Numerical Solution of Two-Dimensional Laminar Circular Jet in a Variable Magnetic Field

Manisha Patel and M. G. Timol

***Abstract*—The flow of a laminar compressible electrically conducting fluid issuing from circular orifices in the presences of radial magnetic field is analyzed. Similarity solutions for the velocity distribution are obtained. The similarity equations are solved numerically and presented graphically.**

***Index Terms*—boundary layer, circular jet, dorodnitsyn transformations, magnetic field, similarity solutions**

***MSC 2010 Codes* — 76B10 , 76N20 , 76W99**

I. Introduction

JETS are used in various industrial applications because of producing high heat and mass transfer coefficients. The flow structures of unconfined impinging jets are divided to three main regions: free jet region, stagnation region and wall jet region.

When a fluid moving in a pipe or tank crosses a slit or a nozzle there is a sudden decrease in cross section area and consequently there is a considerable increase in velocity. It gives rise to the flow of a jet. A fluid flows in the absence of rigid boundaries and, therefore, it is a free flow. These flows are important in technology, metrology etc.. A radial jet is obtained when fluid is blown out of a slit arranged over the circumferences of a pipe. If the pipe is rotates, a swirling radial jet is firmed.

Many researchers have worked on laminar jet flow in the past. The velocity distribution in a laminar circular jet of viscous incompressible fluid has been investigated by Schlichting [1] and Bickly [2]. And for the compressible fluid the results were given by Howarth [3], Illingworth [4] and Pack [5].

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In swirling flows, besides axial and radial velocity components, the tangential component of velocity should also be considered. It is both the linear momentum and the angular momentum of developing swirling flow which play an important role in determining its ultimate form. Mishra et al. [10] studied the effect of circular magnetic field on the decay of a weak swirl in the axially-symmetrical circular free jet of an electrically conducting, viscous fluid and found that the swirl velocity increases due to the presence of the magnetic field, though its decay is inversely proportional to the distance square along the axis of jet, as is the case n the non-magnetic flow.

The past development of the boundary layer theory has given rise to a number of investigations for the possibilities of obtaining similar solutions of boundary equations and also the solution of the specific problems which are of great practical importance in various fields of engineering. It has been observed by many researchers that introduction of Newtonian behavior to the equations of electrically conducting flows severely restricts the possibility of obtaining free jet of an electrically conducting viscous fluid in presence of transverse magnetic field. Peskin [9] has claimed that similarity solution e.g. two-dimensional compressible free jet of an electrically conducting viscous fluid in presence of transverse magnetic field. Peskin [9] has claimed that similarity solution does not exist. Under some restricted interaction parameters, Bansal et al [11] have obtained similarity solution for equation of motion of viscous, compressible and electrically conducting field in the presence of a radial magnetic field for which similarity solutions exist.

The effects of compressibility and variable magnetic field on a laminar circular jet of an electrically conducting fluid by taking the Prandtl number of the fluid as unity was studied by Bansal et al. [11]. The study of Magnetohydrodynamic laminar circular jets also been made by Akerstedt et al. [6] and Cinalli et al.[7]. The study of swirling circular jet have been made by Gallaire et al. [12] and Facciolo et al. [13]. Recently, Jhankal et al. [8] have studied the effect of compressibility and magnetic field on a laminar circular free jet of an electrically conducting gas issuing in the presence of variable radial magnetic field . They have also studied [14] the effects of an axial magnetic field on the temperature distribution and velocity in swirling jet of a viscous incompressible electrically conducting fluid originates from circular slit. The flow field charecteristics of a confined annular jet governing the steady laminar flow have been discussed by Khaled et al. [15].

In the present paper, the flow of a laminar compressible electrically conducting fluid issuing from circular orifices in the presences of radial magnetic field is analyzed. Similarity solutions for the velocity distribution are obtained. The similarity equations are solved numerically using collocation method and trial and error method. Results are shown graphically.

II. GOVERNING EQUATIONS

The fundamental boundary layer equations due to Pai [16] of the electrically conducting, viscous and compressible fluids issuing from circular orifice or a two-dimensional slit in the presence of radial or transverse magnetic field and mixing with surrounding medium of same fluid can be written as follow:

$$\frac{\partial}{\partial x}\left(\rho r^{p} u\right)+\frac{\partial}{\partial r}\left(\rho r^{p} v\right)=0 \tag{1}$$

$$\rho u \frac{\partial u}{\partial x}+\rho v \frac{\partial u}{\partial r}=\frac{1}{r^{p}} \frac{\partial}{\partial r}\left(\mu r^{p} \frac{\partial u}{\partial r}\right)-\sigma B^{2} u \tag{2}$$

subject to the boundary conditions

$$r=0 \Rightarrow \frac{\partial u}{\partial r}=0 \, , \, v=0 \tag{3}$$

$$r=\infty \Rightarrow u=\sigma \frac{B^{2}}{\rho} x^{\alpha} \tag{4}$$

The condition on momentum flux, in absence of the magnetic field is

$$\int_0^\infty \rho r^{p} u^{2}\, dr=\frac{M_{0}}{2 \pi} \tag{5}$$

Here u and v are velocity components in a distance along the axis of the jet ,***r*** is the perpendicular distance to the axis of jet, σ is the electrical conductivity of the fluid, μ is the coefficient of viscosity, ρ is the density of fluid, α is a constant to be determined and p is a real number.

Introducing stream function ψ such that

$$\rho r^{p} u=\frac{\partial \psi}{\partial r} \quad \text{and} \quad \rho r^{p} v=-\frac{\partial \psi}{\partial x} \tag{6}$$

and transforming the independent variables from (x,r) to (X,R) by defing modified Dorodnitsyn [17] transformations we have

$$X=x \ \text{ and } \ \rho_{1} R^{p+1}=\int_{0}^{r}(p+1) \rho\, r^{P} d r \tag{7}$$

Where ρ_1 is the density outside and on the boundary of the jet. Combining (6) and (7) and employing in the equations (1) – (4) ,the continuity equation is satisfied identically and equation of motion is reduced to

$$\frac{p}{R} \frac{\partial \psi}{\partial X} \frac{\partial \psi}{\partial R}+\frac{\partial \psi}{\partial R} \frac{\partial^{2} \psi}{\partial X\, \partial R}-\frac{\partial \psi}{\partial X} \frac{\partial^{2} \psi}{\partial R^{2}} =R^{p} \frac{\partial}{\partial R}\left[\mu_{1} R^{p} \frac{\partial}{\partial R}\left(\frac{1}{R^{p}} \frac{\partial \psi}{\partial R}\right)\right] -\sigma \frac{B^{2}}{\rho} \rho_{1} R^{p} \frac{\partial \psi}{\partial R} \tag{8}$$

Where $\mu=\mu_1$ is kinetic viscosity of the fluid and

$\rho r^{2p}=\rho_1 R^{2p}$

And the condition of momentum flux is

$$\frac{M_{0} p_{1}}{2 \pi}=\int_{0}^{\infty} \frac{1}{R^{p}}\left[\frac{\partial \psi}{\partial R}\right]^{2} d R \tag{9}$$

We try to find the possible form of imposed magnetic field for which similarity solution exists through a group theoretic approach.

III. GROUP THEORETIC ANALYSIS

Let the one-parameter group of transformations be

$$G \quad \begin{cases} X=A^{\alpha_{1}} \bar{X} \ ; \ R=A^{\alpha_{2}} \bar{R} \\ \psi=A^{\alpha_{3}} \bar{\psi} \ ; \ B=A^{\alpha_{4}} \bar{B} \end{cases} \tag{10}$$

where A and α_i (i = 1, 2, 3, 4) are real constants. For equations (8) and (9) the invariant condition gives the relations

$$2 \alpha_{3}-2 \alpha_{2}-\alpha_{1}=\alpha_{3}-3 \alpha_{2}+P \alpha_{2} =2 \alpha_{4}+\alpha_{3}+(p+1) \alpha_{2} \tag{11}$$

$$2 \alpha_{3}-(p+1) \alpha_{2}=0 \tag{12}$$

$$\alpha_{3}-2 \alpha_{2}=2 \alpha_{4}+2 \alpha_{1} \tag{13}$$

It follows that

$$\frac{\alpha_{2}}{\alpha_{3}}=\frac{\alpha}{3-p} \ ; \ \frac{\alpha_{3}}{\alpha_{1}}=\frac{p+1}{3-p} \ ; \quad \frac{\alpha_{4}}{\alpha_{1}}=\frac{-2}{3-p} \ ; \ \alpha=\frac{p+1}{3-p} \tag{14}$$

Necessary similarity transformations can be selected as

$$\eta=\frac{a R}{X^{2 /(3-p)}} \ ;$$

$$f(\eta)=\frac{\psi}{b \mathrm{X}^{(p+1) /(3-p)}} ; \tag{15}$$

$$B=B_{0} X^{-2 /(3-p)}$$

where η is similarity independent variable, a and b are constants.

Under the similarity transformation (15) governing partial differential equations reduce to

$$\frac{p(p+1)}{3-p}\frac{ff'}{\eta^p}+\frac{3p-1}{3-p}f'^2-\frac{p+1}{3-p}f f''$$
$$=\frac{1}{b a^{p-1}}\eta^p\frac{d}{d\eta}\left[\mu_1\eta^p\frac{d}{d\eta}(f'/\eta^p)\right]-$$
$$\frac{1}{b a^{p-1}}\eta^p\frac{\sigma B_0^2\rho_1}{\rho}f' \quad (16)$$

With the condition of constant momentum flux

$$\frac{M_0\rho_1}{2\pi}=a^{p+1}b^2\int_0^\infty\frac{f'^2}{\eta^p}d\eta \quad (17)$$

The associated boundary conditions are

$$\eta=0\Rightarrow f=0\ ;\ \frac{d}{d\eta}(f'/p)=0 \quad (18)$$

$$\eta=\infty\Rightarrow f'/\eta^p=\frac{\sigma B_0^2 p_1}{a^{p+1}b} \quad (19)$$

where prime denotes differentiation with respect to η .

Now setting

$$a^{p-1}b=\mu_1\ \ and\ \ a^{p+1}b=1 \quad (20)$$

We get

$$a=\mu_1{}^{-1/2}\ ;\ b=\mu_1{}^{(p+1)/2} \quad (21)$$

The similarity transformations (15) are

$$\eta=\frac{R}{\mu_1^{1/2}X^{2/(3-p)}}\ ;$$

$$f(\eta)=\frac{\psi}{\mu_1{}^{(p+1)/2}\ X^{(p+1)/(3-p)}}; \quad (22)$$

$$B=\frac{B_0}{X^{2/(3-p)}}$$

In case of two-dimensional compressible jet p = 0 and hence similarity variable (22) becomes

$$\eta=\frac{R}{\mu_1{}^{1/2}X^{2/3}}\ ;$$

$$f(\eta)=\frac{\psi}{\mu_1^{1/2}X^{1/3}}\ ; \quad (23)$$

$$B=B_0X^{-2/3}$$

Writing p=0 in equation (16) and using (15) we get

$$3f'''+f'^2+f f''-m f'=0 \quad (24)$$

$$\int_0^\infty f'^2\,d\eta=C_1 \quad (25)$$

$$\eta=0\Rightarrow f=0\ ,\ f''=0 \quad (26)$$

$$\eta=\infty\Rightarrow f'=m \quad (27)$$

Where $m=\frac{\sigma B_0^2\ \rho_1}{\rho}$ is magnetic interaction parameter and

$C_1=\frac{M_0}{2\pi}\left[\frac{\rho_1}{\nu}\right]^{1/2}$ a constant

IV. SOLUTION BY ORTHOGONAL COLLOCATION

While discussing the various methods of weighted residuals, it was mentined that in orthogonal collection technique the trial solutions are constructed with the help of orthogonal polynomials and roots of the orthogonal polynomials are selected as the collection points. Therefore the choice of the collection points is not arbitrary and hence the application of this method demands a careful attention to the fact whether orthogonality of the polynomial to be employed is valid or not over the entire domain of the problem . Some transformations are required in some cases where the boundary of the problem does not agree with the range of orthogonal polynomial to be used.

The equation of two dimensional jet derived in above article are rearranged as

$$3f''+f\ f'-m f'=0 \quad (28)$$

With associated boundary conditions

$$\eta=0\Rightarrow f=0 \quad (29)$$

$$\eta\to\infty\Rightarrow f'=m \quad (30)$$

In order to proceed with orthogonal colloction technique, let us consider Lagaurre's polynomials to construct the trial functions.

Case I: Collocation Technique: First Approximation

The trial function is assumed in the form

$$f=m\eta+e^{-\eta}(1-e^{-\eta})C_0 \quad (31)$$

f and its derivatives are substituted in the equation (28) . The residual is equated to zero at the collocation point. The collocation point is taken as the root of the Lagaurre polynomial. In this case it is '1' . Simple mathematical exercise will lead to

$$C_0=-\frac{\left(\begin{array}{r}3(e^{-\eta}-4e^{-2\eta})+m\eta(2e^{-2\eta}-e^{-\eta})\\+m(e^{-2\eta}-e^{-\eta})\end{array}\right)}{(e^{-\eta}-e^{-2\eta})(e^{-\eta}+2e^{-2\eta})} \quad (32)$$

With the help of equations (31) and (32) velocity distribution is plotted for different values of m and eta are shown in figure 1. It can be seen from the table 1 that a reasonably close agreement for satisfaction of boundary condition is obtained . The above method can be extended by assuming f in the form :

$$f = m\eta + e^{-\eta}(1-e^{-\eta})C_0 + e^{-2\eta}(1-e^{-\eta})C_1 \qquad (33)$$

f and its derivatives are substituted in the equation (28) , when residual will be equated to zero it will give rise to two non-linear algebraic equations in C_0 *and* C_1 . These equations can be solved for C_0 *and* C_1 . The process of assumptions can be continued, but in that case method of obtaining solutions for constants become more lengthy and difficult.

Case II: Trial and Error Method

We obtain the solution of equation (28) by another method known as ‘ trial and error ‘ method. Here the problem is solved completely by computing through standard methods for solving I.V.P If some initial condition is missing, it is assumed for a moment and the solution is obtained. If this solution does satisfy boundary condition initial guess is adjusted. Thus the method is totally dependent on initial guess. Here the solution has been obtained by converting second order differential equation into two first order differential equations and applying Runge-Kutta’s standard procedure. Every time initial guess for f' is chosen and in each case $f' \to m$ *as* $\eta \to \infty$ is achieved for $\eta = 9$.We had some guidelines for this guess from previous discussed collocation method.

V. RESULT AND DISCUSSION

The results obtained through above two methods are represented in table 1. Following points may be observed for these methods

(i) Both the methods are approximate methods and each has its own limitations.

(ii) In both the cases, the condition $f' \to m$ *as* $\eta \to \infty$ is satisfied, though η is different in beginning in both the cases. However boundary condition gets satisfied for same value of η in both the cases. Convergence is faster in the first case. This could happen when the assumed solution is very near to correct solution . Second method is ‘trial and error‘ method. By name itself it is implied that we have to go on changing our initial guess till all conditions match property.

VI. CONCLUSION

The velocity profile for various values of m is drawn in the figures 1 and 2. It is noted that the effect of magnetic fields is to accelerate the fluid, which is more prominent near the axis of the jet. It may be noted that the present solution will be applicable only at a large distance from the orifice and for the values of m less than or equal to one.

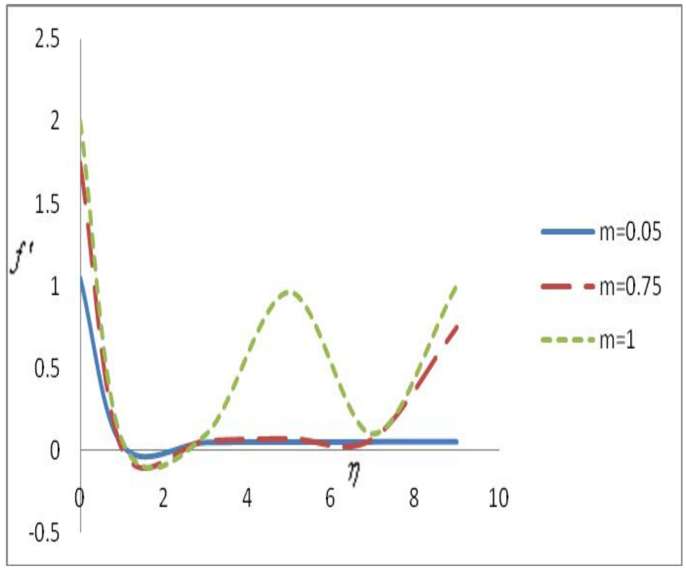

Figure 1 : Velocity profiles against eta for 2-D jet by first approximation

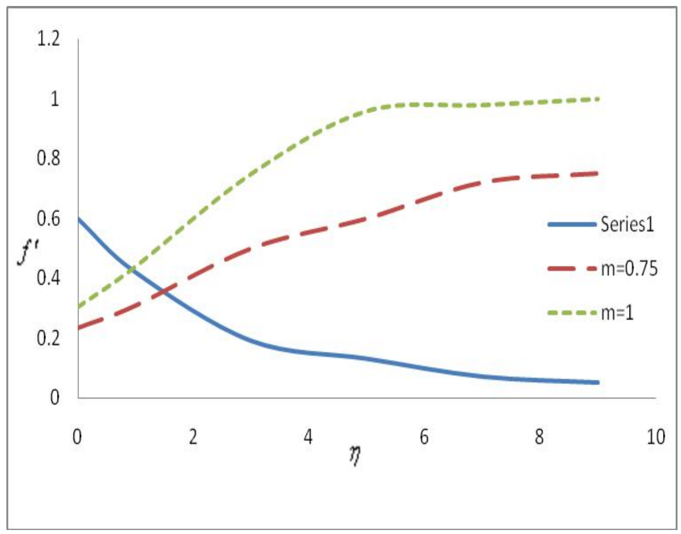

Figure 2: Velocity profiles against eta for 2-D jet by Trial and Error

Table 1: Velocity profile f’ for different eta ,different m by FA & TM

eta	m=0.05		m=0.75		m=1.0	
	FA	TM	FA	TM	FA	TM
0	1.05	0.6	1.75	0.235	2.0	0.304
1	0.03	0.42	0.01	0.31	0.06	0.44
3	0.046	0.19	0.06	0.50	0.094	0.75
5	0.0495	0.13	0.0743	0.60	0.096	0.96
7	0.0499	0.07	0.0749	0.72	0.099	0.98
9	0.05	0.05	0.75	0.75	1.0	1.0

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