Shrinkage Bayesian Approach in Item - Failure Gamma Data In Presence of Prior Point Guess Value

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***Abstract*—Present article investigated some properties of the Bayes Shrinkage estimators for Two - Parameter Gamma distribution. Based on both the symmetric and asymmetric loss functions, properties in terms of relative efficiency have studied for Item - failure censored data.**

***Index Terms*—Bayes estimator, Bayes Shrinkage estimator, Squared error loss function (SELF), LINEX loss function (LLF), Item - failure censored data.**

***MSC 2010 Codes* – 62C10, 62C12**

I. Introduction

LET a random variable x follows the two - parameter Gamma distribution, then the probability density function is

$$f(x;\lambda,\theta)=\frac{1}{\Gamma(\lambda)}\left(\frac{x^{\lambda-1}}{\theta^{\lambda}}\right)exp\left(-\frac{x}{\theta}\right);x>0,\theta>0,\lambda>0. \tag{1}$$

The parameter λ and θ are known as the shape and scale parameter respectively. For $\lambda=1$, two - parameter Gamma distribution is a Negative Exponential distribution.

It is well known that, the Bayes estimate of unknown parameter under SELF is simply the posterior mean. The SELF is often used also because it does not lead to extensive numerical computations but several authors like Varian (1975), Berger (1980), Zellner (1986) and others have recognized that the inappropriateness of using the symmetric loss function in various estimation and prediction problems. The example of a well - known asymmetric loss function is LINEX loss function (LLF) and is proposed first by Varian (1975). The invariant version of LLF (Singh et. al. (2007)) is defined for any parameter θ as

$$L(\Delta)=e^{a\Delta}-a\Delta-1\,;\,a\neq 0\,and\,\Delta=\frac{\hat{\theta}-\theta}{\theta}. \tag{2}$$

The sign and magnitude of $'a'$ represents the direction and degree of asymmetry respectively. The positive (negative) value of $'a'$ is used when overestimation is more (less) serious than underestimation. $L(\Delta)$ is approximately square error and almost symmetric if $|a|$ near to zero.

Let us assume here that $x_{(1)},x_{(2)},...,x_{(n)}$ are n items are put to life test and test is terminated when first $r\,(\leq n)$ items have failed. This censoring criterion is known as Item - failure censoring criterion. Suppose that $x_{(1)},x_{(2)},...x_{(r)}\equiv \underline{x}$ be the first r ordered observations, then the joint probability density function is obtained as

$$f(\underline{x}|\theta)=\frac{n!}{(n-r)!\,(\Gamma(\lambda))^{r}}\left(\frac{1}{\theta^{r\lambda}}\right)\prod_{i=1}^{r}x_{(i)}^{\lambda-1}exp\left(-\frac{rT_{(r)}}{\theta}\right); \tag{3}$$

where $T_{(r)}=\frac{1}{r}\left\{\sum_{i=1}^{r}x_{(i)}+(n-r)\,x_{(r)}\right\}$.

The maximum likelihood (ML) estimator for the parameter θ is

$$\hat{\theta}_{(ML)}=\frac{T_{(r)}}{\lambda}.$$

When prior information about parameter is available in form of a prior point guess value, a procedure makes uses of this prior information by shrinking the usual estimators towards a guess value of parameter with the help of a shrinkage factor $k\,(0\leq k\leq 1)$. According to his belief in guess value, an experimenter specifies the values of shrinkage factor. Let θ_0 be the prior point guess value of unknown parameter θ, then shrinkage estimator (Thompson (1968)) is defined as

$$\hat{S}=k\hat{\theta}+(1-k)\,\theta_0. \tag{4}$$

Here, $\hat{\theta}$ be any estimator of the parameter θ.

In present paper, we proposed some Bayes shrinkage estimators for unknown scale parameter θ under both symmetric and asymmetric loss function. The properties of the estimators have studied in terms of relative efficiencies. A simulation has also been carried out for numerical interpretation.

II. Bayes Shrinkage Estimators under Conjugate Prior

The likelihood function for a given sample $x_{(1)},x_{(2)},...,x_{(n)}$ of size n form (1), is given as

$$L(\underline{x};\theta)=g(\theta;T_n)\,.h(\underline{x})\,; \tag{5}$$

where $g(\theta;T_n)=\theta^{-n\lambda}e^{-\hat{T}/\theta}$, $h(\underline{x})=\left(\frac{1}{\Gamma\lambda}\right)^{n}\left(\prod_{i=1}^{n}x_{(i)}^{\lambda-1}\right)$ and $\hat{T}=\sum_{i=1}^{n}x_{(i)}$.

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Hence, $\hat{T}$ is a sufficient statistic for the considered underlying model. Hence, there exists a family of conjugate prior and which can be obtained by looking at $g(\theta; T_n)$. Hence, the inverted Gamma distribution (Raiffa & Schlaifer (1961)) is taken as the conjugate prior density for parameter θ and is defined for the parameters (α, β) as

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{\theta}\right)^{\alpha+1} exp\left(-\frac{\beta}{\theta}\right);\ \theta > 0, \alpha > 0, \beta > 0. \tag{6}$$

The prior mean and prior variance are $\beta(\alpha-1)^{-1}; \alpha > 1$ and $\beta^2(\alpha-1)^{-2}(\alpha-2)^{-1}; \alpha > 2$ respectively.

Based on Bayes theorem, the posterior density for unknown parameter θ when prior information about θ is considered as $\pi(\theta)$, is given by

$$\pi^*(\theta) = \frac{f(\underline{x}|\theta).\pi(\theta)}{\int_\theta f(\underline{x}|\theta).\pi(\theta)\,d\theta}$$

$$= \frac{\left(\frac{1}{\theta^{r\lambda}}\right)\prod_{i=1}^r x_{(i)}^{\lambda-1} exp\left(-\frac{rT_{(r)}}{\theta}\right).\left(\frac{1}{\theta}\right)^{\alpha+1} exp\left(-\frac{\beta}{\theta}\right)}{\int_\theta \left(\frac{1}{\theta^{r\lambda}}\right)\prod_{i=1}^r x_{(i)}^{\lambda-1} exp\left(-\frac{rT_{(r)}}{\theta}\right).\left(\frac{1}{\theta}\right)^{\alpha+1} exp\left(-\frac{\beta}{\theta}\right)d\theta}$$

$$\Rightarrow \pi^*(\theta) = \frac{\left(rT_{(r)}+\beta\right)^{\alpha_1}}{\Gamma(\alpha_1)}\theta^{-(\alpha_1+1)}.$$

$$exp\left(-\frac{\left(rT_{(r)}+\beta\right)}{\theta}\right); \alpha_1 = r\lambda + \alpha, \theta > 0. \tag{7}$$

The Bayes estimator $\hat{\theta}_S$ under squared error loss function (SELF) for the parameter θ is simply the posterior mean and obtained as

$$\hat{\theta}_S = \int_\theta \theta.\pi^*(\theta)\,d\theta$$

$$= \int_\theta \frac{\left(rT_{(r)}+\beta\right)^{\alpha_1}}{\Gamma(\alpha_1)}\theta^{-\alpha_1} exp\left(-\frac{\left(rT_{(r)}+\beta\right)}{\theta}\right)d\theta$$

$$\Rightarrow \hat{\theta}_S = \frac{rT_{(r)}+\beta}{\alpha_1 - 1};\ r\lambda > 1-\alpha. \tag{8}$$

The guess value of unknown parameter θ_0 is involved in shrinkage estimator (4). There are three different choices available in literature for selecting prior point guess value for any given parameter θ. One may use a prior point guess value ad hock, from previous experiments or some reliable sources. Second choice is, considered a prior distribution having prior mean around θ_0, instead of prior point guess value θ_0. Shirke & Nalawade (2003) present a new criterion for obtaining optimum value of parameter in terms of guess value θ_0 based on Bayes estimation. Following Shirke & Nalawade (2003) we have

$$E\left(\hat{\theta}_S\right) = \theta_0$$

$$\Rightarrow \beta = (\alpha_1 - 1)\theta_0 - rE\left(T_{(r)}\right)$$

$$\Rightarrow \beta = \theta_0(r_1 - r);\ r_1 = \alpha_1 - 1. \tag{9}$$

It is noted here that $2rT_{(r)}\theta^{-1}$ distributed as chi - square with $2r$ degree of freedom. Substituting β from (9) in (8), we get an estimator of the form (4), named as Bayes Shrinkage estimator and is given by

$$\bar{\theta}_{SH} = \lambda_1 T_{(r)} + (1-\lambda_1)\theta;\ \lambda_1 = \frac{r}{r_1}. \tag{10}$$

Now, the Bayes estimator $\hat{\theta}_L$ under LLF for parameter θ is obtained by simplifying following equality

$$\int_\theta \theta^{-(\alpha_1+2)} exp\left(-\frac{rT_{(r)}+\beta-a\hat{\theta}_L}{\theta}\right)d\theta$$

$$= e^a \int_\theta \theta^{-(\alpha_1+2)} exp\left(-\frac{rT_{(r)}+\beta}{\theta}\right)d\theta$$

$$\Rightarrow \left(\frac{rT_{(r)}+\beta-a\hat{\theta}_L}{rT_{(r)}+\beta}\right)^{\alpha_1+1} = e^{-a}$$

$$\Rightarrow \hat{\theta}_L = \phi\left(rT_{(r)}+\beta\right); \phi = \frac{1}{a}\left(1 - exp\left(-\frac{a}{\alpha_1+1}\right)\right). \tag{11}$$

On similar line the parameter β, is optimized by Bayes estimator as

$$E\left(\hat{\theta}_L\right) = \theta_0$$

$$\Rightarrow \beta = \frac{\theta_0}{\phi} - rE\left(T_{(r)}\right)$$

$$\Rightarrow \beta = \theta_0\frac{(1-r\phi)}{\phi}. \tag{12}$$

Hence, the Bayes shrinkage estimator under LLF is

$$\bar{\theta}_{LH} = \lambda_2 T_{(r)} + (1-\lambda_2)\theta_0\ ; \lambda_2 = r\phi. \tag{13}$$

The risk under SELF and LLF for both estimators are summarized in following lines

Risk under SELF

$$R_{SELF}\left(\bar{\theta}_{SH}\right) = \theta^2\left\{\lambda_1^2\left(\frac{r+1}{r} + \delta(\delta-2)\right) + (\delta-1)^2(1-2\lambda_1)\right\}; \delta = \frac{\theta_0}{\theta}$$

$$R_{SELF}\left(\bar{\theta}_{LH}\right)=\theta^2\left\{\lambda_2^2\left(\frac{r+1}{r}+\delta\left(\delta-2\right)\right)+\left(\delta-1\right)^2\left(1-2\lambda_2\right)\right\}$$

Risk under LLF

$$R_{LLF}\left(\bar{\theta}_{SH}\right)=\left(\frac{r}{r-a\lambda_1}\right)^r e^{a(\delta(1-\lambda_1)-1)}-1+a\left(1-\delta\right)\left(1-\lambda_1\right)$$

$$R_{LLF}\left(\bar{\theta}_{LH}\right)=exp\left\{\frac{ar}{\alpha_1+1}+a\left(\left(1-\lambda_2\right)\delta-1\right)\right\}-1+a\left(1-\delta\right)\left(1-\lambda_2\right)$$

The expressions of relative efficiencies for $\bar{\theta}_{SH}$ with respect to $\bar{\theta}_{LH}$ under SELF and LLF are given as

$$RE_{SELF}\left(\bar{\theta}_{LH},\bar{\theta}_{SH}\right)=\frac{R_{SELF}\left(\bar{\theta}_{SH}\right)}{R_{SELF}\left(\bar{\theta}_{LH}\right)}$$

and

$$RE_{LLF}\left(\bar{\theta}_{LH},\bar{\theta}_{SH}\right)=\frac{R_{LLF}\left(\bar{\theta}_{SH}\right)}{R_{LLF}\left(\bar{\theta}_{LH}\right)}.$$

The expressions of relative efficiencies are the functions of r, a, λ, δ and prior parameters α and β. The values of α and β are chosen so as to keep the prior variance unity. Thus pre assumed set of prior parameters are $(\alpha,\beta)=(3,2),(6,10),(11,30)$. Gamma shape parameter is considered to be fixed at $\lambda=1.00$. The censored sample size is considered here $r=04(02)10$ and the value of shape parameter of LLF is $a=0.25,0.50$.

By the help of prior parameters α and β, generate θ from (6). Based on generated θ and some pre assumed prior point guess value, a ratio of δ obtained under 10,000 simulation run. Using δ and other pre assumed values, the relative efficiencies have been obtained and presented in Tables 01 - 02 respectively.

Using Tables 01 and 02, the shrinkage estimator $\bar{\theta}_{LH}$ performs uniformly better than the shrinkage estimator $\bar{\theta}_{SH}$ under both loss criterion for all selected parametric set of values for small prior parameters. For large prior parametric values the shrinkage estimator $\bar{\theta}_{LH}$ performs well only in vicinity of the true value. Further, the relative efficiencies decrease as censored sample size r increases. It is also noted here that the prior parameters α and β increase, similar trend has also been seen. An increasing trend has seen also here, when shape parameter of LLF increases. The efficiency attain maximum at $\delta=1.00$.

III. Bayes Shrinkage Estimators under Quasi Prior

The situation where the life researchers have no prior information about unknown parameter θ, one may use uniform, quasi or improper prior. We considered here a class of quasi prior defined as

$$\pi_1\left(\theta\right)=\frac{1}{\theta^d}exp\left(-\frac{p}{\theta}\right);\theta>0,p>0,d>0. \quad (14)$$

It is noted that for

d	p	
0	0	$\pi_1\left(\theta\right)=1\Rightarrow\ Non\,Informative\,Prior$
0	1	$\pi_1\left(\theta\right)=e^{-1/\theta}$
1	0	$\pi_1\left(\theta\right)=\frac{1}{\theta}\Rightarrow\ Diffuse\,Prior$

The prior mean and prior variance are given respectively as

$$\frac{\Gamma(d-2)}{p^{d-2}}\ ;\ d\geq 2$$

and

$$\frac{\Gamma(d-3)}{p^{2d-4}}\left(p^{d-1}-(d-3)\Gamma(d-2)\right);d>3.$$

Using prior density $\pi_1\left(\theta\right)$, the posterior density for θ is obtained as

$$\pi_1^*\left(\theta\right)=\frac{f\left(\underline{x}|\theta\right).\pi_1\left(\theta\right)}{\int_\theta f\left(\underline{x}|\theta\right).\pi_1\left(\theta\right)d\theta}$$

$$=\frac{\left(\frac{1}{\theta^{r\lambda}}\right)\prod_{i=1}^r x_{(i)}^{\lambda-1}exp\left(-\frac{rT_{(r)}}{\theta}\right).\frac{1}{\theta^d}\,exp\left(-\frac{p}{\theta}\right)}{\int_\theta\left(\frac{1}{\theta^{r\lambda}}\right)\prod_{i=1}^r x_{(i)}^{\lambda-1}exp\left(-\frac{rT_{(r)}}{\theta}\right).\frac{1}{\theta^d}\,exp\left(-\frac{p}{\theta}\right)d\theta}$$

$$\Rightarrow\pi_1^*\left(\theta\right)=\frac{\left(rT_{(r)}+p\right)^{d_1}}{\Gamma\left(d_1\right)}\theta^{-(d_1+1)}.\ exp\left(-\frac{rT_{(r)}+p}{\theta}\right);\ d_1=r\lambda+d-1,\theta>0. \quad (15)$$

On Similar line the Bayes estimators under SELF and LLF for parameter θ are obtained as

$$\hat{\theta}_{S1}=\frac{rT_{(r)}+p}{d_1-1}\ ;\ r\lambda+d>2 \quad (16)$$

and

$$\hat{\theta}_{L1}=\phi_1\left(rT_{(r)}+p\right);\phi_1=\frac{1}{a}\left(1-exp\left(-\frac{a}{r\lambda+d}\right)\right). \quad (17)$$

Similarly, the Bayes shrinkage estimators under SELF and LLF are obtained as by minimizing the Bayes risk

$$\bar{\theta}_{SH1}=\lambda_3T_{(r)}+\left(1-\lambda_3\right)\theta_0;\lambda_3=\frac{r}{d_1-1} \quad (18)$$

and

$$\bar{\theta}_{LH1}=\lambda_4T_{(r)}+\left(1-\lambda_4\right)\theta_0;\lambda_4=r\phi_1. \quad (19)$$

The risk under SELF and LLF for both estimators are summarized in following lines

Risk under SELF

$$R_{SELF}\left(\bar{\theta}_{SH1}\right)=\theta^2\left\{\lambda_3^2\left(\frac{r+1}{r}+\delta\left(\delta-2\right)\right)\right.$$
$$\left.+\left(\delta-1\right)^2\left(1-2\lambda_3\right)\right\}$$

$$R_{SELF}\left(\bar{\theta}_{LH1}\right)=\theta^2\left\{\lambda_4^2\left(\frac{r+1}{r}+\delta\left(\delta-2\right)\right)\right.$$
$$\left.+\left(\delta-1\right)^2\left(1-2\lambda_4\right)\right\}$$

Risk under LLF

$$R_{LLF}\left(\bar{\theta}_{SH1}\right)=\left(\frac{r}{r-a\lambda_3}\right)^r e^{a(\delta(1-\lambda_3)-1)}$$
$$-1+a\left(1-\delta\right)\left(1-\lambda_3\right)$$

$$R_{LLF}\left(\bar{\theta}_{LH1}\right)=exp\left\{\frac{ar}{r\lambda+d}+a\left(\left(1-\lambda_4\right)\delta-1\right)\right\}$$
$$-1+a\left(1-\delta\right)\left(1-\lambda_4\right)$$

The expressions of relative efficiencies for $\bar{\theta}_{SH1}$ with respect to $\bar{\theta}_{LH1}$ under SELF and LLF are given as

$$RE_{SELF}\left(\bar{\theta}_{LH1},\bar{\theta}_{SH1}\right)=\frac{R_{SELF}\left(\bar{\theta}_{SH1}\right)}{R_{SELF}\left(\bar{\theta}_{LH1}\right)}$$

and

$$RE_{LLF}\left(\bar{\theta}_{LH1},\bar{\theta}_{SH1}\right)=\frac{R_{LLF}\left(\bar{\theta}_{SH1}\right)}{R_{LLF}\left(\bar{\theta}_{LH1}\right)}.$$

The expressions of relative efficiencies are the functions of $r,\ a,\ \lambda,\ \delta$ and prior parameters d and p. The selected quasi families of prior density are converted back into the conjugate family of prior when $\alpha=d-1$ and $\beta=p$ are substituted. Thus, all the results are valid as obtained in previous section.

In present case the prior variance should not be equated to unity. Hence, we equated prior mean as unity and estimate the prior parameters for simulation. The selected prior parametric set of values are $(d,p)=(3,1),(8.9,2),(14.63,4)$. In present section similar set of parametric values selected as considered earlier and apply the similar steps of simulation run. The numerical findings are presented here in Tables 03-04.

From, Tables 03 and 04, the shrinkage estimator $\bar{\theta}_{LH1}$ performs uniformly well than shrinkage estimator $\bar{\theta}_{SH1}$ under both risk criterions for small prior parametric values. When prior parametric values increase the well performances of the shrinkage estimator $\bar{\theta}_{LH1}$ restricted in close vicinity of the true value. All the properties have been seen similar and efficiency attain maximum at $\delta=1.00$.

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Table 1: Relative Efficiency between Bayes Shrinkage Estimators $\bar{\theta}_{LH}$ & $\bar{\theta}_{SH}$ under SELF

$RE_{SELF}\,(\bar{\theta}_{LH}, \bar{\theta}_{SH})$

		$\leftarrow a = 0.25 \rightarrow$				$\leftarrow a = 0.50 \rightarrow$			
		$\leftarrow r \rightarrow$				$\leftarrow r \rightarrow$			
α, β	δ	4	6	8	10	4	6	8	10
3, 2	0.50	1.1108	1.0935	1.0798	1.0672	1.1109	1.0948	1.0891	1.0686
	0.60	1.2657	1.2194	1.1846	1.1587	1.2734	1.2262	1.1907	1.1641
	0.70	1.4455	1.3527	1.2895	1.2448	1.4657	1.3694	1.3035	1.2568
	0.80	1.6300	1.4771	1.3821	1.3180	1.6662	1.5047	1.4042	1.3364
	0.90	1.7768	1.5682	1.4468	1.3679	1.8281	1.6049	1.4752	1.3910
	1.00	1.8341	1.6020	1.4703	1.3856	1.8918	1.6423	1.5011	1.4105
	1.10	1.7768	1.5682	1.4468	1.3679	1.8281	1.6049	1.4752	1.3910
	1.20	1.6300	1.4771	1.3821	1.3180	1.6662	1.5047	1.4042	1.3364
	1.30	1.4455	1.3527	1.2895	1.2448	1.4657	1.3694	1.3035	1.2568
	1.40	1.2657	1.2194	1.1846	1.1587	1.2734	1.2262	1.1907	1.1641
6, 10	0.50	0.9343	0.9285	0.9275	0.9267	0.9483	0.9330	0.9322	0.9314
	0.60	1.0072	1.0069	1.0063	1.0057	1.0083	1.0081	1.0075	1.0069
	0.70	1.1145	1.1113	1.1039	1.0960	1.1186	1.1156	1.1080	1.0998
	0.80	1.2669	1.2410	1.2146	1.1918	1.2804	1.2537	1.2261	1.2021
	0.90	1.4407	1.3674	1.3127	1.2715	1.4677	1.3899	1.3318	1.2880
	1.00	1.5281	1.4238	1.3537	1.3034	1.5630	1.4513	1.3764	1.3227
	1.10	1.4407	1.3674	1.3127	1.2715	1.4677	1.3899	1.3318	1.2880
	1.20	1.2669	1.2410	1.2146	1.1918	1.2804	1.2537	1.2261	1.2021
	1.30	1.1145	1.1113	1.1039	1.0960	1.1186	1.1156	1.1080	1.0998
	1.40	1.0072	1.0069	1.0063	1.0057	1.0083	1.0081	1.0075	1.0069
11, 30	0.50	0.9311	0.9268	0.9176	0.9116	0.9440	0.9305	0.9217	0.9161
	0.60	0.9620	0.9537	0.9489	0.9461	0.9640	0.9563	0.9517	0.9492
	0.70	1.0105	1.0085	1.0055	1.0004	1.0108	1.0099	1.0079	1.0009
	0.80	1.0776	1.0684	1.0647	1.0528	1.0841	1.0801	1.0787	1.0769
	0.90	1.2126	1.2025	1.1880	1.1736	1.2248	1.2143	1.1990	1.1839
	1.00	1.3267	1.2833	1.2501	1.2238	1.3475	1.3012	1.2658	1.2378
	1.10	1.2126	1.2025	1.1880	1.1736	1.2248	1.2143	1.1990	1.1839
	1.20	1.0776	1.0684	1.0647	1.0528	1.0841	1.0801	1.0787	1.0769
	1.30	1.0105	1.0085	1.0055	1.0004	1.0108	1.0099	1.0079	1.0009
	1.40	0.9620	0.9537	0.9489	0.9461	0.9640	0.9563	0.9517	0.9492

Table 2: Relative Efficiency between Bayes Shrinkage Estimators $\bar{\theta}_{LH}$ & $\bar{\theta}_{SH}$ under LLF

$RE_{LLF}\left(\bar{\theta}_{LH}, \bar{\theta}_{SH}\right)$

		$\leftarrow a = 0.25 \rightarrow$				$\leftarrow a = 0.50 \rightarrow$			
		$\leftarrow r \rightarrow$				$\leftarrow r \rightarrow$			
α, β	δ	4	6	8	10	4	6	8	10
3, 2	0.50	1.1321	1.1129	1.0972	1.0848	1.1569	1.1352	1.1169	1.1024
	0.60	1.2910	1.2406	1.2026	1.1742	1.3297	1.2730	1.2300	1.1979
	0.70	1.4737	1.3742	1.3066	1.2590	1.5294	1.4173	1.3415	1.2881
	0.80	1.6579	1.4963	1.3966	1.3296	1.7306	1.5485	1.4368	1.3622
	0.90	1.8000	1.5826	1.4570	1.3756	1.8827	1.6384	1.4986	1.4086
	1.00	1.8494	1.6101	1.4753	1.3890	1.9294	1.6621	1.5133	1.4187
	1.10	1.7854	1.5707	1.4471	1.3672	1.8510	1.6127	1.4774	1.3906
	1.20	1.6358	1.4764	1.3792	1.3144	1.6830	1.5057	1.3997	1.3299
	1.30	1.4517	1.3512	1.2853	1.2396	1.4828	1.3687	1.2962	1.2470
	1.40	1.2733	1.2186	1.1804	1.1531	1.2927	1.2268	1.1834	1.1535
6, 10	0.50	0.9492	0.9360	0.9329	0.9314	0.9495	0.9369	0.9353	0.9323
	0.60	1.0123	1.0122	1.0102	1.0009	1.0187	1.0168	1.0165	1.0164
	0.70	1.1195	1.1171	1.1099	1.1018	1.1296	1.1283	1.1211	1.1127
	0.80	1.2745	1.2482	1.2212	1.1978	1.2975	1.2698	1.2407	1.2154
	0.90	1.4494	1.3742	1.3182	1.2761	1.4874	1.4053	1.3443	1.2985
	1.00	1.5341	1.4277	1.3564	1.3055	1.5770	1.4605	1.3829	1.3276
	1.10	1.4451	1.3692	1.3132	1.2714	1.4783	1.3947	1.3337	1.2883
	1.20	1.2725	1.2432	1.2149	1.1911	1.2934	1.2592	1.2274	1.2012
	1.30	1.1210	1.1146	1.1051	1.0958	1.1332	1.1232	1.1111	1.1000
	1.40	1.0146	1.0119	1.0094	1.0074	1.0210	1.0154	1.0108	1.0082
11, 30	0.50	0.9414	0.9274	0.9185	0.9129	0.9441	0.9308	0.9222	0.9167
	0.60	0.9618	0.9541	0.9498	0.9475	0.9639	0.9564	0.9521	0.9498
	0.70	1.0019	1.0017	1.0013	1.0005	1.0027	1.0021	1.0014	1.0008
	0.80	1.0823	1.0814	1.0807	1.0749	1.0882	1.0868	1.0862	1.0793
	0.90	1.2151	1.2050	1.1904	1.1759	1.2302	1.2198	1.2042	1.1888
	1.00	1.3289	1.2850	1.2514	1.2249	1.3525	1.3051	1.2688	1.2403
	1.10	1.2152	1.2040	1.1888	1.1740	1.2305	1.2178	1.2010	1.1848
	1.20	1.0825	1.0823	1.0798	1.0778	1.0892	1.0882	1.0872	1.0855
	1.30	1.0035	1.0033	1.0028	1.0023	1.0063	1.0056	1.0045	1.0034
	1.40	0.9659	0.9576	0.9523	0.9489	0.9660	0.9581	0.9533	0.9504

Table 3: Relative Efficiency between Bayes Shrinkage Estimators $\bar{\theta}_{LH1}$ & $\bar{\theta}_{SH1}$ under SELF

$RE_{SELF}\left(\bar{\theta}_{LH1}, \bar{\theta}_{SH1}\right)$

		$\leftarrow a = 0.25 \rightarrow$				$\leftarrow a = 0.50 \rightarrow$			
		$\leftarrow r \rightarrow$				$\leftarrow r \rightarrow$			
d, p	δ	4	6	8	10	4	6	8	10
3,1	0.50	1.3398	1.2582	1.2073	1.1729	1.3457	1.2631	1.2115	1.1766
	0.60	1.5187	1.3842	1.3039	1.2510	1.5376	1.3990	1.3160	1.2612
	0.70	1.7025	1.5045	1.3923	1.3205	1.7378	1.5304	1.4126	1.3371
	0.80	1.8688	1.6062	1.4642	1.3757	1.9219	1.6428	1.4920	1.3980
	0.90	1.9876	1.6751	1.5116	1.4114	2.0551	1.7196	1.5446	1.4375
	1.00	2.0310	1.6995	1.5281	1.4238	2.1042	1.7470	1.5630	1.4513
	1.10	1.9876	1.6751	1.5116	1.4114	2.0551	1.7196	1.5446	1.4375
	1.20	1.8688	1.6062	1.4642	1.3757	1.9219	1.6428	1.4920	1.3980
	1.30	1.7025	1.5045	1.3923	1.3205	1.7378	1.5304	1.4126	1.3371
	1.40	1.5187	1.3842	1.3039	1.2510	1.5376	1.3990	1.3160	1.2612
8.9,2	0.50	0.9528	0.9393	0.9296	0.9225	0.9552	0.9423	0.9331	0.9264
	0.60	0.9720	0.9683	0.9670	0.9669	0.9740	0.9706	0.9695	0.9694
	0.70	1.0467	1.0462	1.0446	1.0441	1.0481	1.0477	1.0460	1.0454
	0.80	1.1616	1.1583	1.1486	1.1379	1.1695	1.1662	1.1563	1.1451
	0.90	1.3276	1.2893	1.2555	1.2277	1.3470	1.3066	1.2708	1.2413
	1.00	1.4280	1.3567	1.3056	1.2673	1.4558	1.3795	1.3250	1.2841
	1.10	1.3276	1.2893	1.2555	1.2277	1.3470	1.3066	1.2708	1.2413
	1.20	1.1616	1.1583	1.1486	1.1379	1.1695	1.1662	1.1563	1.1451
	1.30	1.0467	1.0462	1.0446	1.0441	1.0481	1.0477	1.0460	1.0454
	1.40	0.9720	0.9683	0.9670	0.9669	0.9740	0.9706	0.9695	0.9694
14.63,4	0.50	0.9283	0.9168	0.9113	0.9092	0.9320	0.9212	0.9161	0.9142
	0.60	0.9651	0.9558	0.9495	0.9452	0.9669	0.9581	0.9522	0.9481
	0.70	0.9888	0.9862	0.9847	0.9837	0.9896	0.9872	0.9857	0.9848
	0.80	1.0477	1.0471	1.0447	1.0389	1.0500	1.0494	1.0468	1.0407
	0.90	1.1528	1.1521	1.1471	1.1394	1.1615	1.1606	1.1555	1.1475
	1.00	1.2719	1.2412	1.2166	1.1966	1.2891	1.2563	1.2301	1.2088
	1.10	1.1528	1.1521	1.1471	1.1394	1.1615	1.1606	1.1555	1.1475
	1.20	1.0477	1.0471	1.0447	1.0389	1.0500	1.0494	1.0468	1.0407
	1.30	0.9888	0.9862	0.9847	0.9837	0.9896	0.9872	0.9857	0.9848
	1.40	0.9651	0.9558	0.9495	0.9452	0.9669	0.9581	0.9522	0.9481

Table 4: Relative Efficiency between Bayes Shrinkage Estimators $\bar{\theta}_{LH1}$ & $\bar{\theta}_{SH1}$ under LLF

$RE_{LLF}\left(\bar{\theta}_{LH1}, \bar{\theta}_{SH1}\right)$

		← $a = 0.25$ →				← $a = 0.50$ →			
		← r →				← r →			
d, p	δ	4	6	8	10	4	6	8	10
	0.50	1.3888	1.2943	1.2358	1.1965	1.4554	1.3429	1.2740	1.2278
	0.60	1.5703	1.4198	1.3309	1.2727	1.6547	1.4785	1.3756	1.3087
	0.70	1.7535	1.5373	1.4161	1.3391	1.8553	1.6045	1.4658	1.3783
	0.80	1.9145	1.6335	1.4832	1.3901	2.0293	1.7056	1.5350	1.4303
	0.90	2.0236	1.6948	1.5245	1.4208	2.1416	1.7660	1.5746	1.4591
3,1	1.00	2.0548	1.7106	1.5345	1.4279	2.1641	1.7748	1.5789	1.4615
	1.10	2.0002	1.6782	1.5118	1.4105	2.0905	1.7302	1.5472	1.4370
	1.20	1.8739	1.6033	1.4594	1.3706	1.9410	1.6405	1.4840	1.3885
	1.30	1.7042	1.4980	1.3841	1.3122	1.7493	1.5206	1.3976	1.3212
	1.40	1.5200	1.3763	1.2939	1.2408	1.5475	1.3863	1.2973	1.2413
	0.50	0.9532	0.9398	0.9302	0.9233	0.9553	0.9425	0.9334	0.9268
	0.60	0.9734	0.9718	0.9712	0.9711	0.9746	0.9720	0.9717	0.9714
	0.70	1.0492	1.0491	1.0477	1.0456	1.0539	1.0533	1.0529	1.0487
	0.80	1.1650	1.1621	1.1526	1.1418	1.1770	1.1748	1.1650	1.1537
	0.90	1.3328	1.2938	1.2594	1.2311	1.3588	1.3168	1.2797	1.2491
8.9,2	1.00	1.4319	1.3594	1.3076	1.2689	1.4647	1.3859	1.3298	1.2878
	1.10	1.3310	1.2909	1.2562	1.2278	1.3550	1.3107	1.2727	1.2419
	1.20	1.1664	1.1609	1.1498	1.1382	1.1800	1.1722	1.1592	1.1460
	1.30	1.0499	1.0489	1.0482	1.0455	1.0559	1.0553	1.0521	1.0483
	1.40	0.9780	0.9738	0.9716	0.9707	0.9807	0.9752	0.9718	0.9699
	0.50	0.9294	0.9189	0.9144	0.9129	0.9325	0.9222	0.9175	0.9159
	0.60	0.9649	0.9558	0.9497	0.9457	0.9668	0.9581	0.9523	0.9483
	0.70	0.9884	0.9861	0.9850	0.9845	0.9893	0.9871	0.9858	0.9852
	0.80	1.0486	1.0478	1.0451	1.0390	1.0520	1.0509	1.0477	1.0409
	0.90	1.1554	1.1543	1.1486	1.1410	1.1655	1.1648	1.1589	1.1509
14.63,4	1.00	1.2735	1.2424	1.2176	1.1975	1.2925	1.2590	1.2324	1.2107
	1.10	1.1542	1.1542	1.1479	1.1399	1.1651	1.1646	1.1575	1.1487
	1.20	1.0412	1.0466	1.0486	1.0488	1.0456	1.0510	1.0527	1.0525
	1.30	0.9912	0.9888	0.9871	0.9861	0.9923	0.9897	0.9878	0.9864
	1.40	0.9679	0.9592	0.9532	0.9490	0.9673	0.9581	0.9517	0.9472