Mehta is a Reader in the Department of Mathematics, S. P. University, Vallabh Vidyanagar-388120, Gujarat, India. (E-mail: himalimehta63@gmail.com).

III. $P_m \otimes_2 P_n$

In this section, we discuss $G_1 \otimes_2 G_2$ for two path graphs. First we recall the definition of usual tensor product of graphs.

Definition 3.1 [6] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The tensor product of G_1 and G_2, denoted by $G = G_1 \otimes G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') in V are adjacent in the tensor product $G_1 \otimes G_2$ if $uu' \in E_1$ and $vv' \in E_2$. Equivalently, $d_{G_1}(u, u') = 1$ and $d_{G_2}(v, v') = 1$.

We extend this concept and define 2- tensor product of graphs.

Definition 3.2 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The 2 - tensor product of G_1 and G_2, denoted by $G_1 \otimes_2 G_2$ is the graph with vertex set $V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in the 2 - tensor product $G_1 \otimes_2 G_2$ if $d_{G_1}(u, u') = 2$ and $d_{G_2}(v, v') = 2$.

Remarks 3.3

(i) If $|V_1| = m$ and $|V_2| = n$ then $|V| = m \cdot n$.

(ii) The graph $G = G_1 \otimes_2 G_2$ is a null graph if $D(G_1) < 2$ or $D(G_2) < 2$.

We fix the following notations. Let P_m and P_n be two path graphs with, $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ respectively. It is clear that $P_2 \otimes_2 P_n$ is a null graph. So, we consider the case when both $m, n \geq 3$.

Proposition 3.4 The graph $P_3 \otimes_2 P_n$ has four non trivial components, each of which is a path graph.

Proof: Let $G = P_3$ and $H = P_n$ be two path graphs. Then $d_G(u_1, u_3) = 2$ and there is no vertex in G at distance two from u_2. Therefore (u_2, v_j) are isolated vertices in $G \otimes_2 H$, for $j = 1, 2, \ldots, n$. Also $d_H(v_j, v_{j+2}) = 2$, for $j = 1, 2, \ldots, n-2$. Therefore (u_1, v_j) will be adjacent to (u_3, v_j); $j = 1, 2, \ldots, n-2$ in $G \otimes_2 H$. These give four vertex disjoint path graphs, which are in the following form:

For $i = 1$, starting with $j = 1$ we get,
$(u_i, v_j) \rightarrow (u_{i+2}, v_{j+2}) \rightarrow (u_i, v_{j+4}) \rightarrow \ldots \rightarrow P$ and $(u_{i+2}, v_j) \rightarrow (u_i, v_{j+2}) \rightarrow (u_{i+2}, v_{j+4}) \rightarrow \ldots \rightarrow P_0$.

where $P = \begin{cases} (u_1, v_{4k+1}), & \text{if n = 4k +1 or 4k +2;} \\ (u_3, v_{4k+3}), & \text{if n = 4k +3} \\ (u_3, v_{4k-1}), & \text{if n = 4k} \end{cases}$ and

$$P_0 = \begin{cases} (u_3, v_{4k+1}), & \text{if n = 4k +1 or 4k +2;} \\ (u_1, v_{4k+3}), & \text{if n = 4k +3} \\ (u_1, v_{4k-1}), & \text{if n = 4k} \end{cases}$$

In this case, length of the path will be $\frac{n-1}{2}$ or $\frac{n}{2} - 1$ as n is odd or even, i.e., we get path graph $P_{\frac{n+1}{2}}$ or $P_{\frac{n}{2}}$ as n is odd or n is even respectively.

Now starting with $j = 2$, in above two paths, the vertices P and P_0 will be as follows:

$$P = \begin{cases} (u_1, v_{4k+2}), & \text{if n = 4k +2 or 4k +3;} \\ (u_3, v_{4k}), & \text{if n = 4k or 4k+1.} \end{cases}$$

$$P_0 = \begin{cases} (u_3, v_{4k+2}), & \text{if n = 4k +2 or 4k +3;} \\ (u_1, v_{4k}), & \text{if n = 4k or 4k+1.} \end{cases}$$

In this case, we get two path graphs $P_{\frac{n-1}{2}}$ or $P_{\frac{n}{2}}$ as n is odd or even respectively in $G \otimes_2 H$.

Finally, the following types of four components we obtain in the resultant product graph $G \otimes_2 H$:

If n is odd, then

$$P_3 \otimes_2 P_n = \left[\overset{2}{\underset{i=1}{\mathring{\bigcup}}} \left(P_{(\frac{n+1}{2})} \right)^{(i)} \right] \cup \left[\overset{2}{\underset{j=1}{\mathring{\bigcup}}} \left(P_{(\frac{n-1}{2})} \right)^{(j)} \right] \cup \cup \left[\overset{n}{\underset{t=1}{\mathring{\bigcup}}} (P_1)^{(t)} \right]$$

If n is even, then

$$P_3 \otimes_2 P_n = \left[\overset{4}{\underset{i=1}{\mathring{\bigcup}}} \left(P_{(\frac{n}{2})} \right)^{(i)} \right] \cup \left[\overset{n}{\underset{t=1}{\mathring{\bigcup}}} (P_1)^{(t)} \right]$$

Note that $P_3 \otimes_2 P_3$ has only two nontrivial components.

Proposition 3.5

(a) If n is odd, then

$$P_4 \otimes_2 P_n = \left[\overset{4}{\underset{i=1}{\mathring{\bigcup}}} \left(P_{(\frac{n+1}{2})} \right)^{(i)} \right] \cup \left[\overset{4}{\underset{j=1}{\mathring{\bigcup}}} \left(P_{(\frac{n-1}{2})} \right)^{(j)} \right]$$

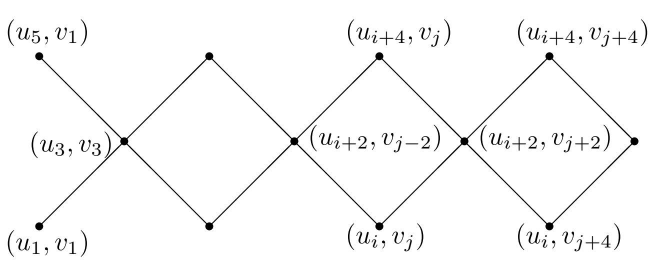
(b) If n is even, then $P_4 \otimes_2 P_n = \left[\overset{8}{\underset{i=1}{\mathring{\bigcup}}} \left(P_{(\frac{n}{2})} \right)^{(i)} \right]$

Proof: Let $G = P_4$ with $V(P_4) = \{u_1, u_2, u_3, u_4\}$. So, $d_G(u_1, u_3) = 2$ and $d_G(u_2, u_4) = 2$. As in Proposition 3.4 we have four components with respect to u_1 and u_3 and another four similar components with respect to u_2 and u_4. So, here we have eight components in $G \otimes_2 H$.

It is known that the usual tensor product $P_m \otimes P_n$ $(m, n \geq 2)$ is a disconnected graph with exactly two components [6]. Next we prove that $P_m \otimes_2 P_n$ $(m, n \geq 4)$ is also a disconnected graph but with exactly eight components.

Proposition 3.6 Let P_m and P_n be two path graphs with $m, n \geq 4$. Then $P_m \otimes_2 P_n$ has eight bipartite components.

Proof As in Proposition 3.5, we have eight vertex disjoint path graphs in $P_4 \otimes_2 P_n$. We fixed second graph P_n and extend the graph P_4. If we add one edge $(u_4 \longleftrightarrow u_5)$ in P_4, the following way, one of the component enlarge in upward direction. Here $i = 1$ with $j = 1, 5, \ldots, 4k-3$.



Similarly, in other components, the graph will also enlarge in upward direction. Thus the number of components remain same, i.e. eight in the graph $P_m \otimes_2 P_n$.

Fix any one component, say H, with vertex set $V(H) = W_1 \cup W_2$ in $P_m \otimes_2 P_n$, where $W_1 = \{(u_i, v_j)/i = 1, 5, 9, \ldots$ with $j = 1, 5, 9, \ldots\}$ and $W_2 = \{(u_i, v_j)/i = 3, 7, 11, \ldots$ with $j = 3, 7, 11, \ldots\}$. Then, it is clear that H is a bipartite graph with partite sets W_1 and W_2.

Corollary 3.7

(1) The degree of each vertex in $P_m \otimes_2 P_n$ is as follows: For $(u_i, v_j) \in V(P_m \otimes_2 P_n)$, deg$(u_i, v_j)$ is equal to

$$\begin{cases} 1, & \text{if i = 1, 2, m-1, or m with j = 1, 2, n-1 or n;} \\ 2, & \text{if i = 1, 2, m-1, or m with } 3 \leq j \leq n-2 \text{ or } 3 \leq i \leq m-2 \text{ with j = 1, 2, n-1 or n;} \\ 4, & \text{if } 3 \leq i \leq m-2 \text{ with } 1 \leq j \leq n \end{cases}$$

(2) $|E(P_m \otimes_2 P_n)| = 2\left[(m-2)(n-2)\right]$

Proof: (1) Consider $(u_i, v_j) \in V(P_m \otimes_2 P_n)$ with $i \in \{1, 2, m-1, m\}$ and $j \in \{1, 2, n-1, n\}$. Then (u_i, v_j) is adjacent with only one vertex, (u_{i+2}, v_{j+2}) or (u_{i+2}, v_{j-2}).

Also if $i \in \{1, 2\}$ and $3 \leq j \leq n-2$. Then (u_i, v_j) is adjacent with (u_{i+2}, v_{j-2}) and (u_{i+2}, v_{j+2}). So degree of (u_i, v_j) is 2.

Similarly if $i \in \{3, 4, \ldots, m-2\}$ with $j \in \{3, 4, \ldots, n-2\}$, then (u_i, v_j) is adjacent with $(u_{i\pm2}, v_{j\pm2})$. So degree of (u_i, v_j) is 4.

(2) The result follows from (1) and by degree sum formula.□

Remark 3.8 Each component of $P_m \otimes_2 P_n$ contain a vertex of degree one. So, it is clear that any component of $P_m \otimes_2 P_n$ is not regular, not Eulerian as well as not Hamiltonian.

IV. $P_m \otimes_2 C_n$ AND $C_m \otimes_2 C_n$

In this section, we discuss 2- tensor product of path graph P_m and cycle graph C_n, when both $m, n \geq 4$. Note that if $n = 4$ then $P_m \otimes_2 C_4$ is isomorphic to $P_m \otimes_2 P_4$.

First we consider the case when $m = 4$ and $n \geq 5$.

Proposition 4.1

(a) Suppose n is even. Then,

$$(1) P_4 \otimes_2 C_n = \left[\overset{8}{\underset{i=1}{\mathring{\bigcup}}} \left(C_{\frac{n}{2}} \right)^{(i)} \right]; \text{ if } n = 4k, k \geq 1$$

$$(2)\ P_4 \otimes_2 C_n = \left[\overset{4}{\underset{i=1}{\mathring{\bigcup}}} \left(C_n \right)^{(i)} \right]; \text{ if } n = 4k+2, k \geq 1$$

(b) Suppose n is odd. Then $P_4 \otimes_2 C_n = \left[\overset{2}{\underset{i=1}{\mathring{\bigcup}}} \left(C_{2n} \right)^{(i)} \right]$

Proof: (a) Suppose n is even. As we have seen in Proposition 3.5,

$$P_4 \otimes_2 P_n = \left[\overset{8}{\underset{i=1}{\mathring{\bigcup}}} \left(P_{\frac{n}{2}} \right)^{(i)} \right]$$

Now, in graph P_n, we add one edge between v_1 to v_n, then it becomes a cycle graph C_n.

Then $d(v_1, v_{n-1}) = 2 = d(v_2, v_n)$ in C_n. So, the edges

$$e_i : (u_i, v_1) \longleftrightarrow (u_{i+2}, v_{n-1}); i = 1, 2,$$
$$e_k : (u_k, v_1) \longleftrightarrow (u_{k-2}, v_{n-1}); k = 3, 4,$$
$$e'_t : (u_t, v_2) \longleftrightarrow (u_{t+2}, v_n); t = 1, 2,$$
$$e'_h : (u_h, v_2) \longleftrightarrow (u_{h-2}, v_n); h = 3, 4.$$

will be added in the resultant respective components.
(1) Suppose $n = 4k,\ k \geq 1$. Then the end vertices (u_1, v_1) and (u_3, v_{n-1}) of edge e_1 are in the same component, in fact these are the pendant vertices in the component $P_{\frac{n}{2}}$. So, by adding the edge e_1, it give a cycle graph $C_{\frac{n}{2}}$.

Similarly, for other seven edges also, the end vertices are the pendant vertices of respective components. So, after adding these edges (e_i, e_k, e'_t, e'_h), all eight components become cycle graphs $C_{\frac{n}{2}}$.

(2) Suppose $n = 4k + 2, k \geq 1$. In this case, end vertices of edge e_1 $[(u_1, v_1) \longleftrightarrow (u_3, v_{n-1})]$ lie in two different components of $P_4 \otimes_2 P_n$. So, when we add the edge e_1 in the product graph $P_4 \otimes_2 P_n$, two components will be joined, and give path graph P_n, whose end vertices are (u_1, v_{n-1}) and (u_3, v_1). But, these are the end vertices of e_3. So, after adding edge e_3, this component become a cycle graph C_n.

Thus by adding other edges, two components of type $P_{\frac{n}{2}}$ will joined and will give a cycle graph C_n.

(b) Suppose n is odd. Then by Proposition 3.5, we have

$$P_4 \otimes_2 P_n = \left[\bigcup_{i=1}^{4} \left(P_{(\frac{n+1}{2})} \right)^{(i)} \right] \bigcup \left[\bigcup_{j=1}^{4} \left(P_{(\frac{n-1}{2})} \right)^{(j)} \right]$$

The end vertices of e_1 lie in two different components of the type $P_{\frac{n+1}{2}}$ and $P_{\frac{n-1}{2}}$. By adding edge e_1 we get path graph P_n. Similarly by adding edge e_3, another $P_{\frac{n+1}{2}}$ and $P_{\frac{n-1}{2}}$ will joined and gives P_n. Then by adding e'_1, first we get P_{2n} and then by adding e'_3 we get the cycle graph C_{2n}. Thus four components $P_{\frac{n+1}{2}}$, $P_{\frac{n+1}{2}}$, $P_{\frac{n-1}{2}}$ and $P_{\frac{n-1}{2}}$ will combined and give only one component C_{2n}.

By similar arguments, by adding the four edges (e_2, e_4, e'_2 and e'_4) we get another cycle graph C_{2n}.

Note that, in above all cases we get components are cycle graphs with even order. So, the graph $P_4 \otimes_2 C_n$ has all bipartite components.

Now, we consider the general case with $m > 4$ and $n \geq 5$.

Proposition 4.2: Let P_m and C_n be path graph and cycle graph with m and n vertices respectively. Then

(a) $P_m \otimes_2 C_n$ has eight components if $n = 4k, k \geq 1$.
(b) $P_m \otimes_2 C_n$ has four components if $n = 4k + 2, k \geq 1$.
(c) $P_m \otimes_2 C_n$ has two components if $n = 4k + 1$ or $(4k + 3), k \geq 1$.

Proof:
(a) Let $n = 4k, k \geq 1$. Then by Proposition 4.1,

$$P_4 \otimes_2 C_n = \left[\bigcup_{i=1}^{8} (C_{\frac{n}{2}})^{(i)} \right]$$

If we add $u_4 \longleftrightarrow u_5$ in P_4, then in the resultant respective cycle, $(u_3, v_j) \longleftrightarrow (u_5, v'_j)$ edges will be added, as $d_{C_n}(v_j, v'_j) = 2$. So, respective components enlarge by additional edges. Also note that the end point of each edge, e.g., the vertices (u_3, v_j) and (u_5, v'_j), lie in different partite sets. So, the component remains bipartite.

Similarly if we increase first graph P_4 up to P_m, then respective bipartite components will be enlarge. However the number of components remain same but the nature of components will depend on m, as we have seen in Proposition 3.5.

(b) Suppose $n = 4k + 2, k \geq 1$. Then by Proposition 4.1(a)

$$P_4 \otimes_2 C_n = \left[\bigcup_{i=1}^{4} (C_n)^{(i)} \right]$$

If we increase first graph P_4 up to P_m, then resultant bipartite components will be enlarge as per case (a).

(c) By similar arguments, for $n = 4k+1$, $n = 4k+3, k \geq 1$, the graph $P_m \otimes_2 C_n$ has two bipartite components.

As in Corollary 3.7, we get the following result:

Corollary 4.3 Let $(u_i, v_j) \in V(P_m \otimes_2 C_n)$.
(1) $\deg(u_i, v_j)$ is equal to

$$\begin{cases} 2, & \text{if i = 1, 2, m-1, or m with } 1 \leq j \leq n \\ 4, & \text{if } 3 \leq i \leq m-2 \text{ with } 1 \leq j \leq n \end{cases}$$

(2)$|E(P_m \otimes_2 C_n)| = 2\,[n(m-2)]$

Remark 4.4 Since each component of the graph $P_m \otimes_2 C_n$ contain vertices with degree 2 as well as 4, it can not be regular. However each component will be Eulerian graph.

Note that each component of $P_4 \otimes_2 C_n$ will be Hamiltonian graph. Further in case of $P_5 \otimes_2 C_n$, only some components will be Hamiltonian, whereas for $P_m \otimes_2 C_n$ with $m > 5$, no component will be Hamiltonian.

Next, we discuss 2 - tensor product of cycle graphs. It is clear that $C_3 \otimes_2 C_n$ is a null graph. So, we consider the case when both $m, n \geq 4$. As usual, we consider cycle graphs C_m and C_n with $V(C_m) = \{u_1, u_2, \ldots u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots v_n\}$ respectively.

It can be checked that $C_4 \otimes_2 C_n$ is isomorphic to $P_4 \otimes_2 C_n$ and we have discuss $P_4 \otimes_2 C_n$ in Proposition 4.1. So, we consider $C_m \otimes_2 C_n$ for $m, n \geq 5$.

Proposition 4.5

(a) Suppose m and n both are even. Then $C_m \otimes_2 C_n$ has eight components if $m = 4k$ and $n = 4l, k, l \geq 1$, otherwise it has four components.
(b) If one of m and n is even and other is odd, then $C_m \otimes_2 C_n$ has two components.
(c) If m and n both are odd, then $C_m \otimes_2 C_n$ is a connected graph.

The number of components in $C_m \otimes_2 C_n$ will be as follows:

m / n	4l	4l+2	4l+1/4l+3
4k	8	4	2
4k+2	4	4	2
4k+1/ 4k+3	2	2	1

Proof: We have seen in Proposition 3.6 that $P_m \otimes_2 P_n$ has eight components. By adding edge $(u, v_1) \longleftrightarrow (u', v_{n-1})$ and $(u, v_2) \longleftrightarrow (u', v_n)$ with $d_{P_m}(u, u') = 2$ in $P_m \otimes_2 P_n$, we have the resultant product $P_m \otimes_2 C_n$. Again, as we known in Proposition 4.2, $P_m \otimes_2 C_n$ has eight components, four components or two components for $n = 4l,\ 4l+2$ or $4l+1\ /\ 4l+3,\ l \geq 1$ respectively.

Now, we add two edges $e_1 : (u_1, v) \longleftrightarrow (u_{m-1}, v')$ and $e_2 : (u_2, v) \longleftrightarrow (u_m, v')$ with $d_{C_m}(v, v') = 2$ in $P_m \otimes_2 C_n$, we have the resultant product $C_m \otimes_2 C_n$.

Fixed $n = 4l,\ l \geq 1$. By similar arguments of Proposition 4.2 we have eight, four or two components for $m = 4k, 4k+2$ or $4k+1/4k+3,\ k \geq 1$ in the resultant product graph $C_m \otimes_2 C_n$.

For $n = 4l+2,\ l \geq 1$, we know that $P_m \otimes_2 C_n$ has four components.

If $m = 4k,\ 4k+2,\ k \geq 1$, then end vertices of additional edge e_1 lies in same component and end vertices of edge e_2 also lies in other same component. So, number of components in the resultant product graph will remain same. If $m = 4k+1/4k+3,\ k \geq 1$, i.e. m is odd and $m-1$ is even, then end vertices $(u_1, v)\,[(u_2, v)]$ and $(u_{m-1}, v')\,[(u_m, v')]$ lie in two different components of $P_m \otimes_2 C_n$. So, two components joined by edges $e_1(e_2)$. Therefore $C_m \otimes_2 C_n$ will have two components.

Lastly, if $n = 4l+1/4l+3, l \geq 1$, then $P_m \otimes_2 C_n$ has two components. By similar arguments, for $m = 4k, 4k+2, k \geq 1$, $C_m \otimes_2 C_n$ has two components.

For $m = 4k+1/4k+3, k \geq 1$, end vertices of additional edges lie in two different components of $P_m \otimes_2 C_n$. So, these two components joined by these edges and it give one component, i.e. $C_m \otimes_2 C_n$ is a connected graph.

As in Corollary 3.7, we get the following result:

Corollary 4.6

(1) The degree of each vertex in the graph $C_m \otimes_2 C_n$ is 4. Consequently, each components of $C_m \otimes_2 C_n$ is a regular, Euler graph and hence Hamiltonian.

(2) $|E(C_m \otimes_2 C_n)| = 2mn$.

In case of usual tensor product, it is known that $C_m \otimes C_n$ is connected graph if m or n is odd, otherwise the usual tensor product gives two components [8]. But in case of 2 - tensor product, it follows from the above result that $C_m \otimes_2 C_n$ is connected if and only if m and n both are odd, otherwise it is disconnected graph and the number of components in $C_m \otimes_2 C_n$ will depend on both m and n.

The derived graph $G^!$ of G is defined in terms of $d(u, v) = 2$ in [2]. Also, the graph G^2 is obtained by considering $d(u, v) \leq 2$. The 2 - tensor product can be obtained with the help of $G^!$ and G^2 or combination of G and G^2 using usual tensor product. But here we consider 2 - tensor product separately as we have obtained $G_1 \otimes_2 G_2$ and their results directly in terms of the factor graphs G_1 and G_2, without computing the graphs $G^!$ or G^2.

Acknowledgment

The research is supported by the SAP programme, Department of Mathematics, S. P. University, Vallabh Vidyanagr, by University Grants Commission (UGC), India.

References

[1] U. P. Acharya and H. S. Mehta, "2- cartesian product of special graphs", *International Journal of Mathematics and Soft Computing,* vol.4, no. 1, pp. 139-144, 2014.

[2] A. Alwardi, B. Arsic, I. Gutman and N. D. Soner, "The common neighborhood graphs and its energy," *Iran. J. of Math. Sci. and Inf.,* vol.7, no. 1, pp. 1-8, 2012.

[3] N.L. Biggs, *Algebraic Graph Theory*, Cambridge University Press, 2^{nd} edition, Cambridge, 1993.

[4] F. Harary, *Graph Theory*, Addison Wesley, Massachusetts, 1969.

[5] S. Moradi, "A note on tensor product of graphs," *Iran. J. of Math. Sci. and Inf.,* vol.7, no. 1, pp. 73-81, 2012.

[6] K. K. Parthasarathy, *Basic Graph Theory*, Tata McGraw Hill, 1994.

[7] I. H. Naga Raja Rao and K. V. S. Sarma, "On tensor product of standard graphs," *International Journal of Computational Cognition,* vol.8. no. 3, pp. 99-103, 2010.

[8] G. Sabidussi, "Graph multiplication," *Math. Z.,* vol. 72, pp. 446-457, 1960.

[9] Y. Shibata and Y. Kikuchi, "Graph products based on the distance in graphs," *IEICE Trans. Fundamentals,* vol.E83-A, 2000.

[10] Z. Lukic, "A note on graphs representable as product graphs," *Publications de l'Institut Mathématique*, vol. 32, no. 52, pp. 115-118, 1982.