On the Eigenvalues and the Determinant of the Right Circulant Matrices with Pell and Pell-Lucas Numbers

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***Abstract*—In this study, we derive the explicit forms of the eigenvalues and the determinant of two right circulant matrices, one with Pell numbers as entries and the other with Pell-Lucas numbers.**

***Index Terms*—Right circulant matrix, Pell numbers, Pell-Lucas numbers, eigenvalue, determinant.**

***MSC 2010 Codes* – 05C50, 11B50**

I. Introduction

THE right circulant matrix with Pell numbers take the form

$$C_R(\vec{P}) = \begin{pmatrix} P_0 & P_1 & P_2 & \ldots & P_{n-2} & P_{n-1} \\ P_{n-1} & P_0 & P_1 & \ldots & P_{n-3} & P_{n-2} \\ P_{n-2} & P_{n-1} & P_0 & \ldots & P_{n-4} & P_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P_2 & P_3 & P_4 & \ldots & P_0 & P_1 \\ P_1 & P_2 & P_3 & \ldots & P_{n-1} & P_0 \end{pmatrix}$$

where P_n is the n^{th} Pell number given by

$$P_n = \frac{\sigma^n - \tau^n}{2\sqrt{2}}$$

with $\sigma = 1 + \sqrt{2}$ and $\tau = 1 - \sqrt{2}$.

On the other hand, the right circulant with Pell-Lucas numbers is given by

$$C_R(\vec{Q}) = \begin{pmatrix} Q_0 & Q_1 & Q_2 & \ldots & Q_{n-2} & Q_{n-1} \\ Q_{n-1} & Q_0 & Q_1 & \ldots & Q_{n-3} & Q_{n-2} \\ Q_{n-2} & Q_{n-1} & Q_0 & \ldots & Q_{n-4} & Q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_2 & Q_3 & Q_4 & \ldots & Q_0 & Q_1 \\ Q_1 & Q_2 & Q_3 & \ldots & Q_{n-1} & Q_0 \end{pmatrix}$$

where Q_n is the n^{th} Pell-Lucas number given by

$$Q_n = \sigma^n + \tau^n$$

Bozkurt [1] solved the determinants of these matrices using matrix decompositions.

Our goal in this study is to find the eigenvalues of these matrices and use them to find their determinants.

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II. Main Results

Theorem 2.1: The eigenavlues of $C_R(\vec{P})$ are given by

$$\lambda_m = \frac{-P_n - (1 + P_{n-1})\omega^{-mk}}{(1 - \sigma\omega^{-mk})(1 - \tau\omega^{-mk})}$$

where m=0, 1, ..., n-1 and $\omega = e^{2\pi i/n}$.

Proof:

The eigenvalue of a right circulant matrix is just the discrete Fourier transform of its first row, hence we have

$$\begin{aligned} \lambda_m &= \sum_{k=0}^{n-1} P_k \omega^{-mk} \\ &= \frac{1}{2\sqrt{2}} \sum_{k=0}^{n-1} [\sigma^k - \tau^k]\omega^{-mk} \\ &= \frac{1}{2\sqrt{2}} \left[\frac{1 - \sigma^n}{1 - \sigma\omega^{-mk}} - \frac{1 - \tau^n}{1 - \tau\omega^{-mk}} \right] \\ &= \frac{1}{2\sqrt{2}} \left[\frac{-(\sigma^n - \tau^n) - (\sigma - \tau)\omega^{-mk} - (\sigma^{n-1} - \tau^{n-1})\omega^{-mk}}{(1 - \sigma\omega^{-mk})(1 - \tau\omega^{-mk})} \right] \\ &= \frac{-P_n - (1 + P_{n-1})\omega^{-mk}}{(1 - \sigma\omega^{-mk})(1 - \tau\omega^{-mk})} \end{aligned}$$

Theorem 2.2: The eigenavlues of $C_R(\vec{Q})$ are given by

$$\mu_m = \frac{2 - Q_n - (2 + Q_{n-1})\omega^{-mk}}{(1 - \sigma\omega^{-mk})(1 - \tau\omega^{-mk})}$$

Proof:

Following the proof of the previous theorem, we obtain

$$\begin{aligned} \mu_m &= \sum_{k=0}^{n-1} Q_k \omega^{-mk} \\ &= \sum_{k=0}^{n-1} [\sigma^k + \tau^k]\omega^{-mk} \\ &= \left[\frac{1 - \sigma^n}{1 - \sigma\omega^{-mk}} + \frac{1 - \tau^n}{1 - \tau\omega^{-mk}} \right] \\ &= \frac{2 - (\sigma^n + \tau^n) - (\sigma + \tau)\omega^{-mk} - (\sigma^{n-1} + \tau^{n-1})\omega^{-mk}}{(1 - \sigma\omega^{-mk})(1 - \tau\omega^{-mk})} \\ &= \frac{2 - Q_n - (2 + Q_{n-1})\omega^{-mk}}{(1 - \sigma\omega^{-mk})(1 - \tau\omega^{-mk})} \end{aligned}$$

Theorem 2.3: The determinant of $C_R(\vec{P})$ is given by

$$|C_R(\vec{P})| = \frac{(-P_n)^n - (1 + P_{n-1})^n}{1 - Q_n + (-1)^n}$$

Proof:

Since the determinant is just the product of the eigenvalues, we must have

$$\begin{aligned} |C_R(\vec{P})| &= \prod_{m=0}^{n-1} \lambda_m \\ &= \prod_{m=0}^{n-1} \frac{-P_n - (1+P_{n-1})\omega^{-mk}}{(1-\sigma\omega^{-mk})(1-\tau\omega^{-mk})} \\ &= \frac{\prod_{m=0}^{n-1}(-P_n - (1+P_{n-1})\omega^{-mk})}{\prod_{m=0}^{n-1}(1-\sigma\omega^{-mk})\prod_{m=0}^{n-1}(1-\tau\omega^{-mk})} \end{aligned}$$

From [2], it has been established that for any x and y,

$$\prod_{m=0}^{n-1}(x - y\omega^{-mk}) = x^n - y^n$$

Using this leads to

$$\begin{aligned} \frac{(-P_n)^n - (1+P_{n-1})^n}{(1-\sigma^n)(1-\tau^n)} &= \frac{(-P_n)^n - (1+P_{n-1})^n}{1-(\sigma^n+\tau^n)+(-1)^n} \\ &= \frac{(-P_n)^n - (1+P_{n-1})^n}{1-Q_n+(-1)^n} \end{aligned}$$

which is as desired.

Theorem 2.4: The determinant of $C_R(\vec{Q})$ is given by

$$|C_R(\vec{Q})| = \frac{(2-Q_n)^n - (2+Q_{n-1})^n}{1-Q_n+(-1)^n}$$

Proof:

$$\begin{aligned} |C_R(\vec{Q})| &= \prod_{m=0}^{n-1} \mu_m \\ &= \prod_{m=0}^{n-1} \frac{2-Q_n - (2+Q_{n-1})\omega^{-mk}}{(1-\sigma\omega^{-mk})(1-\tau\omega^{-mk})} \\ &= \frac{\prod_{m=0}^{n-1}(2-Q_n - (2+Q_{n-1})\omega^{-mk})}{\prod_{m=0}^{n-1}(1-\sigma\omega^{-mk})\prod_{m=0}^{n-1}(1-\tau\omega^{-mk})} \end{aligned}$$

Using similar process as the previous theorem, we obtain

$$|C_R(\vec{Q})| = \frac{(2-Q_n)^n - (2+Q_{n-1})^n}{1-Q_n+(-1)^n}$$

III. Conclusion

We have derived the eigenvalues and the determinant of the matrices $C_R(\vec{P})$ and $C_R(\vec{Q})$. For $C_R(\vec{P})$, the eigenvalues and the determinant are functions of the Pell numbers P_{n-1} and P_n, the Pell-Lucas number Q_n and the number n. On the other hand, the eigenvalues and the determinant of $C_R(\vec{Q})$ is dependent on the Pell-Lucas numbers Q_{n-1} and Q_n and the number n.

IV. Recommendation

A possible venture on this study is to look on the Euclidean norm, spectral norm and inverse of the matrices $C_R(\vec{P})$ and $C_R(\vec{Q})$.

References

[1] D. Bozkurt, "On the determinants and inverses of circulant matrices with Pell and Pell-Lucas numbers," arXiv: 1201.6061v1 [math.NA] 29 Jan 2012.

[2] D. Lind, "A Fibonacci circulant", *Fibonacci Quarterly*, vol. 8, pp. 449-455, 1970.