Generalization of Contractive Mappings with Two Auxiliary Functions in Ordered Metric Spaces

G. V. R. Babu, K. K. M. Sarma and B. K. Leta

***Abstract*—In this paper, we extend the results of Su(2014) to a more general contractive condition and prove a new fixed point theorem in complete metric spaces endowed with a partial order by using two auxiliary functions in which one is a generalized altering distance function and the other one is a non-negative real valued function with certain properties and discuss the importance of using two auxiliary functions. Moreover, we establish examples to substantiate the validity of our theorems. Our theorems generalize the results of Su(2014).**

***Index Terms*—Fixed points, contractive mappings, partially ordered metric space, generalized altering distance function.**
***MSC 2010 Codes* – 47H10, 54H25**

I. Introduction

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the Banach contraction principle which is one of the most powerful results in fixed point theory. Banach proved existence and uniqueness of fixed point for contraction mappings in the metric space setting.

In 2004, Ran and Reurings[9] proved an analogue of Banach contraction principle in partially ordered sets in which the key concept is that the contraction condition on the non-linear map is assumed to hold on elements that are comparable in the partial order and the map is assumed to be monotone. For more investigations on the existence of fixed points in partially ordered metric spaces, we refer [1], [4], [5], [8]. In recent times, existence and uniqueness of fixed points of contraction maps in ordered metric spaces is extensively studied by many researchers, some of which are [7], [13], [12].

In 2005, Nieto and Rodriguez-Lopez [7] proved the following theorem in partially ordered metric space setting.

Theorem 1.1.[7] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous and non-decreasing mapping such that there exists $k \in [0, 1)$ with $d(f(x), f(y)) \leq kd(x, y)$ for all $x \succeq y$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point.

Definition 1.2. Let $(X, \preceq)$ be a partially ordered set. Let $T : X \to X$ be a self map of X. We say T is monotonically non-decreasing if for any $x, y \in X$, $x \preceq y$ implies $T(x) \preceq T(y)$.

Yan, Su and Feng[13], proved some fixed point theorems of contraction type mappings in a complete metric spaces endowed with a partial order by using two auxiliary functions in which one is an altering distance function and the other one is a continuous real valued function on $[0, \infty)$.

Theorem 1.3.[13] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that

$$\psi(d(T(x), T(y))) \leq \phi(d(x, y)) \text{ for all } x \succeq y,$$

where ψ is an altering distance function $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$.
If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Recently, Su [12] proved some fixed point theorems by using generalized altering distance functions .

Definition 1.4. [6] An altering distance function is a function $\psi : [0, \infty) \to [0, \infty)$ which satisfies:
(*i*) ψ is continuous (*ii*) ψ is non-decreasing and
(*iii*) $\psi(t) = 0$ if and only if $t = 0$.

We denote the class of all altering distance functions by Ψ_1.

Definition 1.5.[12] A function $\psi : [0, \infty) \to [0, \infty)$ is said to be a generalized altering distance function if it satisfies (*i*) ψ is non-decreasing and (*ii*) $\psi(t) = 0$ if and only if $t = 0$.

We denote the class of all generalized altering distance functions by Ψ. Clearly $\Psi \subseteq \Psi_1$.

For more works on altering distance functions, we refer [3], [6] and [10].

Theorem 1.6.[12] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that

$$\psi(d(T(x), T(y))) \leq \phi(d(x, y)) \text{ for all } x \succeq y, \tag{1.6.1}$$

where ψ is a generalized altering distance function, and $\phi : [0, \infty) \to [0, \infty)$ is a right upper semi-continuous function

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with the condition $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Su[12], proved the validity of Theorem 1.6 for T being not necessarily continuous, by assuming the following hypothesis in X:
If $\{x_n\}$ is a non-decreasing sequence in X such that

$$x_n \to z, \text{then } x_n \preceq z \text{ for all } n \in \mathbb{N}. \tag{1.6.2}$$

Theorem 1.7.[12] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that X satisfies (1.6.2). Let $T : X \to X$ be a non-decreasing mapping such that

$$\psi(d(T(x), T(y))) \leq \phi(d(x, y)) \text{ for all } x \succeq y, \tag{1.7.1}$$

where ψ is a generalized altering distance function, and $\phi : [0, \infty) \to [0, \infty)$ is a right upper semi-continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Remark 1.8. In Theorem 1.7, Su(Theorem 2.4, [12]) mentioned that ψ is a generalized altering distance function. But, in the actual proof of the theorem, an altering distance function $\psi \in \Psi_1$ is used.

Definition 1.9. Let $(X, \preceq)$ be a partially ordered set. Any two elements $x, y \in X$ are said to be comparable elements of X if either $x \preceq y$ or $y \preceq x$.
We state the following lemma, which we use in our main results.

Lemma 1.10.[2] Suppose that (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k}, x_{n_k-1}) < \epsilon$ and

(i) $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k+1}) = \epsilon$
(ii) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon$
(iii) $\lim_{k\to\infty} d(x_{m_k-1}, x_{m_k}) = \epsilon$
(iv) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon$.

Definition 1.11. Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is said to be upper semi-continuous at a point $x \in X$ if

$$\limsup_{n\to\infty} f(x_n) \leq f(x)$$

whenever $x_n \to x$ as $n \to \infty$. f is said to be upper semi-continuous on X if it is upper semi-continuous at each point of X.

We consider the following class of functions in our study:
$\Phi = \{\phi : [0, \infty) \to [0, \infty) \mid$ (i) ϕ is upper semi-continuous, non-decreasing and continuous at 0,
(ii) $\phi(2t) \leq 2\phi(t)$ for all $t > 0$ and
(iii) $\phi(t) = 0$ if and only if $t = 0\}$.

Sastry, Naidu, Babu and Naidu [11] studied the class of functions Φ and established that every element $\phi \in \Phi$ need not be sub-additive. (Example 2.2 (ii), [11]).

In this paper, we extend the results of Su [12] to a more general contractive condition and prove a new fixed point theorem in a complete metric space endowed with a partial order by using two auxiliary functions in which one is a generalized altering distance function and the other one is a non-negative real valued function with certain properties. Furthermore, we justify the use of two functions instead of using one function.

In the following, we prove our main results.

II. Main Results

Theorem 2.1. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ and $\psi(t) > \phi(t)$ for all $t > 0$ such that for all comparable x and y

$$\psi(d(T(x), T(y))) \leq \max\{\phi(d(x, y)), \phi(d(x, Tx)), \phi(d(y, Ty)), \frac{1}{2}[\phi(d(x, Ty)) + \phi(d(y, Tx))]\} \tag{2.1.1}$$

If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Proof Let $x_0 \in X$ with $x_0 \preceq Tx_0$. If $x_0 = Tx_0$, then x_0 is a fixed point and hence we are through.
Now we assume $x_0 \preceq Tx_0$ with $x_0 \neq Tx_0$.
Since T is non-decreasing, by mathematical induction, we have

$$x_0 \preceq Tx_0 \preceq \ldots \preceq T^n x_0 \preceq T^{n+1} x_0 \preceq \ldots \; . \tag{2.1.2}$$

We define a sequence $\{x_n\}$ in X by $x_n = T^n x_0$ for n = 1, 2, 3, … . From (2.1.2), we have $x_n \preceq x_{n+1}$ for n = 1, 2, 3, … .
If there exists $n \geq 1$ such that $x_n = x_{n+1}$,
then $x_n = x_{n+1} = Tx_n$ so that x_n is a fixed point.
Hence, without loss of generality, we assume that $x_n \preceq x_{n+1}$ with $x_n \neq x_{n+1}$ for all $n \geq 1$.
By (2.1.1) we have

$$\begin{aligned}
\psi(d(x_n, x_{n+1})) = \psi(d(T(x_{n-1}), T(x_n))) &\leq \max\{\phi(d(x_{n-1}, x_n)), \\
&\phi(d(x_{n-1}, Tx_{n-1})), \phi(d(x_n, Tx_n)), \\
&\frac{1}{2}[\phi(d(x_n, Tx_{n-1})) + \phi(d(x_{n-1}, Tx_n))]\}, \\
&= \max\{\phi(d(x_{n-1}, x_n)), \phi(d(x_{n-1}, x_n)), \\
&\phi(d(x_n, x_{n+1})), \frac{1}{2}[\phi(d(x_n, x_n)) + \phi(d(x_{n-1}, x_{n+1}))]\} \\
&= \max\{\phi(d(x_{n-1}, x_n)), \phi(d(x_{n-1}, x_n)), \\
&\phi(d(x_n, x_{n+1})), \frac{1}{2}\phi(d(x_{n-1}, x_{n+1}))\} \\
&\leq \max\{\phi(d(x_{n-1}, x_n)), \phi(d(x_n), x_{n+1})),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\phi(d(x_{n-1},x_n)+d(x_n,x_{n+1}))\} \\
&\leq \max\{\phi(d(x_{n-1},x_n)),\phi(d(x_n,x_{n+1})), \\
&\frac{1}{2}\phi(2\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\})\} \\
&\leq \max\{\phi(d(x_{n-1},x_n)),\phi(d(x_n,x_{n+1})), \\
&\frac{1}{2}(2\phi(\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}))\} \\
&= \max\{\phi(d(x_{n-1},x_n)),\phi(d(x_n,x_{n+1}))\}.
\end{aligned}$$

If $\phi(d(x_n,x_{n+1}))>\phi(d(x_{n-1},x_n))$ for some n, then $\psi(d(x_n,x_{n+1}))\leq\phi(d(x_n,x_{n+1}))<\psi(d(x_n,x_{n+1}))$, a contradiction.

Consequently, $\phi(d(x_{n-1},x_n))$ is the maximum which implies that $\psi(d(x_n,x_{n+1}))\leq\phi(d(x_{n-1},x_n))<\psi(d(x_{n-1},x_n))$ for all $n\geq 1$. By the property of ψ, it follows that $d(x_n,x_{n+1})\leq d(x_{n-1},x_n)$ for all $n\geq 1$. Therefore $\{d(x_n,x_{n+1})\}$ is a decreasing sequence of non-negative real numbers.

Hence there exists $r\geq 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1})=r$. If $r>0$, then $\psi(r)\leq\limsup_{n\to\infty}\phi(d(x_{n-1},x_n))\leq\phi(r)<\psi(r)$, a contradiction.

Hence $r = 0$.

We now show that $\{x_n\}$ is a Cauchy sequence in X. If possible, suppose that $\{x_n\}$ is not Cauchy. Then there exists an $\epsilon>0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k>m_k>k$ such that $d(x_{m_k},x_{n_k})\geq\epsilon$ and $d(x_{m_k},x_{n_k-1})<\epsilon$. Now by Lemma 1.10, it follows that

$$\lim_{k\to\infty} d(x_{m_k-1},x_{n_k-1})=\epsilon \text{ and } \lim_{k\to\infty} d(x_{m_k},x_{n_k})=\epsilon.$$

By using the inequality (2.1.1), we have

$$\begin{aligned}
\psi(\epsilon)\leq\psi(d(x_{m_k},x_{n_k}))&\leq\max\big\{\phi(d(x_{m_k-1},x_{n_k-1})),\\
&\phi(d(x_{m_k-1},x_{m_k})),\phi(d(x_{n_k-1},x_{n_k}))\\
&\frac{1}{2}\big[\phi(d(x_{m_k-1},x_{n_k}))+\phi(d(x_{n_k-1},x_{m_k}))\big]\big\}.
\end{aligned}$$

On taking the limit supremum as $k\to\infty$, we have

$$\psi(\epsilon)\leq\max\{\phi(\epsilon),\phi(0),\phi(0),\frac{1}{2}[\phi(\epsilon)+\phi(\epsilon)]\}=\phi(\epsilon)<\psi(\epsilon),$$

a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there exists $z\in X$ such that $x_n\to z$ as $n\to\infty$.

Thus $z=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}Tx_n=Tz$.

Hence z is a fixed point of T.

We prove that Theorem 2.1 is still valid for T being not necessarily continuous, which is shown in the following theorem:

Theorem 2.2. Let $(X,\preceq)$ be a partially ordered set, suppose that there exists a metric d on X such that (X,d) is a complete metric space. Assume that X has the following property:

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n\to z$, then $x_n\preceq z$ for all $n\in\mathbb{N}$. (2.2.1)

Let $T:X\to X$ be a non-decreasing mapping. Suppose that there exist $\psi\in\Psi_1$ and $\phi\in\Phi$ with $\psi(t)>\phi(t)$ for all $t>0$ such that for all comparable x and y

$$\begin{aligned}
\psi(d(T(x),T(y)))&\leq\max\{\phi(d(x,y)),\phi(d(x,Tx)),\\
&\phi(d(y,Ty)),\frac{1}{2}[\phi(d(x,Ty))+\phi(d(y,Tx))]\} \qquad (2.2.2)
\end{aligned}$$

If there exists $x_0\in X$ with $x_0\preceq Tx_0$, then T has a fixed point.

Proof For $x_0\in X$ and $x_0\neq Tx_0$ with $x_0\preceq Tx_0$ the sequence $\{x_n\}$ defined by $x_n=T^nx_0$, $n=1,2,\ \ldots$ is non-decreasing and it is Cauchy, which follows as in the proof of Theorem 2.1. Since X is complete, we have $x_n\to z$ for some $z\in X$.

Hence by the hypothesis (2.2.1), we have $x_n\preceq z$ for all n and by the inequality (2.2.2) we have

$$\begin{aligned}
\psi(d(x_{n+1},Tz))&=\psi(d(Tx_n,Tz))\\
&\leq\max\big\{\phi(d(x_n,z)),\phi(d(x_n,Tx_n)),\phi(d(z,Tz)),\\
&\frac{1}{2}\big[\phi(d(x_n,Tz))+\phi(d(z,Tx_n))\big]\big\}\\
&=\max\big\{\phi(d(x_n,z)),\phi(d(x_n,x_{n+1})),\phi(d(z,z)),\\
&\frac{1}{2}\big[\phi(d(x_n,z))+\phi(d(z,x_{n+1}))\big]\big\}.
\end{aligned}$$

On letting $n\to\infty$ it follows that

$$\begin{aligned}
\psi(d(z,Tz))&=\lim_{n\to\infty}\psi(d(x_{n+1},Tz))\\
&\leq\max\big\{\limsup_{n\to\infty}\phi(d(x_n,z)),\limsup_{n\to\infty}\phi(d(x_n,x_{n+1})),\\
&\phi(d(z,z)),\frac{1}{2}\big[\limsup_{n\to\infty}\phi(d(x_n,z))+\limsup_{n\to\infty}\phi(d(z,x_{n+1}))\big]\big\}\\
&\leq\max\big\{\phi(d(z,z)),\phi(d(z,z)),\phi(d(z,Tz)),\\
&\frac{1}{2}\big[\phi(d(z,Tz))+\phi(d(z,z))\big]\big\}=\phi(d(z,Tz)).
\end{aligned}$$

Which implies that

$$\psi(d(z,Tz))\leq\phi(d(z,Tz))<\psi(d(z,Tz)),$$

a contradiction. Hence $z=Tz$.

Theorem 2.3. Let $(X,\preceq)$ be a partially ordered set, suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T:X\to X$ be a non-decreasing mapping. Assume that there exist $\psi\in\Psi$ and $\phi\in\Phi$ and $\psi(t)>\phi(t)$ for all $t>0$ such thatfor all comparable x and y

$$\begin{aligned}
\psi(d(T(x),T(y)))&\leq\max\big\{\phi(d(x,y)),\\
&\frac{1}{2}[\phi(d(x,Tx))+\phi(d(y,Ty))],\\
&\frac{1}{2}[\phi(d(x,Ty))+\phi(d(y,Tx))]\big\}. \qquad (2.3.1)
\end{aligned}$$

Further, assume that either (i) T is continuous or (ii) X satisfies condition (2.2.1).

If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

proof: Since the inequality (2.3.1) implies the inequality (2.1.1) the conclusion of this theorem follows from Theorem 2.1, when T is continuous; when X satisfies condition (2.2.1) the conclusion follows from Theorem 2.2.

Lemma 2.4. Under the hypotheses of Theorem 2.3 suppose that z is a fixed point of T and z is comparable to some $x \in X$, then $T^n x \to z$ as $n \to \infty$.

proof: Without loss of generality, we suppose that $z \preceq x$ with $z \neq x$, then $Tz \preceq Tx$.
By mathematical induction it follows that $z \preceq T^n x$ for all n.
Now by the inequality (2.3.1), we have

$$\psi(d(z,T^{n+1}x)) = \psi(d(T^{n+1}z,T^{n+1}x))$$
$$= \psi(d(T(T^n z),T(T^n x))) \leq \max\{\phi(d(T^n z,T^n x)), \frac{1}{2}[\phi(d(T^n z,T^{n+1}z))+\phi(d(T^n x,T^{n+1}x))], \frac{1}{2}[\phi(d(T^n z,T^{n+1}x))+\phi(d(T^n x,T^{n+1}z))]\},$$
$$= \max\{\phi(d(z,T^n x)), \frac{1}{2}[\phi(d(z,z))+ \phi(d(T^n x,T^{n+1}x))], \frac{1}{2}[\phi(d(z,T^{n+1}x))+\phi(d(T^n x,z))]\}$$
$$= \max\{\phi(d(z,T^n x)), \frac{1}{2}[\phi(d(T^n x,T^{n+1}x))], \frac{1}{2}[\phi(d(z,T^{n+1}x))+\phi(d(T^n x,z))]\}$$
$$\leq \max\{\phi(d(z,T^n x)), \frac{1}{2}\phi[d(T^n x,z)+ d(z,T^{n+1}x)], \frac{1}{2}[\phi(d(z,T^{n+1}x))+\phi(d(T^n x,z))]\}$$
$$\leq \max\{\phi(d(z,T^n x)), \frac{1}{2}\phi[2 \max\{d(T^n x,z), d(z,T^{n+1}x)\}], \frac{1}{2}[\phi(d(z,T^{n+1}x))+\phi(d(T^n x,z))]\}$$
$$\leq \max\{\phi(d(z,T^n x)), \frac{1}{2}[2\,\phi[\max\{d(T^n x,z), d(z,T^{n+1}x)\}], \frac{1}{2}[\phi(d(z,T^{n+1}x))+\phi(d(T^n x,z))]\}$$
$$= \max\{\phi(d(z,T^n x)), d(z,T^{n+1}x)\}.$$

If $\phi(d(z,T^n x)) < \phi(d(z,T^{n+1}x))$, then

$$\psi(d(z,T^{n+1}x)) \leq \phi(d(z,T^{n+1}x)) < \psi(d(z,T^{n+1}x)),$$

a contradiction.
Therefore, $\phi(d(z,T^n x))$ is the maximum. Hence

$$\psi(d(z,T^{n+1}x)) \leq \phi(d(z,T^n x)) < \psi(d(z,T^n x)) \text{ for all } n.$$

Since ψ is non-decreasing, it follows that
$d(z,T^{n+1}x) \leq d(z,T^n x)$.
Thus, $d(z,T^n x)$ is a decreasing sequence of non-negative real numbers.
Then there exists $r \geq 0$ such that $\lim_{n\to\infty} d(z,T^n x) = r$.
We show that $r = 0$.
If possible, suppose that $r > 0$. In this case, $\psi(r) \leq \psi(d(z,T^{n+1}x)) \leq \phi(d(z,T^n x))$.
On taking limit supremum as $n \to \infty$,
we have $\psi(r) \leq \phi(r) < \psi(r)$,
a contradiction.
Hence $r = 0$.
Thus $T^n x \to z$.

Theorem 2.5.: In addition to the hypotheses of Theorem 2.3, we assume the following condition (H):
(H): for each $y, z \in X$, there exists $x \in X$ which is comparable to y and z. Then T has a unique fixed point in X.

proof Suppose z and y are two fixed points of T with $z \neq y$. Here we consider two cases

<u>*Case*</u> (*i*): z is comparable to y. In this case, it is clear that $T^n z = z$ is comparable to $T^n y = y$. Now from the inequality (2.3.1), it follows that

$$\psi(d(z,y)) = \psi(d(T^n z,T^n y)) = \psi(d(T(T^{n-1}z),T(T^{n-1}y))$$
$$\leq \max\{\phi(d(T^{n-1}z,T^{n-1}y)), \frac{1}{2}[\phi(d(T^{n-1}z,T^n z))] +\phi(d(T^{n-1}y,T^n y))], \frac{1}{2}[\phi(d(T^{n-1}z,T^n y))+\phi(d(T^{n-1}y,T^n z))]\}$$
$$= \max\{\phi(d(z,y)), \frac{1}{2}[\phi(d(z,z))+\phi(d(y,y))], \frac{1}{2}[\phi(d(z,y)) +\phi(d(y,z))]\} = \max\{\phi(d(z,y)),\ \phi(0),\ \phi(d(z,y))\}$$
$$= \phi(d(z,y)) < \psi(d(z,y)),$$

a contradiction.
Therefore, $z = y$.

<u>*Case*</u> (*ii*): z is not comparable to y.
In this case, by the hypotheses (H) there exists $x \in X$ such that x is comparable to z and y.
Since z and y are fixed points of T and both are comparable to x, it follows by Lemma 2.4 $T^n x \to z$, and $T^n x \to y$.
Therefore by the uniqueness of limits, we have $z = y$.

Corollary 2.6. Let $(X, \preceq)$ be a partially ordered set, suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ and $\psi(t) > \phi(t)$ for all $t > 0$ such that for all comparable x and y

$$\psi(d(T(x),T(y))) \leq \phi(d(x,y)), \quad (2.6.1)$$

Further, assume that either (*i*) T is continuous or (*ii*) X satisfies condition (2.2.1).
If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

proof Since the inequality (2.6.1) implies the inequality (2.1.1), the conclusion of this corollary follows from Theorem 2.1, when T is continuous; when X satisfies condition (2.2.1) the conclusion follows from Theorem 2.2.

Corollary 2.7. In addition to the hypotheses of Corollary 2.6, we assume the following:
Condition (H): for each $y, z \in X$, there exists $x \in X$ which is comparable to y and z.
Then T has a unique fixed point in X.
proof Since the inequality (2.6.1) implies the inequality (2.3.1) the conclusion of this corollary follows from Theorem 2.3 and Theorem 2.5.

III. Examples

In this section we provide examples in support of the main results of section 2. Further, we provide examples to show that our results generalize Theorem 1.6 and Theorem 1.7 of section 1.
The following example is in support of Theorem 2.1.

Example 3.1. Let $X = l^2$, where $l^2 = \{\bar{x} \mid \bar{x} = (x_1,\ x_2,\ ...) \mid x_i \in \mathbb{R}$ for $i = 1,\ 2,\ ...$ and $\sum_{i=1}^{\infty} |x_i|^2 < \infty\}$. We define a metric d on X by
$d(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\| = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2\right)^{\frac{1}{2}}$, where
$\bar{x} = (x_1,\ x_2,\ ...) \in l^2$ and $\bar{y} = (y_1,\ y_2,\ ...) \in l^2$.
Then (X, d) is a complete metric space. Let

$$Y = \{(\frac{1}{4},\ 0, 0\ ,\ \ldots), (0,\ 0,\ \frac{1}{4},\ 0,\ \ldots), (\frac{1}{4},\ \frac{1}{2^2},\ \frac{1}{2^3},\ \ldots), \ldots,\ (\frac{1}{4},\ \frac{1}{2^n},\ \frac{1}{2^{n+1}},\ \ldots),\ \ldots\} \text{ for } n = 2,\ 3,\ \ldots\ .$$

Then Y is a closed subspace of X so that it is a complete metric space.
We now define a partial order $\preceq$ on Y by $\bar{x} \preceq \bar{y}$ if and only if $x_i \geq y_i$ for all $i = 1,\ 2,\ ...$ in the usual sense, where $\bar{x} = (x_1,\ x_2,\ ...) \in l^2$ and $\bar{y} = (y_1,\ y_2,\ ...) \in l^2$.
Then $(Y, d, \preceq)$ is a partial ordered complete metric space.
We define $T : Y \to Y$ by

$$T(\frac{1}{4},\ 0,\ 0,\ \ldots) = (\frac{1}{4},\ 0,\ 0,\ \ldots),$$

$$T(0,\ 0,\ \frac{1}{4},\ 0,\ \ldots) = (0,\ 0,\ \frac{1}{4},\ 0,\ \ldots),$$

$$T(\frac{1}{4},\ \frac{1}{2^n},\ \frac{1}{2^{n+1}},\ \ldots) = (\frac{1}{4},\ \frac{1}{2^{n+1}},\ \frac{1}{2^{n+2}},\ \ldots) \quad \text{for}$$

$n = 2,\ 3,\ \ldots$.
We choose $\bar{x}_0 = (\frac{1}{4},\ \frac{1}{2^2},\ \frac{1}{2^3},\ \ldots)$.
We have $T\bar{x}_0 = (\frac{1}{4},\ \frac{1}{2^3},\ \frac{1}{2^4},\ \ldots)$ so that $\bar{x}_0 \preceq T\bar{x}_0$. Also T is non-decreasing on Y.
We define functions $\psi, \phi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \begin{cases} 0 & \text{if } t = 0 \\ 2^{-n+2} & \text{if } 2^{-n} \leq t < 2^{-n+1} \quad \text{for } n = 2,\ 3,\ ... \\ 2 & \text{if } t \geq \frac{1}{2}, \end{cases}$$

and

$$\phi(t) = \begin{cases} 0 & \text{if } t = 0 \\ 2^{-n+1} & \text{if } 2^{-n} \leq t < 2^{-n+1} \text{ for } n = 2,\ 3,\ ... \\ 1 & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Then $\psi \in \Psi$ and $\phi \in \Phi$.
We now verify the inequality (2.1.1).
Let $\bar{x} = (\frac{1}{4},\ 0,\ 0,\ \ldots)$ and $\bar{y} = (\frac{1}{4},\ \frac{1}{2^n},\ \frac{1}{2^{n+1}},\ \ldots)$ for $n = 2,\ 3,\ ...,$ so that
$T\bar{x} = (\frac{1}{4},\ 0,\ 0,\ \ldots)$ and $T\bar{y} = (\frac{1}{4},\ \frac{1}{2^{n+1}},\ \frac{1}{2^{n+2}},\ \ldots)$.
Now,

$$\begin{aligned}
\psi(d(T\bar{x}, T\bar{y})) &= \psi(\|T\bar{x} - T\bar{y}\|) \\
&= \psi(\big[(\frac{1}{2^{n+1}})^2 + (\frac{1}{2^{n+2}})^2 + \ldots\big]^{\frac{1}{2}}) \\
&= \psi([\frac{1}{2^{2n+2}} + \frac{1}{2^{2n+4}} + \ldots]^{\frac{1}{2}}) \\
&= \psi([\frac{1}{2^2}\big(\frac{1}{2^{2n}} + \frac{1}{2^{2n+2}} + \ldots\big)]^{\frac{1}{2}}) \\
&= \psi(\frac{1}{2}[\frac{1}{2^n}\big(1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\big)^{\frac{1}{2}}]) \\
&= \psi(\frac{1}{2^{n+1}}[\frac{4}{3}]^{\frac{1}{2}}) = \psi(\frac{1}{\sqrt{3}}\frac{1}{2^n}) = 2^{-n+1} = \phi(\frac{2}{2^n}\frac{1}{\sqrt{3}}) \\
&= \phi([\frac{1}{2^{2n}} + \frac{1}{2^{2n+2}} + \ldots]^{\frac{1}{2}}) = \phi(\|\bar{x} - \bar{y}\|) = \phi(d(\bar{x}, \bar{y})) \\
&\leq \max\big\{\phi(d(\bar{x}, \bar{y})), \phi(d(\bar{x}, T\bar{x})), \phi(d(\bar{y}, T\bar{y})), \\
&\qquad \frac{1}{2}\big[\phi(d(\bar{x}, T\bar{y})) + \phi(d(\bar{y}, T\bar{x}))\big]\big\}
\end{aligned}$$

for all comparable $\bar{x},\ \bar{y} \in Y$.

This implies that the inequality (2.1.1) holds.
Since $\bar{x} = (\frac{1}{4},\ 0,\ 0,\ \ldots)$ and $\bar{y} = (0,\ 0,\ \frac{1}{4},\ 0,\ 0,\ \ldots)$ are not comparable and also
$\bar{x} = (0,\ 0,\ \frac{1}{4},\ 0,\ 0,\ \ldots)$ and $\bar{y} = (\frac{1}{4},\ \frac{1}{2^n},\ \frac{1}{2^{n+1}},\ \ldots)$ are not comparable,
so we need not verify the inequality for these elements.
Hence T, ψ and ϕ satisfy all the hypotheses of Theorem 2.1 on Y.
Therefore, by Theorem 2.1 T has a fixed point in Y.
Here, we observe that $\bar{x} = (\frac{1}{4}, 0, 0, \ldots)$ and $\bar{y} = (0, 0, \frac{1}{4}, \ldots)$ are two fixed points of T in Y.

Remark 3.2. Now the following question is natural. Can we replace ψ and ϕ of Theorem 2.1 by a single function ψ_1 having the properties of ψ and ϕ?
i.e, Can we replace the inequality (2.1.1) by the following inequality:

$$\psi_1(d(T(x), T(y))) < \max\big\{\psi_1(d(x, y)), \psi_1(d(x, Tx)), \psi_1(d(y, Ty)), \frac{1}{2}\big[\psi_1(d(x, Ty)) + \psi_1(d(y, Tx))\big]\big\}$$

for all comparable x and y with $x \neq y$ where $\psi_1 \in \Psi \cup \Phi$. (3.2.1)

The following example suggests that its answer is 'No'.

Example 3.3. Let $C = \{x \mid x = (x_1,\ x_2,\ x_3,\ \ldots) \mid \{x_i\}_{i=1}^{\infty}$ converges$\}$. For arbitrary $x \in C$ we define a norm on C by

$$\|x\|_C = \sup_{i\ \geq\ 1} |x_i| \ .$$

Then C is complete.
Let $C_0 = \{x \mid x = (x_1,\ x_2,\ x_3,\ \ldots) \mid x_n \to 0 \text{ as } n \to \infty\}$.
i.e,
$C_0 = \{(1,\ \frac{1}{2},\ \frac{1}{2^2},\ \ldots),\ (1,\ \frac{1}{3},\ \frac{1}{3^2},\ \ldots),\ \ldots,\ (1,\ \frac{1}{n},\ \frac{1}{n^2},\ \ldots),\ \ldots\ |n = 2,\ 3,\ \ldots\ \}$.
For arbitrary $x \in C_0$ we define a norm on C_0 by

$$\|x\|_{C_0} = \sup_{i\ \geq\ 1} |x_i| \ .$$

Since C_0 is a closed subspace of C, C_0 is complete.
We define a partial order $\preceq$ on C_0 by $x \preceq y$ if and only if $x_i \leq y_i$ for all i in the usual sense, where
$x = (x_1,\ x_2,\ x_3,\ \ldots) \in C_0$ and
$y = (y_1,\ y_2,\ y_3,\ \ldots) \in C_0$.
We now define $T : C_0 \to C_0$ by
$T(x_1,\ x_2,\ x_3,\ \ldots) = (1,\ x_1,\ x_2,\ x_3,\ \ldots)$.
We define a function $\psi_1 : [0, \infty) \to [0, \infty)$ by

$$\psi_1(t) = \begin{cases} 0 & \text{if } t = 0 \\ 2^{-n+2} & \text{if } 2^{-n} \leq t < 2^{-n+1} \text{ for } n = 2,\ 3,\ \ldots \\ 2 & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Choose $x_0 = (1,\ \frac{1}{2},\ \frac{1}{2^2},\ \ldots)$, we have
$Tx_0 = (1,\ 1,\ \frac{1}{2},\ \frac{1}{2^2},\ \ldots)$ and $x_0 \preceq Tx_0$.

We now verify the inequality (3.2.1). The following are the possible cases.
<u>*Case*</u> (*i*): $x = (1,\ \frac{1}{2},\ \frac{1}{2^2},\ \ldots)$, $y = (1,\ \frac{1}{n},\ \frac{1}{n^2},\ \ldots)$
n = 3, 4, ..., .
In this case $Tx = (1,\ 1,\ \frac{1}{2},\ \frac{1}{2^2},\ \ldots)$ and
$Ty = (1,\ 1,\ \frac{1}{n},\ \frac{1}{n^2},\ \ldots)$. Now,

$$\psi_1(d(Tx, Ty)) = \psi_1(sup|Tx - Ty|)$$
$$= \psi_1(\sup\ \{|\frac{1}{2} - \frac{1}{n}|,\ |\frac{1}{2^2} - \frac{1}{n^2}|, \ldots\})$$
$$= \psi_1(\frac{1}{2} - \frac{1}{n}) < \psi_1(\frac{1}{2}) = \psi_1(\sup\ \{|\frac{1}{2} - 1|,\ |\frac{1}{2^2} - \frac{1}{2}|, \ldots\})$$
$$= \psi_1(\ \sup |x - Tx|)\ = \psi_1(d(x, Tx)).$$

This implies

$$\psi_1(d(T(x), T(y))) < \psi_1(d(x, Tx)) \leq \max\big\{\psi_1(d(x, y)),$$
$$\psi_1(d(x, Tx)), \psi_1(d(y, Ty), \frac{1}{2}\big[\psi_1(d(x, Ty)) + \psi_1(d(y, Tx))\big]\big\}$$

for all comparable x and y.
<u>*Case*</u> (*ii*):

$$x_n = (1,\ \frac{1}{n},\ \frac{1}{n^2}, \ldots) \text{ and } x_{n+1} = (1,\ \frac{1}{n+1},\ \frac{1}{(n+1)^2}, \ldots)$$

n = 2, 3, ... Here $Tx_n = (1,\ 1,\ \frac{1}{n},\ \frac{1}{n^2},\ \ldots)$
$Tx_{n+1} = (1,\ 1,\ \frac{1}{n+1},\ \frac{1}{(n+1)^2},\ \ldots)$ $n = 2,\ 3,\ \ldots$.
Here we have,

$$\psi_1(d(Tx_n, Tx_{n+1})) = \psi_1(\sup\ \{|\frac{1}{n} - \frac{1}{n+1}|,$$
$$|\frac{1}{n^2} - \frac{1}{(n+1)^2}|, \ldots\}) = \psi_1(\frac{1}{n} - \frac{1}{n+1}) < \psi_1(1 - \frac{1}{n})$$
$$= \psi_1(\sup\ \{|\frac{1}{n} - 1|, |\frac{1}{n^2} - \frac{1}{2}|, \ldots\}) = \psi_1(d(x_n, Tx_n))$$
$$\leq \max\{\psi_1(d(x_n, x_{n+1})),\ \psi_1(d(x_n, Tx_n)),$$
$$\psi_1(d(x_{n+1}, Tx_{n+1})), \frac{1}{2}[\psi_1(d(x_n, Tx_{n+1}))$$
$$+\psi_1(d(x_{n+1}, Tx_n))]\} \text{for all comparable } x_n \text{ and } x_{n+1}.$$

<u>*Case*</u> (*iii*): $x_n = (1,\ \frac{1}{n},\ \frac{1}{n^2},\ \ldots)$, $n = 2,\ 3,\ \ldots$,
$x_m = (1,\ \frac{1}{m},\ \frac{1}{m^2},\ \ldots)$, $m = 2,\ 3,\ \ldots$ and $n \geq m$.
Here $Tx_n = (1,\ 1, \frac{1}{n},\ \frac{1}{n^2},\ \ldots)$ and
$Tx_m = (1,\ 1,\ \frac{1}{m},\ \frac{1}{m^2},\ \ldots)$.
Therefore

$$\psi_1(d(Tx_n, Tx_m)) = \psi_1(\sup\{|\frac{1}{n} - \frac{1}{m}|, |\frac{1}{n^2} - \frac{1}{m^2}|, \ldots\})$$
$$= \psi_1(\frac{1}{n} - \frac{1}{m} <\ \psi_1(1 - \frac{1}{n})$$
$$= \psi_1(1 - \frac{1}{n}) = \psi_1(\sup\{|\frac{1}{n} - 1|, |\frac{1}{n^2} - \frac{1}{2}|, \ldots\}) = \psi_1(d(x_n, Tx_n)),$$

Then we have

$$\psi_1(d(Tx_n, Tx_m)) < \psi_1(d(x_n, Tx_n)),$$

which implies that

$$\psi_1(d(Tx_n, Tx_m)) < \psi_1(d(x_n, Tx_n)) \leq \max\big\{\psi_1(d(x_n, x_m)),$$
$$\psi_1(d(x_n, Tx_n)), \psi_1(d(x_m, Tx_m)),$$
$$\frac{1}{2}\big[\psi_1(d(x_n, Tx_m)) + \psi_1(d(x_m, Tx_n))\big]\big\}$$

for all comparable x_n and x_m.

From all the above three cases, it follows that T satisfies inequality (3.2.1).

Here we observe that T satisfies the remaining hypotheses of Theorem 2.1. But T has no fixed point in C_0.

The following example suggests that Theorem 2.2 do not guarantee the uniqueness of fixed point.

Example 3.4. Let $X = [0, 1]$. We define a partial order $\preceq$ on X by

$$\preceq := \{(x, y) \in X \times X \mid x = y\} \cup \{(x_n, x_m) \mid x_n = \frac{1}{2^{n+2}}$$
$$n, m = 0,\ 1,\ 2\ \ldots\ ,\ m \geq n\big\} \cup \{(\frac{1}{2^{n+2}}, 0) :\ n = 0,\ 1,\ 2\ \ldots\ \},$$

where $x \preceq y$ if and only if $x \geq y$ in the usual sense.

We define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

We choose $x_0 = \frac{1}{4}$, we have $T(\frac{1}{4}) = \frac{1}{2^3} = x_1$,
$T(\frac{1}{2^3}) = \frac{1}{2^4} = x_2$, ..., and so we define a sequence
$x_n = \frac{1}{2^{n+2}}$ for $n = 0,\ 1,\ 2,\ \ldots$.
By mathematical induction, we have

$$x_0 \preceq Tx_0 \preceq \ldots \preceq T^n x_0 \preceq T^{n+1} x_0 \preceq \ldots \ .$$

We define functions $\psi, \phi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \frac{t}{1+t} \text{ for all } t \geq 0$$

and

$$\phi(t) = \begin{cases} 0 & \text{if } \ t = 0 \\ 2^{-n-3} & \text{if } \ 2^{-n-2} \le t < 2^{-n-1} \text{ for } n = 0,\ 1,\ ... \\ \frac{1}{4} & \text{if } \ t \ge \frac{1}{2}. \end{cases}$$

We now verify the inequality (2.2.2).
Case (*i*): $(x, y) = (x_n, 0)$, where $x_n = \frac{1}{2^{n+2}}$ for $n = 0,\ 1,\ 2,\ \dots$.
In this case, we have

$$\psi(d(Tx_n), T(0))) = \psi(d(x_{n+1}, 0)) = \psi[\frac{1}{2^{n+3}} - 0]$$
$$= \psi[\frac{1}{2^{n+3}}] = \frac{1}{2^{n+3}+1} \le \frac{1}{2^{n+3}}$$
$$= \phi(|\frac{1}{2^{n+2}} - 0|) = \phi(d(x_n, 0))$$

From this we have

$$\psi(d(Tx_n), T(0)) = \phi(d(x_n, 0)) \le \max\ \{\phi(d(x_n, 0)), \phi(d(x_n, Tx_n)), \phi(d(0, T(0))), \frac{1}{2}[\phi(d(x_n, T(0))) + \phi(d(0, Tx_n))]\}.$$

Case (*ii*): $(x, y) = (x_n, x_{n+1})$ $n = 0,\ 1,\ 2,\ \dots$, so that $Tx_n = x_{n+1}$ and $Tx_{n+1} = x_{n+2}$. Now,

$$\psi(d(Tx_n, Tx_{n+1})) = \psi(d(x_{n+1}, x_{n+2})) = \psi\left[\frac{1}{2^{n+3}} - \frac{1}{2^{n+4}}\right]$$
$$= \psi\left[\frac{1}{2^{n+4}}\right] = \frac{1}{2^{n+4}+1} \le \frac{1}{2^{n+4}} = 2^{-n-4}$$
$$= \phi\left[\frac{1}{2^{n+3}}\right] = \phi\left[\frac{1}{2^{n+2}} - \frac{1}{2^{n+3}}\right] = \phi(d(x_n, Tx_n)).$$

In this case, we have

$$\psi(d(Tx_n, Tx_{n+1})) = \phi(d(x_n, Tx_n)) \le \max\ \{\phi(d(x_n, x_{n+1})), \phi(d(x_n, Tx_n)), \phi(d(x_{n+1}, Tx_{n+1})), \frac{1}{2}[\phi(d(x_n, Tx_{n+1})) + \phi(d(x_{n+1}, Tx_n))]\}.$$

Case (*iii*): $(x, y) = (x_n, x_m)$ $n = 0,\ 1,\ 2,\ \dots$, $m = 0,\ 1,\ 2,\ \dots$, $m \ge n$.
Here we have $Tx_n = x_{n+1}$ and $Tx_m = x_{m+1}$. Now,

$$\psi(d(Tx_n, Tx_m)) = \psi(d(x_{n+1}, x_{m+1})) = \psi[\frac{1}{2^{n+3}} - \frac{1}{2^{m+3}}]$$
$$= \psi[\frac{2^m - 2^n}{2^{n+m+3}}] = \psi[\frac{2^{m-3} - 2^{n-3}}{2^{n+m}}].$$

and

$$\phi(d(x_n, x_m)) = \phi[\frac{1}{2^{n+2}} - \frac{1}{2^{m+2}}] = \phi[\frac{2^m - 2^n}{2^{n+m+2}}]$$
$$= \phi[\frac{2^{m-2} - 2^{n-2}}{2^{n+m}}].$$

Let $t = \dfrac{2^{m-3} - 2^{n-3}}{2^{n+m}}$, we have

$$2t = \frac{2^{m-2} - 2^{n-2}}{2^{n+m}}.$$

Since $t \in (0, 1)$, then there exists $k \in \mathbb{N}$ such that

$$2^{-k-2} \le t < 2^{-k-1} \text{ and } 2^{-k-1} \le 2t < 2^{-k}.$$

Therefore,

$$\psi(t) = \frac{t}{1+t} = \frac{2^{-k-2}}{1+2^{-k-2}} \le 2^{-k-1} = \phi(2t) = 2^{-k-1},$$

$$\phi(t) = 2^{-k-2} \text{ and } \phi(2t) = 2^{-k-1} = 2\phi(t). \tag{3.4.1}$$

From (3.4.1), we have

$$\phi \in \Phi \text{ and } \psi(d(Tx_n,\ Tx_m)) = \phi(d(x_n,\ x_m)) \le \max\ \{\phi(d(x_n,\ x_m)), \phi(d(x_n,\ Tx_n)), \phi(d(x_m,\ Tx_m)), \frac{1}{2}[\phi(d(x_n,\ Tx_m)) + \phi(d(x_m,\ Tx_n))]\}$$

for all comparable x_n and x_m.
Now we define a sequence $x_n = \frac{1}{2^{n+3}}$ $n = 1,\ 2\ \dots$.
Since $x_n \ge x_{n+1}$, we have $x_n \preceq x_{n+1}$ so that $\{x_n\}$ is a non-decreasing sequence and $x_n \preceq 0$ for all $n \ge 1$.

Hence $T,\ \phi$ and ψ satisfy all the hypotheses of Theorem 2.2 and T has fixed points 0 and $\frac{1}{2}$.

Remark 3.5. As in Remark 3.2, if we replace the inequality (2.2.2) by (3.2.1) in Theorem 2.2, then the conclusion of Theorem 2.2 may fail to hold.
i.e in this case T may not have a fixed point. To justify this remark we provide an example in the following:

Example 3.6. Let $X = l^2$, where
$l^2 = \{\bar{x} \mid \bar{x} = (x_1,\ x_2,\ ...) \mid x_i \in \mathbb{R}$ for $i = 1,\ 2,\ ...$ and $\sum_{i=1}^{\infty} |x_i|^2 < \infty\}$.
We define a metric d on X by
$d(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\| = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2\right)^{\frac{1}{2}}$, where
$\bar{x} = (x_1,\ x_2,\ ...) \in l^2$
and $\bar{y} = (y_1,\ y_2,\ ...) \in l^2$. Then (X, d) is a complete metric space.
Let $Y = \{(0,\ 0,\ \dots)\} \cup \{(\frac{1}{2^n},\ \frac{1}{2^{n+1}}.\ \dots) \mid n = 2,\ 3,\ ,\ \dots\}$.
Then Y is a complete metric space.
We define a partial ordered $\preceq$ on Y by $x \preceq y$ if and only if $x_i \ge y_i$ for all i in the usual sense.
Then $(Y, d, \preceq)$ is a partial ordered complete metric space.
We now define $T : Y \to Y$ by

$$T(0,\ 0,\ \dots) = (\frac{1}{2^2},\ \frac{1}{2^3},\ \frac{1}{2^4},\ \dots),$$

$$T(\frac{1}{2^n},\ \frac{1}{2^{n+1}},\ \dots) = (\frac{1}{2^{n+1}},\ \frac{1}{2^{n+2}},\ \dots) \text{ for } n = 2,\ 3,\ \dots\ .$$

We define a function $\psi_1 : [0, \infty) \to [0, \infty)$ by

$$\psi_1(t) = \frac{t}{1+t}.$$

We choose $x_0 = (\frac{1}{2^2},\ \frac{1}{2^3},\ \dots)$ we have $Tx_0 = (\frac{1}{2^3},\ \frac{1}{2^4},\ \dots)$. This implies $x_0 \preceq Tx_0$
Now we verify the inequality (3.2.1)

Case (i): $x = (0,\ 0,\ \ldots)$ and $y = (\frac{1}{2^2},\ \frac{1}{2^2},\ \ldots)$, then we have

$$\psi_1\big(d(T(x), T(y))\big) = \psi_1\big(\ |\frac{1}{2^2} - \frac{1}{2^3}|^2 + |\frac{1}{2^3} - \frac{1}{2^4}|^2 + \ldots\big)^{\frac{1}{2}}$$

$$= \psi_1\big(\frac{1}{2^6} + \frac{1}{2^8} + \ldots\big)^{\frac{1}{2}} = \psi_1(\frac{1}{4\sqrt{3}}) = \frac{1}{1+4\sqrt{3}} < \frac{1}{1+2\sqrt{3}}$$

$$= \psi_1(\frac{1}{2\sqrt{3}}) = \psi_1\big(\frac{1}{2^4} + \frac{1}{2^6} + \ldots\big)^{\frac{1}{2}} = \psi_1(d(x,y))$$

Case (ii): $x = (\frac{1}{2^n},\ \frac{1}{2^{n+1}},\ \ldots)$, $y = (\frac{1}{2^m},\ \frac{1}{2^{m+1}},\ \ldots)$, $n,\ m = 2,\ 3,\ \ldots$ with $m \geq n$.

In this case,

$$\psi_1[d(T(x), T(y))]$$

$$= \psi_1[(|\frac{1}{2^{n+1}} - \frac{1}{2^{m+1}}|)^2 + (|\frac{1}{2^{n+2}} - \frac{1}{2^{m+2}}|)^2 + \ldots]^{\frac{1}{2}}$$

$$= \psi_1\big[\big(\frac{2^m - 2^n}{2^{n+m+1}}\big)^2 + \big(\frac{2^m - 2^n}{2^{n+m+2}}\big)^2 + \ldots\big]^{\frac{1}{2}}$$

$$= \psi_1\big[\big(\frac{2^m - 2^n}{2^{n+m}}\big)\big((\frac{1}{2})^2 + (\frac{1}{2^2})^2 + \ldots\big)^{\frac{1}{2}}\big]$$

$$= \psi_1\big[\big(\frac{2^m - 2^n}{2^{n+m}}\big)\big(\frac{1}{\sqrt{3}}\big)\big] < \psi_1\big[\big(\frac{2^m - 2^n}{2^{n+m}}\big)\big(\frac{2}{\sqrt{3}}\big)\big]$$

$$= \psi_1\big[\big(\frac{2^m - 2^n}{2^{n+m}}\big)\big((1)^2 + (\frac{1}{2})^2 + \ldots\big)^{\frac{1}{2}}\big]$$

$$= \psi_1\big[\big(\frac{2^m - 2^n}{2^{n+m}}\big)^2 + \big(\frac{2^m - 2^n}{2^{n+m+1}}\big)^2 + \ldots\big]^{\frac{1}{2}}$$

$$= \psi_1[(|\frac{1}{2^n} - \frac{1}{2^m}|)^2 + (|\frac{1}{2^{n+1}} - \frac{1}{2^{m+1}}|)^2 + \ldots]^{\frac{1}{2}}$$

$$= \psi_1[d(x,y)].$$

From case (i) and case (ii), we have

$$\psi_1(d(Tx,\ Ty)) < \psi_1(d(x,\ y)) \leq \max\{\psi_1(d(x,\ y)), \psi_1(d(x,\ Tx)), \psi_1(d(y,\ Ty)), \frac{1}{2}[\psi_1(d(x,\ Ty)) + \psi_1(d(y,\ Tx))]\}$$

for all comparable x and y.

We define a sequence $x_n = (\frac{1}{2^n}, \frac{1}{2^{n+1}}, \ldots)$ $n = 0,\ 1,\ 2,\ \ldots$. Thus $x_n \preceq x_{n+1}$ for all $n \geq 0$ and $x_n \preceq \bar{0}$ for all $n \geq 0$ where $\bar{0} = (0,\ 0,\ 0,\ \ldots)$.

Hence T and ψ_1 satisfy all the hypotheses of Theorem 2.2, but T has no fixed point. This example justifies why we used two distinct functions.

The following example is in support of Theorem 2.5.

Example 3.7.
Let $X = \{\frac{1}{2^{n+2}} \mid n = 0,\ 1,\ 2,\ \ldots\ .\} \cup \{0,\ 1,\ 2\}$.
We define a partial order $\preceq$ on X by

$$\preceq := \{(x,y) \in X \times X \mid x = y\} \cup \{(x_n, x_m) \mid x_n = \frac{1}{2^{n+2}}\ \ n = 0,\ 1,\ 2,\ \ldots,\ m = 0,\ 1,\ 2,\ \ldots,\ m \geq n\} \cup \{(\frac{1}{2^{n+2}}, 0) \mid n = 0,\ 1,\ 2,\ \ldots\} \cup \{(1,\ 0), (2,\ 0)\},$$

where $x \preceq y$ if and only if $x \geq y$ in the usual sense.
We define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \in \{\frac{1}{2^{n+2}}\ \ n = 0,\ 1,\ 2, \ldots,\ \} \cup \{0\} \\ \frac{1}{4} & \text{if } x \in \{1, 2\}. \end{cases}$$

We choose $x_0 = \frac{1}{4}$ and we have $T(\frac{1}{4}) = \frac{1}{2^3}$. This implies $x_0 \preceq Tx_0$.

We define functions $\psi, \phi : [0, \infty) \to [0, \infty)$ as in Example 3.1. We now verify the inequality (2.3.1)

Case (i): $(x,\ y) = (1,\ 0)$, we have

$$\psi(d(T(1),\ T(0))) = \psi(d(\frac{1}{4},\ 0)) = \psi(\frac{1}{4}) = 1 = \phi(1) = \phi(d(1,\ 0)).$$

Case (ii): $(x,\ y) = (2,\ 0)$, so that

$$\psi(d(T(2),\ T(0))) = \psi(d(\frac{1}{4},\ 0)) = \psi(\frac{1}{4}) = 1 = \phi(2) = \phi(d(2, 0)).$$

Case (iii): $(x, y) = (x_n, 0)$,where $x_n = \frac{1}{2^{n+2}}|$ $n = 0,\ 1,\ 2,\ \ldots$.

In this case, we have

$$\psi(d(Tx_n), T(0))) = \psi(d(x_{n+1}, 0)) = \psi\big[\frac{1}{2^{n+3}} - 0]$$

$$= \psi\big[\frac{1}{2^{n+3}}\big] = 2^{-n-3+2} = 2^{-n-1} = 2^{-n-2+1}$$

$$= \phi(d(\frac{1}{2^{n+2}} - 0)) = \phi(d(x_n, 0).$$

From this we have,

$$\psi(d(Tx_n), T(0)) = \phi(d(x_n, 0)) \leq \max\{\phi(d(x_n, 0)), \frac{1}{2}[\phi(d(x_n, Tx_n)) + \phi(d(0, T(0)))], \frac{1}{2}[\phi(d(x_n, T(0))) + \phi(d(0, Tx_n))]\}.$$

Case (iv): $(x, y) = (x_n, x_{n+1})$ $n = 0,\ 1,\ 2,\ \ldots$.

In this case, we have

$$\psi(d(Tx_n, Tx_{n+1})) = \psi(d(x_{n+1}, x_{n+2})) = \psi[\frac{1}{2^{n+3}} - \frac{1}{2^{n+4}}]$$

$$= \psi\big[\frac{1}{2^{n+4}}\big] = 2^{-n-4+2} = 2^{-n-2} = 2^{-n-3+1}$$

$$= \phi\big[\frac{1}{2^{n+3}}\big] = \phi\big[\frac{1}{2^{n+2}} - \frac{1}{2^{n+3}}\big] = \phi(d(x_n, x_{n+1}))$$

Case (v) $(x, y) = (x_n, x_m)$ $n = 0,\ 1,\ 2,\ \ldots,$ $m = 0,\ 1,\ 2,\ \ldots,$ $m \geq n$.

In this case, we have

$$\psi(d(Tx_n, Tx_m)) = \psi(d(x_{n+1}, x_{m+1})) = \psi\big[\frac{1}{2^{n+3}} - \frac{1}{2^{m+3}}\big]$$

$$= \psi\big[\frac{2^m - 2^n}{2^{n+m+3}}\big] = \psi\big[\frac{2^{m-3} - 2^{n-3}}{2^{n+m}}\big].$$

and

$$\phi(d(x_n, x_m)) = \phi\big[\frac{1}{2^{n+2}} - \frac{1}{2^{m+2}}\big] = \phi\big[\frac{2^m - 2^n}{2^{n+m+2}}\big]$$

$$= \phi\big[\frac{2^{m-2} - 2^{n-2}}{2^{n+m}}\big].$$

Let $t = \frac{2^{m-3}-2^{n-3}}{2^{n+m}}$, we have $2t = \frac{2^{m-2}-2^{n-2}}{2^{n+m}}$.
Since $t \in (0,1)$, then there exists $k \in \mathbb{N}$ such that $2^{-k} \leq t < 2^{-k+1}$ and $2^{-k+1} \leq 2t < 2^{-k+2}$.
Therefore

$$\psi(t) = 2^{-k+2},\ \phi(2t) = 2^{-k+2},\ \phi(t) = 2^{-k+1} \text{ and}$$
$$\phi(2t) = 2^{-k+2} = 2\phi(t). \tag{3.7.1}$$

From (3.7.1) we have $\psi\big[\frac{2^{m-3}-2^{n-3}}{2^{n+m}}\big] = \phi\big[\frac{2^{m-2}-2^{n-2}}{2^{n+m}}\big]$ and $\phi \in \Phi$.
From all the above five cases, we have

$$\psi(d(T(x),T(y))) < \phi(d(x,y)) \leq \max\ \{\phi(d(x,y)), \frac{1}{2}[\phi(d(x,Tx))+\phi(d(y,Ty))], \frac{1}{2}[\phi(d(x,Ty))+\phi(d(y,Tx))]\}$$

for all comparable x and y.

If we take $x = 1,\ y = 2$ which are not comparable, then there exists $z = 0$ such that all the hypotheses of Theorem 2.5 are satisfied, then T has a unique fixed point $z = 0$.

Remark 3.8. Since the inequality (1.6.1) implies the inequality (2.1.1), by Theorem 2.1 the conclusion of Theorem 1.6 holds so that Theorem 1.6 follows as a corollary to Theorem 2.1. In the following, we provide an example for which the inequality (1.6.1) of Theorem 1.6 fails to hold.

Example 3.9. Let $X = [0,\ \infty)$.
We define a partial order $\preceq$ on X by

$$\preceq := \big\{(x,y) \in X \times X \mid x = y\big\} \cup \big\{(x_n, x_m), (\frac{1}{3\ 2^n}, x_n), (\frac{1}{3\ 2^n}, x_{n+1}) \mid x_n = \frac{1}{2^{n+2}}, n, m = 0,\ 1,\ 2,\ \ldots,\quad m \geq n\big\} \cup \big\{(\frac{13}{12}, 1), (\frac{2}{3}, \frac{1}{2})\big\},$$

where $x \preceq y$ if and only if $x \geq y$ in the usual sense.
We define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,1], \\ 2x - \frac{3}{2}, & \text{if } x \in [1, \frac{13}{12}] \\ \frac{2}{3}, & \text{if } x \geq \frac{13}{12}. \end{cases}$$

Clearly, T is non-decreasing and continuous on X.
We choose $x_0 = \frac{1}{4}$, and we have $T(\frac{1}{4}) = \frac{1}{2^3}$ which implies that, $x_0 \preceq Tx_0$.
We define functions $\psi, \phi : [0,\infty) \to [0,\infty)$ as in Example 3.1. Then $\psi \in \Psi$ and $\phi \in \Phi$.
We now verify the inequality (2.1.1).
Case (*i*): $(x,y) = (\frac{13}{12}, 1)$.

$$\psi(d(Tx,Ty)) = \psi(d(T(\frac{13}{12}), T(1))) = \psi(|\frac{2}{3} - \frac{1}{2}|) = \psi(\frac{1}{6}) = \frac{1}{2} = \phi(\frac{5}{12}) = \phi(|\frac{13}{12} - \frac{2}{3}|) = \phi(d(x,Tx,)).$$

In this case, we have

$$\psi(d(T(x),T(y))) = \phi(d(x,Tx)) \leq \max\ \{\phi(d(x,y)), \phi(d(x,Tx)), \phi(d(y,Ty)), \frac{1}{2}[\phi(d(x,Ty)) + \phi(d(y,Tx))]\},$$

where $(x,y) = (\frac{13}{12}, 1)$.
Case (*ii*): $(x,y) = (\frac{2}{3}, \frac{1}{2})$.

$$\begin{aligned}\psi(d(Tx,Ty)) &= \psi(d(T(\frac{2}{3}), T(\frac{1}{2}))) = \psi(d(\frac{1}{3}, \frac{1}{4})) \\ &= \psi(\frac{1}{12}) = \frac{1}{4} = \phi(\frac{1}{6}) = \phi(d(\frac{2}{3}, \frac{1}{2})) \\ &\leq \max\big\{\phi(d(\frac{2}{3}, \frac{1}{2}))\ , \phi(d(\frac{2}{3}, T(\frac{2}{3}))), \phi(d(\frac{1}{2}, T(\frac{1}{2})), \\ &\quad \frac{1}{2}\big[\phi(d(\frac{1}{2}, T(\frac{2}{3}))) + \phi(d(\frac{2}{3}, T(\frac{1}{2})))\big]\big\}.\end{aligned}$$

Case (*iii*): $(x,y) = (x_n, x_{n+1})$ $n = 0,\ 1,\ \ldots$. Here $Tx_n = x_{n+1}$ and $Tx_{n+1} = x_{n+2}$. Now,

$$\begin{aligned}\psi(d(Tx_n, Tx_{n+1})) &= \psi(d(x_{n+1}, x_{n+2})) = \psi\big[\frac{1}{2^{n+3}} - \frac{1}{2^{n+4}}\big] \\ &= \psi\big[\frac{1}{2^{n+4}}\big] = 2^{-n-4+2} = 2^{-n-2} = 2^{-n-3+1} = \phi\big[\frac{1}{2^{n+3}}\big] \\ &= \phi\big[\frac{1}{2^{n+2}} - \frac{1}{2^{n+3}}\big] = \phi(d(x_n, Tx_n)).\end{aligned}$$

In this case, we have

$$\psi(d(Tx_n, Tx_{n+1})) = \phi(d(x_n, Tx_n)) \leq \max\ \{\phi(d(x_n, x_{n+1})), \phi(d(x_n, Tx_n)), \phi(d(x_{n+1}, Tx_{n+1})), \frac{1}{2}[\phi(d(x_n, Tx_{n+1})) + \phi(d(x_{n+1}, Tx_n))]\}.$$

Case (*iv*): $(x,y) = (x_n, x_m)$ $n = 0,\ 1,\ 2,\ \ldots$, $m = 0,\ 1,\ 2,\ \ldots$, $m \geq n$, $Tx_n = x_{n+1}$ and $Tx_m = x_{m+1}$.
Now,

$$\begin{aligned}\psi(d(Tx_n, Tx_m)) &= \psi(d(x_{n+1}, x_{m+1})) = \psi[\frac{1}{2^{n+3}} - \frac{1}{2^{m+3}}] \\ &= \psi[\frac{2^m - 2^n}{2^{n+m+3}}] = \psi\big[\frac{2^{m-3} - 2^{n-3}}{2^{n+m}}].\end{aligned}$$

and

$$\begin{aligned}\phi(d(x_n, x_m)) &= \phi[\frac{1}{2^{n+2}} - \frac{1}{2^{m+2}}] = \phi[\frac{2^m - 2^n}{2^{n+m+2}}] \\ &= \phi[\frac{2^{m-2} - 2^{n-2}}{2^{n+m}}].\end{aligned}$$

$$\text{Let } t = \frac{2^{m-3} - 2^{n-3}}{2^{n+m}}, \text{ we have } 2t = \frac{2^{m-2} - 2^{n-2}}{2^{n+m}}.$$

Since $t \in (0,1)$, then there exists $k \in \mathbb{N}$ such that

$$2^{-k} \leq t < 2^{-k+1} \text{ and } 2^{-k+1} \leq 2t < 2^{-k+2}.$$

Therefore

$$\psi(t) = 2^{-k+2},\ \phi(2t) = 2^{-k+2},\ \phi(t) = 2^{-k+1} \text{ and}$$
$$\phi(2t) = 2^{-k+2} = 2\phi(t). \tag{3.9.1}$$

From (3.9.1), we have $\phi \in \Phi$ and

$$\psi(d(Tx_n,\ Tx_m)) = \phi(d(x_n,\ x_m)) \leq \max\ \{\phi(d(x_n,\ x_m)), \phi(d(x_n,\ Tx_n)),\ \phi(d(x_m,\ Tx_m)), \frac{1}{2}[\phi(d(x_n,\ Tx_m)) + \phi(d(x_m,\ Tx_n))]\}$$

for all comparable $x_n$and x_m.
Case (v): $(x,y)=(\frac{1}{3\ 2^n},\frac{1}{2^{n+2}})$ $n=0,\ 1,\ 2,\ \dots$.
Now,

$$\psi(d(T(\frac{1}{3\ 2^n}),T(\frac{1}{2^{n+2}})))=\psi\big[|\frac{1}{3\ 2^{n+1}}-\frac{1}{2^{n+3}}|\big]$$
$$=\psi\big[\frac{1}{3\ 2^{n+3}}\big]=2^{-n-5+2}=2^{-n-3}=2^{-n-4+1}$$
$$=\phi\big[\frac{1}{3\ 2^{n+2}}\big]=\phi\big[|\frac{1}{3\ 2^n}-\frac{1}{2^{n+2}}|\big]=\phi(d(\frac{1}{3\ 2^n},\frac{1}{2^{n+2}})).$$

This implies that

$$\psi(d(T(\frac{1}{3\ 2^n}),T(\frac{1}{2^{n+2}})))=\phi(d(\frac{1}{3\ 2^n},\frac{1}{2^{n+2}}))$$
$$\leq \max\ \{\phi(d(\frac{1}{3\ 2^n},\frac{1}{2^{n+2}})),\phi(d(\frac{1}{3\ 2^n},T(\frac{1}{3\ 2^n}))),$$
$$\phi(d(\frac{1}{2^{n+2}},T(\frac{1}{2^{n+2}}))),\frac{1}{2}[\phi(d(\frac{1}{2^{n+2}},T(\frac{1}{3\ 2^n})))$$
$$+\phi(d(\frac{1}{3\ 2^n},T(\frac{1}{2^{n+2}})))]\}.$$

Case (vi): $(x,y)=(\frac{1}{3\ 2^n},\frac{1}{2^{n+3}})$ $n=0,\ 1,\ 2,\ \dots$. Now,

$$\psi(d(T(\frac{1}{3\ 2^n}),T(\frac{1}{2^{n+3}})))=\psi\big[|\frac{1}{3\ 2^{n+1}}-\frac{1}{2^{n+4}}|\big]$$
$$=\psi\big[\frac{5}{3\ 2^{n+4}}\big]=2^{-n-4+2}=2^{-n-2}=2^{-n-3+1}$$
$$=\phi\big[\frac{5}{3\ 2^{n+3}}\big]=\phi\big[|\frac{1}{3\ 2^n}-\frac{1}{2^{n+3}}|\big]=\phi(d(\frac{1}{3\ 2^n},\frac{1}{2^{n+3}})).$$

This implies that

$$\psi(d(T(\frac{1}{3\ 2^n}),T(\frac{1}{2^{n+3}})))=\phi(d(\frac{1}{3\ 2^n},\frac{1}{2^{n+3}}))$$
$$\leq \max\ \{\phi(d(\frac{1}{3\ 2^n},\frac{1}{2^{n+3}})),\phi(d(\frac{1}{3\ 2^n},T(\frac{1}{3\ 2^n}))),$$
$$\phi(d(\frac{1}{2^{n+3}},T(\frac{1}{2^{n+3}}))),\frac{1}{2}[\phi(d(\frac{1}{2^{n+3}},T(\frac{1}{3\ 2^n})))$$
$$+\phi(d(\frac{1}{3\ 2^n},T(\frac{1}{2^{n+3}})))]\}.$$

Hence T, ϕ and ψ satisfy all the hypotheses of Theorem 2.1 and T has a fixed point 0 in X.

Now we show that the inequality of Theorem 1.6 fails to hold. For, let $(x,y)=(\frac{13}{12},1)$.

$$\psi(d(Tx,Ty))=\psi(d(T(\frac{13}{12}),T(1)))=\psi(|\frac{2}{3}-\frac{1}{2}|)$$
$$=\psi(\frac{1}{6})\nleq\phi(\frac{1}{12})=\phi(|\frac{13}{12}-1|)=\phi(d(x,y))$$

for any $\psi\in\Psi$ and $\phi\in\Phi$ with $\phi(t)<\psi(t)$ for all $t>0$.

Hence inequality (1.6.1) does not hold. Therefore Theorem 1.6 is not applicable here.

Hence by Remark 3.8 it follows that Theorem 2.1 is a generalization of Theorem 1.6.

Remark 3.10. Since the inequality (1.7.2) implies the inequality (2.2.2), by Theorem 2.2 the conclusion of Theorem 1.7 holds so that Theorem 1.7 follows as a corollary to Theorem 2.2. In the following, we provide an example for which the inequality (1.7.2) of Theorem 1.7 fails to hold.

Example 3.11. Let $X=[0,\ \infty)$.
We define a partial order $\preceq$ on X by

$$\preceq:=\big\{(x,y)\in X\times X\mid x=y\big\}\cup\big\{(x_n,x_m),\ (\frac{7}{3\ 2^{n+3}},\frac{1}{2^{n+2}}),$$
$$(\frac{7}{3\ 2^{n+3}},\frac{1}{2^{n+3}}),(\frac{1}{2^{n+2}},0),(\frac{7}{3\ 2^{n+3}},0),\ \ |\ x_n=\frac{1}{2^{n+2}},$$
$$n,m=0,\ 1,\ 2,\ \dots,\quad m\geq n\big\},$$

where $x\preceq y$ if and only if $x\geq y$ in the usual sense.
We define $T:X\to X$ by

$$T(x)=\begin{cases}\frac{x}{2}, & \text{if } x\in[0,1],\\ 2x-\frac{3}{2}, & \text{if } x\in[1,\frac{25}{24}]\\ \frac{7}{11}, & \text{if } x>\frac{25}{24}.\end{cases}$$

Clearly, T is non-decreasing and discontinuous on X.
We choose $x_0=\frac{1}{4}$, we have $T(\frac{1}{4})=\frac{1}{2^3}$ which implies that $x_0\preceq Tx_0$.
We define functions $\psi,\phi:[0,\infty)\to[0,\infty)$ as defined in Example 3.4.
We now verify the inequality (2.2.2).
Case (i): $(x,y)=(\frac{25}{24},1)$.

$$\psi(d(Tx,Ty))=\psi(d(T(\frac{25}{24}),T(1)))=\psi(|\frac{7}{12}-\frac{1}{2}|)$$
$$=\psi(\frac{1}{12})=\frac{1}{13}\leq\frac{1}{8}=\phi(\frac{11}{24})=\phi(|\frac{25}{24}-\frac{7}{12}|)=\phi(d(x,Tx))$$
$$\leq\max\{\phi(d(x,y)),\phi(d(x,Tx)),\phi(d(y,Ty)),$$
$$\frac{1}{2}[\phi(d(x,Ty))+\phi(d(y,Tx))]\},$$

where $(x,y)=(\frac{25}{24},1)$.

Case (ii): $(x,y)=(\frac{7}{12},\frac{1}{2})$.

$$\psi(d(Tx,Ty))=\psi(d(T(\frac{7}{12}),T(\frac{1}{2})))=\psi(d(\frac{7}{24},\frac{1}{4}))$$
$$=\psi(\frac{1}{24})=\frac{1}{25}\leq\frac{1}{8}=\phi(\frac{7}{24})=\phi(d(x,Tx))$$
$$\leq\max\{\phi(d(x,y)),\phi(d(x,Tx)),\phi(d(y,Ty)),$$
$$\frac{1}{2}[\phi(d(x,Ty))+\phi(d(y,Tx))]\},$$

where $(x,y)=(\frac{7}{12},\frac{1}{2})$.
Case (iii) : $(x,y)=(x_n,x_{n+1})$ $n=0,\ 1,\ \dots$, then $Tx_n=x_{n+1}$ and $Tx_{n+1}=x_{n+2}$.
Now,

$$\psi(d(Tx_n,Tx_{n+1}))=\psi(d(x_{n+1},x_{n+2}))=\psi\big[\frac{1}{2^{n+3}}-\frac{1}{2^{n+4}}\big]$$
$$=\psi\big[\frac{1}{2^{n+4}}\big]=\frac{1}{1+2^{n+4}}\leq 2^{-n-4}$$
$$=\phi\big[\frac{1}{2^{n+3}}\big]=\phi\big[\frac{1}{2^{n+2}}-\frac{1}{2^{n+3}}\big]=\phi(d(x_n,Tx_n)).$$

In this case, we have

$$\psi(d(Tx_n,Tx_{n+1}))=\phi(d(x_n,Tx_n))\le \max\{\phi(d(x_n,x_{n+1})),\ \phi(d(x_n,Tx_n)),\phi(d(x_{n+1},Tx_{n+1})),\ \frac{1}{2}[\phi(d(x_n,Tx_{n+1}))+\phi(d(x_{n+1},Tx_n))]\}.$$

<u>*Case*</u> (*iv*): $(x,y)=(x_n,x_m)$ $n=0,\ 1,\ \dots,$ $m=0,\ 1,\ \dots,$ $m\ge n,$ so that $Tx_n=x_{n+1}$ and $Tx_m=x_{m+1}$. Here

$$\psi(d(Tx_n,Tx_m))=\psi(d(x_{n+1},x_{m+1}))=\psi[\frac{1}{2^{n+3}}-\frac{1}{2^{m+3}}]$$
$$=\psi[\frac{2^m-2^n}{2^{n+m+3}}]=\psi[\frac{2^{m-3}-2^{n-3}}{2^{n+m}}].$$

and

$$\phi(d(x_n,x_m))=\phi[\frac{1}{2^{n+2}}-\frac{1}{2^{m+2}}]=\phi[\frac{2^m-2^n}{2^{n+m+2}}]$$
$$=\phi[\frac{2^{m-2}-2^{n-2}}{2^{n+m}}].$$

Let $t=\dfrac{2^{m-3}-2^{n-3}}{2^{n+m}}$, we have $2t=\dfrac{2^{m-2}-2^{n-2}}{2^{n+m}}$.

Since $t\in(0,1)$, then there exists $k\in\mathbb{N}$ such that

$$2^{-k}\le t<2^{-k+1} \text{ and } 2^{-k+1}\le 2t<2^{-k+2}.$$

Therefore

$$\psi(t)=2^{-k+2},\ \phi(2t)=2^{-k+2},\ \phi(t)=2^{-k+1} \text{ and}$$
$$\phi(2t)=2^{-k+2}=2\phi(t). \qquad (3.11.1)$$

From (3.11.1), we have $\phi\in\Phi$ and

$$\psi(d(Tx_n,\ Tx_m))=\phi(d(x_n,\ x_m))\le \max\{\phi(d(x_n,\ x_m)),\ \phi(d(x_n,\ Tx_n)),\ \phi(d(x_m,\ Tx_m)),\ \frac{1}{2}[\phi(d(x_n,\ Tx_m))+\phi(d(x_m,\ Tx_n))]\}$$

for all comparable $x_n$and x_m.

<u>*Case*</u> (*v*): $(x,y)=(\frac{7}{3\ 2^{n+3}},\frac{1}{2^{n+2}})$ $n=0,\ 1,\ 2,\dots$. Now,

$$\psi(d(T(x),T(y)))=\psi(d(T(\frac{7}{3\ 2^{n+3}}),T(\frac{1}{2^{n+2}})))$$
$$=\psi[|\frac{7}{3\ 2^{n+4}}-\frac{1}{2^{n+3}}|]=\psi[\frac{1}{3\ 2^{n+4}}]=\frac{1}{1+3\ 2^{n+4}}$$
$$\le 2^{-n-5}=\phi[\frac{1}{3\ 2^{n+3}}]=\phi[|\frac{7}{3\ 2^{n+3}}-\frac{1}{2^{n+2}}|]$$
$$=\phi(d(\frac{7}{3\ 2^{n+3}},\frac{1}{2^{n+2}}))=\phi(d(x,y)).$$

This implies that

$$\psi(d(T(\frac{7}{3\ 2^{n+3}}),T(\frac{1}{2^{n+2}})))=\phi(d(\frac{7}{3\ 2^{n+3}},\frac{1}{2^{n+2}}))$$
$$\le \max\{\phi(d(\frac{7}{3\ 2^{n+3}},\frac{1}{2^{n+2}})),\phi(d(\frac{7}{3\ 2^{n+3}},T(\frac{7}{3\ 2^{n+3}}))),$$
$$\phi(d(\frac{1}{2^{n+2}},T(\frac{1}{2^{n+2}}))),\frac{1}{2}[\phi(d(\frac{1}{2^{n+2}},T(\frac{7}{3\ 2^{n+3}})))$$
$$+\phi(d(\frac{7}{3\ 2^{n+3}},T(\frac{1}{2^{n+2}})))]\}.$$

<u>*Case*</u> (*vi*): $(x,y)=(\frac{7}{3\ 2^{n+3}},\frac{1}{2^{n+3}})$ $n=0,\ 1,\ 2,\ \dots$. Now

$$\psi(d(T(x),T(y)))=\psi(d(T(\frac{7}{3\ 2^{n+3}}),T(\frac{1}{2^{n+3}})))$$
$$=\psi[|\frac{7}{3\ 2^{n+4}}-\frac{1}{2^{n+4}}|]=\psi[\frac{4}{3\ 2^{n+4}}]=\frac{4}{1+3\ 2^{n+4}}$$
$$\le 2^{-n-2}=2^{-n-3+1}=\phi[\frac{4}{3\ 2^{n+3}}]$$
$$=\phi[|\frac{7}{3\ 2^{n+3}}-\frac{1}{2^{n+3}}|]=\phi(d(\frac{7}{3\ 2^{n+3}},\frac{1}{2^{n+3}}))$$
$$=\phi(d(x,y)).$$

This implies that

$$\psi(d(T(\frac{7}{3\ 2^{n+3}}),T(\frac{1}{2^{n+3}})))=\phi(d(\frac{7}{3\ 2^{n+3}},\frac{1}{2^{n+3}}))$$
$$\le \max\{\phi(d(\frac{7}{3\ 2^{n+3}},\frac{1}{2^{n+3}})),\ \phi(d(\frac{7}{3\ 2^{n+3}},T(\frac{7}{3\ 2^{n+3}}))),$$
$$\phi(d(\frac{1}{2^{n+3}},T(\frac{1}{2^{n+3}}))),$$
$$\frac{1}{2}[\phi(d(\frac{1}{2^{n+3}},T(\frac{7}{3\ 2^{n+3}})))+\phi(d(\frac{7}{3\ 2^{n+3}},T(\frac{1}{2^{n+3}})))]\}.$$

<u>*Case*</u> (*vii*): (x,y) =$(x_n,0)$,where $x_n=\frac{1}{2^{n+2}}|$ $n=0,\ 1,\ 2,\ \dots$. In this case, we have

$$\psi(d(Tx_n),T(0)))=\psi(d(x_{n+1},0))=\psi[\frac{1}{2^{n+3}}-0]$$
$$=\psi[\frac{1}{2^{n+3}}]=\frac{1}{1+2^{n+3}}\le 2^{-n-3}$$
$$=\phi(d(\frac{1}{2^{n+2}}-0))=\phi(d(x_n,0)).$$

Now, we have

$$\psi(d(Tx_n),T(0))=\phi(d(x_n,0))\le \max\{\phi(d(x_n,0)),\ \phi(d(x_n,Tx_n)),\phi(d(0,T(0))),\ \frac{1}{2}[\phi(d(x_n,T(0)))+\phi(d(0,Tx_n))]\}.$$

<u>*Case*</u> (*viii*): $(x,y)=(\frac{7}{3\ 2^{n+3}},0)$,where $n=0,\ 1,\ 2,\ \dots$. In this case, we have

$$\psi(d(T(\frac{7}{3\ 2^{n+3}}),T(0)))=\psi(d(\frac{7}{3\ 2^{n+4}},0))=\psi[|\frac{7}{3\ 2^{n+4}}-0|]$$
$$=\psi[\frac{7}{3\ 2^{n+4}}]=\frac{7}{1+3\ 2^{n+4}}\le 2^{-n-2}=2^{-n-3+1}$$
$$=\phi(\frac{7}{3\ 2^{n+3}})=\phi(d(\frac{7}{3\ 2^{n+3}},0)).$$

Now, we have

$$\psi(d(T(\frac{7}{3\ 2^{n+3}}),T(0))=\phi(d(\frac{7}{3\ 2^{n+3}},0))$$
$$\le \max\{\phi(d(\frac{7}{3\ 2^{n+3}},0)),\phi(d(\frac{7}{3\ 2^{n+3}},T(\frac{7}{3\ 2^{n+3}}))),$$
$$\phi(d(0,T(0))),\frac{1}{2}[\phi(d(\frac{7}{3\ 2^{n+3}},T(0)))+\phi(d(0,T(\frac{7}{3\ 2^{n+3}})))]\}.$$

We define a sequence $x_n = \frac{1}{2^{n+3}}$ $n = 1, 2 \ldots$. Since $x_n \geq x_{n+1}$, we have $x_n \preceq x_{n+1}$ so that $\{x_n\}$ is a non-decreasing sequence and $x_n \preceq 0$ for all $n \geq 1$.
Hence T, ϕ and ψ satisfy all the hypotheses of Theorem 2.2 and T has a fixed point 0 in X. Now we show that the inequality of Theorem 1.7 fails to hold.

$$\psi(d(Tx,Ty)) = \psi(d(T(\frac{25}{24}), T(1))) = \psi(|\frac{7}{12} - \frac{1}{2}|)$$

$$= \psi(\frac{1}{12}) \nleq \phi(\frac{1}{24}) = \phi(|\frac{25}{24} - 1|) = \phi(d(x,y))$$

for any $\psi \in \Psi$ and $\phi \in \Phi$ with $\phi(t) < \psi(t)$ for all $t > 0$.
Hence the inequality of Theorem 1.7 does not hold.
Therefore Theorem 1.7 is not applicable here.

Hence by Remark 3.10 it follows that Theorem 2.2 is a generalization of Theorem 1.7.

An open Problem: Can we prove the uniqueness of fixed point of Theorem 2.1 and Theorem 2.2 under the assumption 'condition (H)' of Theorem 2.5?

IV. Conclusion

In this paper, we proved the existence of fixed points (Theorem 2.1, Theorem 2.2 and Theorem 2.3) and its uniqueness (Theorem 2.5) by using two auxiliary functions in a partial ordered complete metric space. Further, we provided examples in support of our theorems to illustrate the importance of using two auxiliary functions in the contractive condition (Example 3.3 and Example 3.6) and to show our results generalize the theorems of Su[12] (Example 3.9 and Example 3.11).

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