Existence of Coupled Fixed Points by using Generalized Altering Distance Functions in Four Variables under F - Invariant Set

G. V. R. Babu and P. D. Sailaja

***Abstract*—Let (X, d) be a complete metric space. Let $F : X \times X \to X$ be a mapping and M be a non-empty subset of X^4. We prove the existence of coupled fixed points of F using two generalized altering distance functions in four variables under F - invariant set without using mixed monotone property. We deduce the existence of coupled fixed points of F in partially ordered metric spaces from our main results. Our results extend the fixed point results of Babu, Lalitha and Sandhya (2007) to coupled fixed point theorems and generalize the results of Bhaskar and Lakshmikantham (2006). Examples are provided in support of our results.**

***Index Terms*—Coupled fixed point, F-invariant set, mixed monotone property, generalized altering distance function.**

***MSC 2010 Codes* – 47H10, 54H25**

I. INTRODUCTION

The Banach contraction principle is one of the most versatile fundamental result in fixed point theory which has a number of generalizations. One among them is the method of altering distances between the points with the use of a certain continuous control function.

In 1984, Khan, Swaleh and Sessa [1] initiated the study of existence of fixed points in metric spaces using altering distance functions. For more details and literature, we refer Choudhury [2], Choudhury and Dutta [3], Doric [4] and Sastry and Babu [5] and the related references cited in these papers.

Throughout this paper, we write $R^+ = [0, \infty)$.

Definition 1.1 A function $\psi : R^+ \to R^+$ is said to be an *altering distance function* if the following conditions hold:

(i) ψ is continuous,
(ii) ψ is non-decreasing, and
(iii) $\psi(t) = 0$ if and only if $t = 0$.

In 2000, Choudhury and Dutta [3] extended the definition of altering distance function in single variable to two variables, and later this concept was further extended to three variables which is named as *generalized altering distance function in three variables* by Choudhury [2].

Definition 1.2 [2] A function $\psi : (R^+)^3 \to R^+$ is said to be an *generalized altering distance function in three variables* if the following conditions are satisfied:

(i) $\psi(x, y, z)$ is continuous in each of its three variables,
(ii) $\psi(x, y, z)$ is non-decreasing in each of its three variables, and
(iii) $\psi(x, y, z) = 0$ if and only if $x = y = z = 0$.

Choudhury [2] proved the following common fixed point theorem by using generalized altering distance function.

Theorem 1.3. [2] Let (X, d) be a complete metric space and $S, T : X \to X$ be two self mappings such that the following inequality is satisfied:

$$\varphi_1(d(Sx, Ty)) \leq \psi_1(d(x, y), d(x, Sx), d(y, Ty)) - \psi_2(d(x, y), d(x, Sx), d(y, Ty))$$

for each $x, y \in X$, where ψ_1 and ψ_2 are generalized altering distance functions and $\varphi_1(x) = \psi_1(x, x, x)$. Then S and T have a unique common fixed point.

Later, in 2007, Babu, Lalitha and Sandhya [6] extended the definition of generalized altering distance function in three variables to four variables as follows:

Definition 1.4 [6] A function $\psi : (R^+)^4 \to R^+$ is said to be a *generalized altering distance function in four variables* if the following conditions are satisfied:

(i) $\psi(x, y, z, t)$ is continuous in each of its four variables,
(ii) $\psi(x, y, z, t)$ is non-decreasing in each of its four variables, and
(iii) $\psi(x, y, z, t) = 0$ if and only if $x = y = z = t = 0$.

In [6] Babu, Lalitha and Sandhya established the following theorem.

Theorem 1.5. [6]Let (X, d) be a complete metric space and S and T be two selfmaps of X. Assume that there exist two generalized altering distance functions ψ_1 and ψ_2 satisfying

$$\varphi_1(d(Sx, Ty)) \leq \psi_1\big(d(x, y), d(x, Sx), d(y, Ty), \tfrac{d(x,Ty)+d(y,Sx)}{2}\big) - \psi_2\big(d(x, y), d(x, Sx), d(y, Ty), \tfrac{d(x,Ty)+d(y,Sx)}{2}\big)$$

for each $x, y \in X$, where $\varphi_1(t) = \psi_1(t, t, t, t)$. Then S and T have a unique common fixed point.

Further, in [6] the authors mentioned that the technique used in Theorem 1.5 is not possible to apply for an altering distance function of five variables (Remark 3.4 and Example 3.5 of [6]).

Definition 1.6 Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$ is said to be a *coupled fixed point of* F if $F(x, y) = x$ and $F(y, x) = y$.

G. V. R. Babu is a Professor in the Department of Mathematics, Andhra University, Visakhapatnam - 530 003, Andhra Pradesh, India. (E-mail: gvr_babu@hotmail.com)

P. D. Sailaja is an Associate Professor in the Department of Science and Humanities, Lendi Institute of Engineering and Technology, Vizianagaram - 535 005, India. (E-mail: sailajadurga@yahoo.com).

Definition 1.7 [7] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. We say that F satisfies *mixed monotone property* if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y, *i.e.,* for any $x, y \in X$

$x_1, x_2 \in X,\ x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$ and

$y_1, y_2 \in X,\ y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$.

Theorem 1.8. [8] Let $(X, \preceq)$ be a partially ordered set and suppose that d is a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping satisfying mixed monotone property. Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \tfrac{k}{2}[d(x, u) + d(y, v)] \qquad (1.8.1)$$

for each $x \succeq u$ and $y \preceq v$.

Suppose that either F is continuous or the following conditions hold in X:

(i) if a non-decreasing sequence $\{x_n\} \subseteq X$ with $x_n \to x$, then $x_n \preceq x$ for all n and

(ii) if a non-increasing sequence $\{y_n\} \subseteq X$ with $y_n \to y$, then $y_n \succeq y$ for all n.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ then F has a coupled fixed point.

For more works on the existence of fixed points/coupled fixed points in partially ordered sets, we refer ([9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]).

In 2010, Samet and Vetro [21] introduced the notion of F-invariant set and established the existence of coupled fixed points in complete metric spaces. The notion of F-invariant set is more general than the mixed monotone property, *i.e.,* the class of all maps $F : X \times X \to X$ where M is an F-invariant set includes the class of all mixed monotone maps, $F : X \times X \to X$ on a partially ordered set $(X, \preceq)$. For more literature on these works we refer ([22], [23], [24], [25]).

Definition 1.9 (Samet and Vetro [21]) Let (X, d) be a metric space and M be a non-empty subset of X^4. Let $F : X \times X \to X$ be a mapping. We say that M is an *F-invariant subset of X^4* if the following two conditions hold for each $(x, y, u, v) \in X$:

(i) $(x, y, u, v) \in M \Leftrightarrow (v, u, y, x) \in M$,

(ii) $(x, y, u, v) \in M \Rightarrow$

$$(F(x, y), F(y, x), F(u, v), F(v, u)) \in M.$$

Definition 1.10 (Sintunavarat Kumam and Cho [23]) Let X be a non-empty set and M be a non-empty subset of X^4. We say that M satisfies the *transitivity property* if for each $x, y, u, v, z, t \in X$, $(x, y, u, v) \in M$ and $(u, v, z, t) \in M$ implies $(x, y, z, t) \in M$.

The following example shows that the class of all maps $F : X \times X \to X$ where M is an F-invariant set is larger than the class of all mixed monotone maps, $F : X \times X \to X$ on a partially ordered set $(X, \preceq)$.

Example 1.11 (Sintunavarat Kumam and Cho [23]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping satisfying the mixed monotone property. *i.e.,* for any $x, y \in X$

$x_1, x_2 \in X,\ x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$ and

$y_1, y_2 \in X,\ y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$.

We define a subset M of X^4 by

$$M = \{(a, b, c, d) \in X^4 / a \succeq c \ and \ b \preceq d\}.$$

Then M is an F-invariant subset of X^4 which satisfies the transitivity property.

In this paper, we prove the existence of coupled fixed points of F using generalized altering distance functions in four variables under F - invariant set without using mixed monotone property. We deduce the existence of coupled fixed points of F in partially ordered metric spaces from our main results. Our results extend the fixed point results of Babu, Lalitha and Sandhya [6] to coupled fixed point theorems and generalize the results of Bhaskar and Lakshmikantham [8]. Examples are provided in support of our results.

II. Main Results

We start this section with the following lemma which is useful in our subsequent discussion.

Lemma 2.1 Let $(X,\ d)$ be a metric space and $\{x_n\},\ \{y_n\}$ be two sequences in X such that $d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \to 0$ as $n \to \infty$. If at least one of $\{x_n\}$ and $\{y_n\}$ is not a Cauchy sequence then there exist $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) \geq m(k) > k$ such that the following hold:

(i) $\lim\limits_{k\to\infty} [d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})] = \epsilon$

(ii) $\lim\limits_{k\to\infty} [d(x_{m(k)+1}, x_{n(k)}) + d(y_{m(k)+1}, y_{n(k)})] = \epsilon$

(iii) $\lim\limits_{k\to\infty} [d(x_{m(k)}, x_{n(k)+1}) + d(y_{m(k)}, y_{n(k)+1})] = \epsilon$ and

(iv) $\lim\limits_{k\to\infty} [d(x_{m(k)+1}, x_{n(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1})] = \epsilon$.

Proof: Suppose that atleast one of $\{x_n\}$ and $\{y_n\}$ is not a Cauchy sequence in X. Hence there exist $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) \geq n(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)}) \geq \epsilon. \qquad (2.1.1)$$

Choose $m(k)$ the smallest integer such that (2.1.1) holds. Then, we have (2.1.1) together with

$$d(x_{m(k)-1}, x_{n(k)}) + d(y_{m(k)-1}, y_{n(k)}) < \epsilon \qquad (2.1.2)$$

hold.

Let us now prove (i).

(i): From (2.1.1), we have

$$\begin{aligned}\epsilon &\leq d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)}) \\ &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \\ &\qquad + d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)}) \\ &< \epsilon + d(x_{m(k)}, x_{m(k)-1}) + d(y_{m(k)}, y_{m(k)-1}), \text{ using (2.1.2).}\end{aligned}$$

On taking limit supremum as $k \to \infty$, we get

$$\epsilon \leq \limsup_{k\to\infty} [d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})]$$

$$< \limsup_{k\to\infty}[\epsilon + d(x_{m(k)}, x_{m(k)-1}) + d(y_{m(k)}, y_{m(k)-1})]$$
$$= \epsilon.$$

Hence

$$\limsup_{k\to\infty}[d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})] = \epsilon. \quad (2.1.3)$$

On taking limit infimum as $k \to \infty$, we get

$$\epsilon \leq \liminf_{k\to\infty}[d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})]$$
$$< \liminf_{k\to\infty}[\epsilon + d(x_{m(k)}, x_{m(k)-1}) + d(y_{m(k)}, y_{m(k)-1})]$$

$= \epsilon$, since $\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$ and the equality holds if any one of the sequence converges.

Therefore,

$$\liminf_{k\to\infty}[d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})] = \epsilon. \quad (2.1.4)$$

Hence from (2.1.3) and (2.1.4), $\lim_{k\to\infty}[d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})]$ exists and is equal to ϵ. Hence (i) holds.

(ii): We consider

$$d(x_{m(k)+1}, x_{n(k)}) + d(y_{m(k)+1}, y_{n(k)})$$
$$\leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)})$$
$$+ d(y_{m(k)+1}, y_{m(k)}) + d(y_{m(k)}, y_{n(k)})$$
$$\leq d(x_{m(k)+1}, x_{m(k)}) + d(y_{m(k)+1}, y_{m(k)})$$
$$+ d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)}).$$

On taking limit supremum as $k \to \infty$, we get

$$\limsup_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)}) + d(y_{m(k)+1}, y_{n(k)})] \leq \epsilon. \quad (2.1.5)$$

Now, for each $k = 0, 1, 2...$

$$d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})$$
$$\leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)})$$
$$+ d(y_{m(k)}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{n(k)})$$

Hence

$$d(x_{m(k)+1}, x_{n(k)}) + d(y_{m(k)+1}, y_{n(k)})$$
$$\geq [d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})]$$
$$- [d(x_{m(k)}, x_{m(k)+1}) + d(y_{m(k)}, y_{m(k)+1})].$$

On taking limit infimum as $k \to \infty$, we get

$$\liminf_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)}) + d(y_{m(k)+1}, y_{n(k)})]$$
$$\geq \liminf_{k\to\infty}[d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})]$$
$$- \limsup_{k\to\infty}[d(x_{m(k)}, x_{m(k)+1}) + d(y_{m(k)}, y_{m(k)+1})]$$
$$= \epsilon. \quad (2.1.6)$$

Hence from (2.1.5) and (2.1.6), we get

$$\epsilon \leq \liminf_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)}) + d(y_{m(k)+1}, y_{n(k)})]$$
$$\leq \limsup_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)}) + d(y_{m(k)+1}, y_{n(k)})] \leq \epsilon.$$

Hence $\lim_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)}) + d(y_{m(k)+1}, y_{n(k)})]$ exists and is equal to ϵ.

Therefore (ii) holds.

(iii): We consider

$$d(x_{m(k)}, x_{n(k)+1}) + d(y_{m(k)}, y_{n(k)+1})$$
$$\leq d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}) + d(y_{m(k)}, y_{n(k)})$$
$$+ d(y_{n(k)}, y_{n(k)+1})$$
$$\leq d(x_{m(k)}, x_{n(k)}) + d(y_{n(k)}, y_{m(k)}) + d(x_{n(k)}, x_{n(k)+1})$$
$$+ d(y_{n(k)}, y_{n(k)+1}).$$

On taking limit supremum as $k \to \infty$, we get

$$\limsup_{k\to\infty}[d(x_{m(k)}, x_{n(k)+1}) + d(y_{m(k)}, y_{n(k)+1})] \leq \epsilon. \quad (2.1.7)$$

Now, for each $k = 0, 1, 2...$

$$d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})$$
$$\leq d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$
$$+ d(y_{m(k)}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)})$$

Hence

$$d(x_{m(k)}, x_{n(k)+1}) + d(y_{m(k)}, y_{n(k)+1})$$
$$\geq [d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})]$$
$$- [d(x_{n(k)+1}, x_{n(k)}) + d(y_{n(k)+1}, y_{n(k)})].$$

On taking limit infimum as $k \to \infty$, we get

$$\liminf_{k\to\infty}[d(x_{m(k)}, x_{n(k)+1}) + d(y_{m(k)}, y_{n(k)+1})]$$
$$\geq \liminf_{k\to\infty}[d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})]$$
$$- \limsup_{k\to\infty}[d(x_{n(k)+1}, x_{n(k)}) + d(y_{n(k)+1}, y_{n(k)})]$$
$$= \epsilon. \quad (2.1.8)$$

Hence from (2.1.7) and (2.1.8), we get

$$\epsilon \leq \liminf_{k\to\infty}[d(x_{m(k)}, x_{n(k)+1}) + d(y_{m(k)}, y_{n(k)+1})]$$
$$\leq \limsup_{k\to\infty}[d(x_{m(k)}, x_{n(k)+1}) + d(y_{m(k)}, y_{n(k)+1})] \leq \epsilon.$$

Hence $\lim_{k\to\infty}[d(x_{m(k)}, x_{n(k)+1}) + d(y_{m(k)}, y_{n(k)+1})]$ exists and is equal to ϵ.

Therefore (iii) holds.

(iv): For $k = 0, 1, 2, ...$, we consider

$$d(x_{m(k)+1}, x_{n(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1})$$
$$\leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})$$
$$+ d(y_{m(k)+1}, y_{m(k)}) + d(y_{m(k)}, y_{n(k)}) + d(y_{n(k)}, y_{n(k)+1}).$$

On taking limit supremum as $k \to \infty$, we get

$$\limsup_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1})] \leq \epsilon. \quad (2.1.9)$$

Now, for each $k = 0, 1, 2...$

$$d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})$$

$$\leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)}) + d(y_{m(k)}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)}).$$

Therefore,

$$\begin{aligned} &d(x_{m(k)+1}, x_{n(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1}) \\ &\quad \geq [d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})] \\ &\qquad - [d(x_{m(k)+1}, x_{m(k)}) + d(y_{m(k)+1}, y_{m(k)})] \\ &\qquad - [d(x_{n(k)+1}, x_{n(k)}) + d(y_{n(k)+1}, y_{n(k)})]. \end{aligned}$$

On taking limit infimum as $k \to \infty$, we get

$$\begin{aligned} &\liminf_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1})] \\ &\quad \geq \liminf_{k\to\infty}[d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})] \\ &\qquad - \limsup_{k\to\infty}[d(x_{m(k)+1}, x_{m(k)}) + d(y_{m(k)+1}, y_{m(k)})] \\ &\qquad - \limsup_{k\to\infty}[d(x_{n(k)+1}, x_{n(k)}) + d(y_{n(k)+1}, y_{n(k)})] \\ &\quad = \epsilon. \end{aligned} \tag{2.1.10}$$

Hence from (2.1.9) and (2.1.10), we get

$$\begin{aligned} \epsilon &\leq \liminf_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1})] \\ &\leq \limsup_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1})] \leq \epsilon. \end{aligned}$$

Therefore $\lim_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1})]$ exists and is equal to ϵ. Hence (iv) holds. □

Theorem 2.2. Let (X, d) be a complete metric space and M be a non empty subset of X^4. Suppose that $F : X \times X \to X$ is a mapping. Assume that there exist two generalized altering distance functions in four variables ψ_1 and ψ_2 such that

$$\begin{aligned} &\varphi_1\left(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2}\right) \\ &\leq \psi_1\Big(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\ &\qquad \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \\ &\qquad \tfrac{1}{2}\big[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}\big]\Big) \\ &\quad - \psi_2\Big(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\ &\qquad \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \\ &\qquad \tfrac{1}{2}\big[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2}, \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}\big]\Big) \end{aligned} \tag{2.2.1}$$

for each $(x, y, u, v) \in M$ and $\varphi_1(t) = \psi_1(t, t, t, t)$. Further assume that

(i) either (a) F is continuous or

(b) X has the following property:

"for any two sequences $\{x_n\}, \{y_n\}$ in X with $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$, if $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $(x, y, x_n, y_n) \in M$ for all n",

(ii) M is F-invariant subset of X^4 satisfying the transitivity property, and

(iii) there exist $x_0,\ y_0 \in X$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$

then F has a coupled fixed point.

Proof: Let $(x_0, y_0) \in X \times X$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$. Let $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Then $(x_1, y_1, x_0, y_0) \in M$. Since M is F-invariant subset of X^4, we have $(F(x_1, y_1), F(y_1, x_1), F(x_0, y_0), F(y_0, x_0)) \in M$. Again let $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Then $(x_2, y_2, x_1, y_1) \in M$. On continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$x_{n+1} = F(x_n, y_n),\ y_{n+1} = F(y_n, x_n)$, and

$(x_{n+1}, y_{n+1}, x_n, y_n) \in M$ for $n = 0, 1, 2...$.

If $x_n = x_{n+1}$ and $y_n = y_{n+1}$ for some n then (x_n, y_n) is a coupled fixed point of F. So, without loss of generality, we assume that $x_n \neq x_{n+1}$ or $y_n \neq y_{n+1}$ for all n.

Now for $n \in \{0, 1, 2...\}$, we have

$$\begin{aligned} &\varphi_1\left(\tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}\right) \\ &= \varphi_1\left(\tfrac{d(F(x_n,y_n),\ F(x_{n+1},y_{n+1}))\ +\ d(F(y_n,x_n),\ F(y_{n+1},x_{n+1}))}{2}\right) \\ &= \varphi_1\left(\tfrac{d(F(x_{n+1},y_{n+1}),\ F(x_n,y_n))\ +\ d(F(y_{n+1},x_{n+1}),\ F(y_n,x_n))}{2}\right) \\ &\leq \psi_1\Big(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,F(x_n,y_n))+d(y_n,F(y_n,x_n))}{2}, \\ &\qquad \tfrac{d(x_{n+1},F(x_{n+1},y_{n+1}))+d(y_{n+1},F(y_n,x_n))}{2}, \\ &\qquad \tfrac{1}{2}\big[\tfrac{d(x_{n+1},F(x_n,y_n))+d(y_{n+1},F(y_n,x_n))}{2} \\ &\qquad\quad + \tfrac{d(x_n,F(x_{n+1},y_{n+1}))+d(y_n,F(y_{n+1},x_{n+1}))}{2}\big]\Big) \\ &\quad -\psi_2\Big(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,F(x_n,y_n))+d(y_n,F(y_n,x_n))}{2}, \\ &\qquad \tfrac{d(x_{n+1},F(x_{n+1},y_{n+1}))+d(y_{n+1},F(y_n,x_n))}{2}, \\ &\qquad \tfrac{1}{2}\big[\tfrac{d(x_{n+1},F(x_n,y_n))+d(y_{n+1},F(y_n,x_n))}{2} \\ &\qquad\quad + \tfrac{d(x_n,F(x_{n+1},y_{n+1}))+d(y_n,F(y_{n+1},x_{n+1}))}{2}\big]\Big) \\ &\leq \psi_1\Big(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2}, \\ &\qquad \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \\ &\qquad \tfrac{1}{2}\big[\tfrac{d(x_{n+1},x_{n+1})+d(y_{n+1},y_{n+1})}{2} + \tfrac{d(x_n,x_{n+2})+d(y_n,y_{n+2})}{2}\big]\Big) \\ &\quad -\psi_2\Big(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2}, \\ &\qquad \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \\ &\qquad \tfrac{1}{2}\big[\tfrac{d(x_{n+1},x_{n+1})+d(y_{n+1},y_{n+1})}{2} + \tfrac{d(x_n,x_{n+2})+d(y_n,y_{n+2})}{2}\big]\Big) \\ &= \psi_1\Big(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2}, \\ &\qquad \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{1}{2}\big[\tfrac{d(x_n,x_{n+2})+d(y_n,y_{n+2})}{2}\big]\Big) \\ &\quad - \psi_2\Big(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2}, \\ &\qquad \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{1}{2}\big[\tfrac{d(x_n,x_{n+2})+d(y_n,y_{n+2})}{2}\big]\Big) \\ &\leq \psi_1\Big(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2}, \\ &\qquad \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{1}{2}\big[\tfrac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2} \\ &\qquad\quad + \tfrac{d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})}{2}\big]\Big) \end{aligned}$$

$$- \psi_2(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2}, \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{1}{2}[\tfrac{d(x_n,x_{n+2})+d(y_n,y_{n+2})}{2}]). \tag{2.2.2}$$

If $d(x_{n+1}, x_n) + d(y_{n+1}, y_n) < d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})$ for some n =0,1,2..., then since ψ_1 is non-decreasing in each of its variables, from (2.2.2), we get

$$\begin{aligned}
&\varphi_1(\tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2})\\
&\le \psi_1(\tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2})\\
&\quad - \psi_2(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2}, \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{1}{2}[\tfrac{d(x_n,x_{n+2})+d(y_n,y_{n+2})}{2}])\\
&= \varphi_1(\tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2})\\
&\quad - \psi_2(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2}, \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{1}{2}[\tfrac{d(x_n,x_{n+2})+d(y_n,y_{n+2})}{2}]),
\end{aligned}$$

which is a contradiction. Hence

$d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) < d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ for all n = 0,1,2,... .

Therefore, the sequence $\{r_n\}$, $r_n = d(x_{n+1}, x_n)+d(y_{n+1}, y_n)$ for all n = 0,1,2,... is a decreasing sequence of non-negative real numbers. Hence there exists $r \ge 0$ such that $r \le r_n$ for each n = 0,1,2,... and

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty}[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = r.$$

We show that $r = 0$. Suppose $r > 0$. Now since ψ_2 is increasing in each of its four variables, using (2.2.2), we have

$$\begin{aligned}
&0 \le \psi_2(\tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}, 0)\\
&\le \psi_2(\tfrac{d(x_{n+1},x_n)+d(y_{n+1},y_n)}{2}, \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2}, \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{1}{2}[\tfrac{d(x_n,x_{n+2})+d(y_n,y_{n+2})}{2}])\\
&\le \psi_1(\tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}, \tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2})\\
&\quad - \varphi_1(\tfrac{d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})}{2}).
\end{aligned}$$

Now on letting $n \to \infty$, we get

$$\begin{aligned}
0 \le \psi_2(\tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}, 0) &\le \psi_1(\tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}, \tfrac{1}{2}(\tfrac{r}{2} + \tfrac{r}{2})) - \varphi_1(\tfrac{r}{2})\\
&= \psi_1(\tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}) - \varphi_1(\tfrac{r}{2})\\
&= \varphi_1(\tfrac{r}{2}) - \varphi_1(\tfrac{r}{2})\\
&= 0.
\end{aligned}$$

Hence $\psi_2(\frac{r}{2}, \frac{r}{2}, \frac{r}{2}, 0)$ = 0. This implies that $r = 0$. Thus,

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty}[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = 0.$$

Now we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Suppose that at least one of $\{x_n\}$ and $\{y_n\}$ is not a Cauchy sequence in X. Hence there exist $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) \ge n(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)}) \ge \epsilon. \tag{2.2.3}$$

Choose $m(k)$ the smallest positive integer such that (2.2.3) holds. Then, we have (2.2.3) and

$$d(x_{m(k)-1}, x_{n(k)}) + d(y_{m(k)-1}, y_{n(k)}) < \epsilon \tag{2.2.4}$$

hold. Now, from Lemma 2.1, we have the following:

(i) $\lim_{k\to\infty}[d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})] = \epsilon$
(ii) $\lim_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)}) + d(y_{m(k)+1}, y_{n(k)})] = \epsilon$
(iii) $\lim_{k\to\infty}[d(x_{m(k)}, x_{n(k)+1}) + d(y_{m(k)}, y_{n(k)+1})] = \epsilon$ and
(iv) $\lim_{k\to\infty}[d(x_{m(k)+1}, x_{n(k)+1}) + d(y_{m(k)+1}, y_{n(k)+1})] = \epsilon$.

Now, since $m(k) \ge n(k)$ and M satisfies the transitivity property, we have $(x_{m(k)}, y_{m(k)}, x_{n(k)}, y_{n(k)}) \in M$. By using (2.2.1), we get

$$\begin{aligned}
&0 \le \varphi_1(\tfrac{\epsilon}{2}) = \varphi_1(\lim_{k\to\infty} \tfrac{d(x_{m(k)+1},x_{n(k)+1})\ +\ d(y_{m(k)+1},y_{n(k)+1})}{2})\\
&= \lim_{k\to\infty} \varphi_1(\tfrac{1}{2}[d(F(x_{m(k)}, y_{m(k)}),\ F(x_{n(k)}, y_{n(k)}))\\
&\qquad +\ d(F(y_{m(k)}, x_{m(k)}),\ F(y_{n(k)}, x_{n(k)}))])\\
&\le \lim_{k\to\infty}[\psi_1(\tfrac{d(x_{m(k)},x_{n(k)})+d(y_{m(k)},y_{n(k)})}{2}, \tfrac{d(x_{m(k)},x_{m(k)+1})+d(y_{m(k)},y_{m(k)+1})}{2}, \tfrac{d(x_{n(k)},x_{n(k)+1})+d(y_{n(k)},y_{n(k)+1})}{2},\\
&\qquad \tfrac{1}{2}[\tfrac{d(x_{m(k)},x_{n(k)+1})+d(y_{m(k)},y_{n(k)+1})}{2} + \tfrac{d(x_{n(k)},x_{m(k)+1})+d(y_{n(k)},y_{m(k)+1})}{2}])\\
&\quad -\psi_2(\tfrac{d(x_{m(k)},x_{n(k)})+d(y_{m(k)},y_{n(k)})}{2}, \tfrac{d(x_{m(k)},x_{m(k)+1})+d(y_{m(k)},y_{m(k)+1})}{2}, \tfrac{d(x_{n(k)},x_{n(k)+1})+d(y_{n(k)},y_{n(k)+1})}{2}, \tfrac{1}{2}[\tfrac{d(x_{m(k)},x_{n(k)+1})+d(y_{m(k)},y_{n(k)+1})}{2}\\
&\qquad + \tfrac{d(x_{n(k)},x_{m(k)+1})+d(y_{n(k)},y_{m(k)+1})}{2}])]\\
&= \psi_1(\tfrac{\epsilon}{2}, 0, 0, \tfrac{1}{2}(\tfrac{\epsilon}{2} + \tfrac{\epsilon}{2})) - \psi_2(\tfrac{\epsilon}{2}, 0, 0, \tfrac{1}{2}(\tfrac{\epsilon}{2} + \tfrac{\epsilon}{2}))\\
&= \psi_1(\tfrac{\epsilon}{2}, 0, 0, \tfrac{\epsilon}{2}) - \psi_2(\tfrac{\epsilon}{2}, 0, 0, \tfrac{\epsilon}{2})\\
&\le \psi_1(\tfrac{\epsilon}{2}, \tfrac{\epsilon}{2}, \tfrac{\epsilon}{2}, \tfrac{\epsilon}{2}) - \psi_2(\tfrac{\epsilon}{2}, 0, 0, \tfrac{\epsilon}{2})\\
&\le \varphi_1(\tfrac{\epsilon}{2}) - \psi_2(\tfrac{\epsilon}{2}, 0, 0, \tfrac{\epsilon}{2}).
\end{aligned}$$

This implies that $\psi_2(\frac{\epsilon}{2}, 0, 0, \frac{\epsilon}{2}) = 0$. Hence $\epsilon = 0$, a contradiction. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete there exist $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

First we suppose that (i) (*a*) holds. That is F is continuous. Then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x, y) \text{ and}$$
$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(y, x).$$

Therefore (x, y) is a coupled fixed point of F.

Now suppose that condition (i) (*b*) holds. Hence $(x, y, x_n, y_n) \in M$ for all n.

Now, we consider

$$\varphi_1(\tfrac{d(x_{n+1},F(x,y))+d(y_{n+1},F(y,x))}{2})$$

$$
\begin{aligned}
&= \varphi_1(\tfrac{F(x_n,y_n),F(x,y))+d(F(y_n,x_n),F(y,x))}{2}) \\
&\leq \psi_1(\tfrac{d(x,x_n)+d(y,y_n)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\
&\quad \tfrac{d(x_n,F(x_n,y_n))+d(y_n,F(y_n,x_n))}{2}, \\
&\quad \tfrac{1}{2}[\tfrac{d(x,F(x_n,y_n))+d(y,F(y_n,x_n))}{2} + \tfrac{d(x_n,F(x,y))+d(y_n,F(y,x))}{2}]) \\
&\quad -\psi_2(\tfrac{d(x,x_n)+d(y,y_n)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\
&\quad \tfrac{d(x_n,F(x_n,y_n))+d(y_n,F(y_n,x_n))}{2}, \\
&\quad \tfrac{1}{2}[\tfrac{d(x,F(x_n,y_n))+d(y,F(y_n,x_n))}{2} + \tfrac{d(x_n,F(x,y))+d(y_n,F(y,x))}{2}]).
\end{aligned}
$$

On letting $n \to \infty$, we get

$$
\begin{aligned}
&\varphi_1(\tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}) \\
&\leq \psi_1(0, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\
&\quad \tfrac{1}{2}[\tfrac{d(x,F(x,y))+d(y,F(y,x))}{2} + \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}]) \\
&\quad -\psi_2(0, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\
&\quad \tfrac{1}{2}[\tfrac{d(x,F(x,y))+d(y,F(y,x))}{2} + \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}]) \\
&\leq \psi_1(\tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\
&\quad \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}) \\
&\quad -\psi_2(0, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\
&\quad \tfrac{1}{2}[\tfrac{d(x,F(x,y))+d(y,F(y,x))}{2} + \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}]) \\
&= \varphi_1(\tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}) \\
&\quad - \psi_2(0, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\
&\quad \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}).
\end{aligned}
$$

This implies that

$$
\psi_2(0, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}]) = 0.
$$

Therefore $d(x, F(x, y)) + d(y, F(y, x)) = 0$. *i.e.,* $F(x, y) = x$ and $F(y, x) = y$. Hence (x, y) is a coupled fixed point of F.□

We observe that Theorem 2.2 does not guarantee the uniqueness of coupled fixed point (Example 2.4). In order to obtain the uniqueness of coupled fixed point of Theorem 2.2. we impose certain additional condition and prove the following theorem.

Theorem 2.3. In addition to the hypotheses of Theorem *2.2* assume that the following condition holds:

"for each $(x, y), (u, v) \in X \times X$ there exists $(z, t) \in X \times X$ such that $(x, y, z, t) \in M$ and $(z, t, u, v) \in M$". *(2.3.1)*

Proof: By Theorem 2.2, we conclude that F has a coupled fixed point (x, y) (say). *i.e.,*$x = F(x, y)$ and $y = F(y, x)$.

Suppose if possible there exists $(u, v) \in X \times X$ such that $F(u, v) = u$ and $F(v, u) = v$. Assume that $x \neq u$ or $y \neq v$.

Now from (2.3.1) there exists $(z, t) \in X \times X$ such that $(x, y, z, t) \in M$ and $(z, t, u, v) \in M$. Then from (2.2.1), we have

$$
\begin{aligned}
&\varphi_1(\tfrac{d(x,u)+d(y,v)}{2}) = \varphi_1(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2}) \\
&\leq \psi_1(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\
&\quad \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \\
&\quad \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}]) \\
&\quad -\psi_2(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\
&\quad \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \\
&\quad \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}]) \\
&= \psi_1(\tfrac{d(x,u)+d(y,v)}{2}, 0, 0, \tfrac{d(x,u)+d(y,v)}{2}) \\
&\quad - \psi_2(\tfrac{d(x,u)+d(y,v)}{2}, 0, 0, \tfrac{d(x,u)+d(y,v)}{2}) \\
&\leq \psi_1(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,u)+d(y,v)}{2}) \\
&\quad -\psi_2(\tfrac{d(x,u)+d(y,v)}{2}, 0, 0, \tfrac{d(x,u)+d(y,v)}{2}) \\
&= \varphi_1(\tfrac{d(x,u)+d(y,v)}{2}) - \psi_2(\tfrac{d(x,u)+d(y,v)}{2}, 0, 0, \tfrac{d(x,u)+d(y,v)}{2}),
\end{aligned}
$$

which is a contradiction. Hence $x = u$ and $y = v$. Thus uniqueness of coupled fixed point follows. □

The following is an example in support of Theorem 2.2, in which the function F have more than one coupled fixed point.

Example 2.4 Let $X = \{\frac{1}{2}, 1, 2, 4\}$ with the usual metric.

Let $A = \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1), (\frac{1}{2}, 2), (\frac{1}{2}, 4), (1, \frac{1}{2}), (1, 1), (1, 2), (1, 4)\}$
$B = \{(2, \frac{1}{2}), (2, 1), (2, 2), (2, 4)\}$ and
$C = \{(4, \frac{1}{2}), (4, 1), (4, 2), (4, 4)\}$.

We define $F : X \times X \to X$ by

$$
F(x, y) = \begin{cases} \frac{1}{2}, & if \ \ (x, y) \in A \\ 1, & if \ (x, y) \in B \\ 0, & if \ (x, y) \in C. \end{cases}
$$

We define $\psi_1, \psi_2 : (R^+)^4 \to R^+$ by

$\psi_1(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$ and

$\psi_2(t_1, t_2, t_3, t_4) = \frac{t_1+t_2+t_3+t_4}{16}, \ t_1, t_2, t_3, t_4 \geq 0.$

Let

$$
\begin{aligned}
M = \{&(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), (1, 1, 1, 1), (2, 2, 2, 2), (4, 4, 4, 4), \\
&(1, \tfrac{1}{2}, \tfrac{1}{2}, 2), (2, \tfrac{1}{2}, \tfrac{1}{2}, 1), (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 2), (2, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), \\
&(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1), (1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), (1, \tfrac{1}{2}, 1, 2), (2, 1, \tfrac{1}{2}, 1), \\
&(1, \tfrac{1}{2}, 1, 1), (1, 1, \tfrac{1}{2}, 1), (2, \tfrac{1}{2}, 2, 2), (2, 2, \tfrac{1}{2}, 2), \\
&(2, \tfrac{1}{2}, 2, 1), (1, 2, \tfrac{1}{2}, 2), (4, \tfrac{1}{2}, 4, 2), (2, 4, \tfrac{1}{2}, 4), \\
&(4, \tfrac{1}{2}, 4, 1), (1, 4, \tfrac{1}{2}, 4)\}.
\end{aligned}
$$

Then M is an F-invariant set satisfying the transitivity property. We choose $x_0 = 4$ and $y_0 = 1$. Then $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) = (4, \frac{1}{2}, 4, 1) \in M$.

Here we observe that F fails to satisfy mixed monotone property for $(x, y) = (\frac{1}{2}, \frac{1}{2})$ and $(u, v) = (\frac{1}{2}, 4)$.

We now verify the inequality (2.2.1) for the above pairs.

We denote L.H.S of the inequality (2.2.1) be denoted by 'a' and that of the R.H.S by 'b'.

Case (i): $(x, y, u, v) \in M$ such that $(x, y), (u, v), (y, x), (v, u)$ are either in A or B or C.

In this case $a = \varphi_1(\frac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = 0$ so that the inequality (2.2.1) holds trivially.

Case (ii): $(x, y, u, v) = (1, \frac{1}{2}, \frac{1}{2}, 2)$.

Then $a = \frac{1}{4},\ b = \frac{113}{128}$.

Case (iii): $(x, y, u, v) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2)$ or $(2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ or $(4, \frac{1}{2}, 4, 2)$ or $(2, 4, \frac{1}{2}, 4)$.

Here $a = \frac{1}{4},\ b = \frac{41}{64}$.

Case (iv): $(x, y, u, v) = (1, \frac{1}{2}, 1, 2)$.

In this case $a = \frac{1}{4},\ b = \frac{38}{64}$.

Case (v): $(x, y, u, v) = (2, 1, \frac{1}{2}, 1)$.

In this case $a = \frac{1}{4},\ b = \frac{9}{16}$.

Case (vi): $(x, y, u, v) = (2, \frac{1}{2}, 2, 2)$ or $(2, 2, \frac{1}{2}, 2)$.

Then $a = \frac{1}{4},\ b = \frac{51}{64}$.

Case (vii): $(x, y, u, v) = (2, \frac{1}{2}, \frac{1}{2}, 1)$.

Then $a = \frac{1}{4},\ b = \frac{109}{128}$.

Case (viii): $(x, y, u, v) = (2, \frac{1}{2}, \frac{1}{2}, 2)$.

Then $a = \frac{1}{4},\ b = \frac{81}{64}$.

From all the above cases, we conclude that F satisfies the inequality (2.2.1). Hence all the hypotheses of Theorem 2.2 hold good and $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 4)$ and $(4, 4)$ are coupled fixed points of F.

Here we observe that the condition (2.3.1) fails to hold, for if $(x, y) = (\frac{1}{2}, \frac{1}{2})$ and $(u, v) = (\frac{1}{2}, 4)$ there exists no $(z, t) \in x \times X$ such that $(x, y, z, t) \in M$ and $(z, t, u, v) \in M$.

One more example in support of Theorem 2.2 is the following.

Example 2.5 Let $X = \{1, 3, 5, 7\}$ with the usual metric.

Let $A = \{(1, 1), (1, 3), (3, 1), (3, 3), (3, 5), (3, 7), (5, 7), (5, 5), (7, 3), (7, 5), (7, 7)\}$

$B = \{(1, 7), (7, 1)\}$ and

$C = \{(1, 5), (5, 1), (5, 3)\}$.

We define $F : X \times X \to X$ by

$$F(x, y) = \begin{cases} 3, & if \ \ (x, y) \in A \\ 7, & if \ (x, y) \in B \\ 1, & if \ (x, y) \in C. \end{cases}$$

We define $\psi_1, \psi_2 : (R^+)^4 \to R^+$ by

$\psi_1(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$ and

$\psi_2(t_1, t_2, t_3, t_4) = \frac{t_1+t_2+t_3+t_4}{24},\ t_1, t_2, t_3, t_4 \geq 0.$

Let

$M = \{(1, 1, 3, 3), (3, 3, 1, 1), (3, 3, 3, 3), (3, 3, 7, 7), (7, 7, 3, 3), (5, 3, 5, 7)(7, 5, 3, 5), (7, 7, 7, 7), (7, 3, 5, 7), (7, 5, 3, 7), (1, 1, 7, 7), (7, 7, 1, 1), (3, 3, 5, 5), (5, 5, 3, 3), (7, 3, 7, 5), (5, 7, 3, 7)(5, 5, 5, 5), (5, 3, 7, 5), (5, 7, 3, 5), ((1, 1, 5, 5), (5, 5, 1, 1), (1, 1, 1, 1), (1, 3, 3, 3), (3, 3, 3, 1), (1, 1, 3, 1), (1, 3, 1, 1), (7, 7, 3, 1), (1, 3, 7, 7)(5, 5, 3, 1), (1, 3, 5, 5), (7, 7, 5, 5), (5, 5, 7, 7), (5, 5, 5, 5), (5, 3, 3, 7), (7, 3, 3, 5), (5, 3, 3, 5), (7, 3, 3, 7), (1, 3, 3, 1)\}$.

Then it is easy to see that M is an F-invariant set satisfying the transitivity property. We choose $x_0 = 3$ and $y_0 = 1$. Then $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) = (3, 3, 3, 1) \in M$. We now verify the inequality (2.2.1) for $(x, y, u, v) \in M$.

Case (i): $(x, y, u, v) \in M$ such that $(x, y), (u, v), (y, x), (v, u)$ are either in A or B or C.

In this case $\varphi_1(\frac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = 0$ so that the inequality (2.2.1) holds trivially.

Case (ii): $(x, y, u, v) = (5, 3, 5, 7)$

$$\text{In this case } \varphi_1(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \varphi(\tfrac{2+0}{2}) = \varphi(1) = 1.$$

$t_1 = \frac{d(x,u)+d(y,v)}{2} = 2;\ t_2 = \frac{d(x,F(x,y))+d(y,F(y,x))}{2} = 2;$

$t_3 = \frac{d(u,F(u,v))+d(v,F(v,u))}{2} = 3;$

$t_4 = \frac{d(x,F(u,v))+d(y,F(v,u))}{2} = 1;$

$t_5 = \frac{d(u,F(x,y))+d(v,F(y,x))}{2} = 4.$

Hence

$\psi_1(t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)) - \psi_2(t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)) = \frac{29}{16}$ so that the inequality (2.2.1) holds.

Case (iii): $(x, y, u, v) = (7, 5, 3, 5)$

$$\text{Here } \varphi_1(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \varphi(\tfrac{0+2}{2}) = \varphi(1) = 1.$$

$t_1 = \frac{d(x,u)+d(y,v)}{2} = 2;\ t_2 = \frac{d(x,F(x,y))+d(y,F(y,x))}{2} = 2;$

$t_3 = \frac{d(u,F(u,v))+d(v,F(v,u))}{2} = 3;$

$t_4 = \frac{d(x,F(u,v))+d(y,F(v,u))}{2} = 2;$

$t_5 = \frac{d(u,F(x,y))+d(v,F(y,x))}{2} = 1.$

Hence

$\psi_1(t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)) - \psi_2(t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)) = \frac{29}{24}$ so that the inequality (2.2.1) holds.

Case (iv): $(x, y, u, v) = (5, 3, 7, 5)$

$$\text{Then } \varphi_1(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \varphi(\tfrac{2+0}{2}) = \varphi(1) = 1.$$

$t_1 = \frac{d(x,u)+d(y,v)}{2} = 2$; $t_2 = \frac{d(x,F(x,y))+d(y,F(y,x))}{2} = 2$;
$t_3 = \frac{d(u,F(u,v))+d(v,F(v,u))}{2} = 3$;
$t_4 = \frac{d(x,F(u,v))+d(y,F(v,u))}{2} = 4$;
$t_5 = \frac{d(u,F(x,y))+d(v,F(y,x))}{2} = 1$.

Hence

$\psi_1(t_1, t_2, t_3, \frac{1}{2}(t_4+t_5)) - \psi_2(t_1, t_2, t_3, \frac{1}{2}(t_4+t_5)) = \frac{53}{24}$

Hence the inequality (2.2.1) holds.

Case (v): $(x, y, u, v) = (5, 7, 3, 5)$

Here $\varphi_1(\frac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \varphi(\frac{2+0}{2}) = \varphi(1) = 1.$

$t_1 = \frac{d(x,u)+d(y,v)}{2} = 2$; $t_2 = \frac{d(x,F(x,y))+d(y,F(y,x))}{2} = 3$;
$t_3 = \frac{d(u,F(u,v))+d(v,F(v,u))}{2} = 2$;
$t_4 = \frac{d(x,F(u,v))+d(y,F(v,u))}{2} = 4$;
$t_5 = \frac{d(u,F(x,y))+d(v,F(y,x))}{2} = 2$.

Hence

$\psi_1(t_1, t_2, t_3, \frac{1}{2}(t_4+t_5)) - \psi_2(t_1, t_2, t_3, \frac{1}{2}(t_4+t_5)) = \frac{31}{12}$ so that the inequality (2.2.1) holds.

Case (vi): $(x, y, u, v) = (5, 3, 3, 7)$

Here $\varphi_1(\frac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \varphi(\frac{2+0}{2}) = \varphi(1) = 1.$

$t_1 = \frac{d(x,u)+d(y,v)}{2} = 3$; $t_2 = \frac{d(x,F(x,y))+d(y,F(y,x))}{2} = 2$;
$t_3 = \frac{d(u,F(u,v))+d(v,F(v,u))}{2} = 2$;
$t_4 = \frac{d(x,F(u,v))+d(y,F(v,u))}{2} = 2$;
$t_5 = \frac{d(u,F(x,y))+d(v,F(y,x))}{2} = 2$.

Hence

$\psi_1(t_1, t_2, t_3, \frac{1}{2}(t_4+t_5)) - \psi_2(t_1, t_2, t_3, \frac{1}{2}(t_4+t_5)) = \frac{63}{24}$ so that the inequality (2.2.1) holds.

Case (vii): $(x, y, u, v) = (7, 3, 3, 5)$

Then $\varphi_1(\frac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \varphi(\frac{2+0}{2}) = \varphi(1) = 1.$

$t_1 = \frac{d(x,u)+d(y,v)}{2} = 3$; $t_2 = \frac{d(x,F(x,y))+d(y,F(y,x))}{2} = 2$;
$t_3 = \frac{d(u,F(u,v))+d(v,F(v,u))}{2} = 2$;
$t_4 = \frac{d(x,F(u,v))+d(y,F(v,u))}{2} = 2$;
$t_5 = \frac{d(u,F(x,y))+d(v,F(y,x))}{2} = 2$.

Hence

$\psi_1(t_1, t_2, t_3, \frac{1}{2}(t_4+t_5)) - \psi_2(t_1, t_2, t_3, \frac{1}{2}(t_4+t_5)) = \frac{63}{24}$ so that the inequality (2.2.1) holds.

Case (viii): $(x, y, u, v) = (5, 3, 3, 5)$

Then $\varphi_1(\frac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \varphi(\frac{2+2}{2}) = \varphi(2) = 2.$

$t_1 = \frac{d(x,u)+d(y,v)}{2} = 2$; $t_2 = \frac{d(x,F(x,y))+d(y,F(y,x))}{2} = 2$;
$t_3 = \frac{d(u,F(u,v))+d(v,F(v,u))}{2} = 2$;
$t_4 = \frac{d(x,F(u,v))+d(y,F(v,u))}{2} = 2$;
$t_5 = \frac{d(u,F(x,y))+d(v,F(y,x))}{2} = 3$.

Hence

$\psi_1(t_1, t_2, t_3, \frac{1}{2}(t_4+t_5)) - \psi_2(t_1, t_2, t_3, \frac{1}{2}(t_4+t_5)) = \frac{103}{48}$

Hence the inequality (2.2.1) holds.

Hence all the hypotheses of Theorem 2.2 hold and (3,3) is the unique coupled fixed point of F.

Here we observe that the hypotheses of Theorem 2.3 also hold good.

Corollary 2.6 Let (X, d) be a complete metric space and M be a non-empty subset of X^4. Suppose that $F : X \times X \to X$ is a mapping. Assume that there exists $k \in [0, 1)$ such that

$$\frac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2} \le k \max\{\frac{d(x,u)+d(y,v)}{2}, \frac{d(x,F(x,y))+d(y,F(y,x))}{2}, \frac{d(u,F(u,v))+d(v,F(v,u))}{2}, \frac{1}{2}[\frac{d(x,F(u,v))+d(y,F(v,u))}{2} + \frac{d(u,F(x,y))+d(v,F(y,x))}{2}]\} \tag{2.6.1}$$

for each $(x, y, u, v) \in M$. Further assume that

(i) either (*a*) F is continuous (or)

(*b*) X has the following property:

"for any two sequences $\{x_n\}, \{y_n\}$, in X with $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$, $x_n \to x$, , $y_n \to y$ as $n \to \infty$ then $(x, y, x_n, y_n) \in M$ for all n",

(ii) M is F-invariant subset of X^4 satisfying the transitivity property, and

(iii) there exist $x_0, y_0 \in X$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$

then F has a coupled fixed point.

Proof. By choosing $\psi_1(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$ and $\psi_2(t_1, t_2, t_3, t_4) = (1-k)\max\{t_1, t_2, t_3, t_4\}$, $t_1, t_2, t_3, t_4 \ge 0$ in Theorem 2.2, we obtain the conclusion of this corollary. □

Corollary 2.7 Let (X, d) be a complete metric space and M be a non-empty subset of X^4. Suppose that $F : X \times X \to X$ is a mapping. Assume that there exists $k \in [0, 1)$ such that

$d(F(x, y), F(u, v))$

$$\leq k\max\{\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}]\} \quad (2.7.1)$$

for each $(x,y,u,v) \in M$. Further assume that

(i) either (a) F is continuous (or)

(b) X has the following property:

"for any two sequences $\{x_n\},\{y_n\}$, in X with $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$, $x_n \to x$, $,y_n \to y$ as $n \to \infty$ then $(x, y, x_n, y_n) \in M$ for all n",

(ii) M is F-invariant subset of X^4 satisfying the transitivity property, and

(iii) there exist $x_0, y_0 \in X$ such that $(F(x_0,y_0), F(y_0,x_0), x_0, y_0) \in M$

then F has a coupled fixed point.

Proof. Let $(x,y,u,v) \in M$. Since M has the transitivity property, $(v,u,y,x) \in M$. Now from (2.7.1), we get

$$d(F(v,u),F(y,x)) \leq k\max\{\tfrac{d(v,y)+d(u,x)}{2}, \tfrac{d(v,F(v,u))+d(u,F(u,v))}{2}, \tfrac{d(y,F(y,x))+d(x,F(x,y))}{2}, \tfrac{1}{2}[\tfrac{d(v,F(y,x))+d(u,F(x,y))}{2} + \tfrac{d(y,F(v,u))+d(x,F(u,v))}{2}]\}. \quad (2.7.2)$$

Now adding (2.7.1) and (2.7.2), we obtain (2.6.1). Hence the conclusion follows from Corollary 2.6. □

Corollary 2.8 Let (X,d) be a complete metric space and M be a non-empty subset of X^4. Suppose that $F : X \times X \to X$ is a mapping. Assume that there exists a mapping $\varphi : R^+ \to R^+$, φ is continuous, increasing and $\varphi(t) = 0$ if and only if $t = 0$ such that

$$\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2} \leq M(x,y) - \varphi(M(x,y)) \quad (2.8.1)$$

where $M(x,y) = \max\{\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}]\}$

for each $(x,y,u,v) \in M$. Further assume that

(i) either (a) F is continuous (or)

(b) X has the following property:

"for any two sequences $\{x_n\},\{y_n\}$, in X with $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$, $x_n \to x$, $,y_n \to y$ as $n \to \infty$ then $(x, y, x_n, y_n) \in M$ for all n",

(ii) M is F-invariant subset of X^4 satisfying the transitivity property, and

(iii) there exist $x_0, y_0 \in X$ such that $(F(x_0,y_0), F(y_0,x_0), x_0, y_0) \in M$

then F has a coupled fixed point.

Proof: We define $\psi_1, \psi_2 : (R^+)^4 \to R^+$ in Theorem 2.2 by $\psi_1(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$ and $\psi_2(t_1,t_2,t_3,t_4) = \varphi(\max\{t_1,t_2,t_3,t_4\})$, $t_1,t_2,t_3,t_4 \geq 0$, we obtain the inequality (2.8.1). Hence the conclusion follows from Theorem 2.2. □

Remark 2.9 Theorem 2.2 is a coupled fixed point version of Theorem 1.5.

III. Existence of coupled fixed points via an altering distance function in four variables in partially ordered sets

In this section we apply the results of Section 2 to establish the existence of coupled fixed points by using an altering distance function in four variables.

Theorem 3.1 Let $(X,\preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$. Assume that there exist two generalized altering distance functions in four variables ψ_1 and ψ_2 such that

$$\begin{aligned}&\varphi_1(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) \\ &\leq \psi_1(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}]) \\ &-\psi_2(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2}, \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}])\end{aligned} \quad (3.1.1)$$

for each $x,y,u,v \in X$ with $x \succeq u$ and $y \preceq v$ and $\varphi_1(t) = \psi_1(t,t,t,t)$. Further assume that

(i) either (a) F is continuous (or)

(b) X has the following property:

"if a non-decreasing sequence $\{x_n\} \subseteq X$ with $x_n \to x$, then $x_n \preceq x$ for all n and

if a non-increasing sequence $\{y_n\} \subseteq X$ with $y_n \to y$, then $y_n \succeq y$ for all n",

(ii) F satisfies mixed monotone property, and

(iii) if there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$

then F has a coupled fixed point.

Proof: We define a subset M of X^4 by

$$M = \{(a,b,c,d) \in X^4 / a \succeq c \text{ and } b \preceq d\}.$$

Then M is an F-invariant subset of X^4 which satisfies the transitivity property (from Example 1.11).

For $(x,y),\ (u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$, we have $(x,y,u,v) \in M$. Hence (3.1.1) holds for any $(x,y,u,v) \in M$.

Also $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$ implies $(F(x_0,y_0),F(y_0,x_0),x_0,y_0) \in M$. And condition (i) (b) implies condition (i) (b) of Theorem 2.2. Hence all the hypotheses of Theorem 2.2 are satisfied and therefore F has a coupled fixed point. □

Theorem 3.2 In addition to the hypotheses of Theorem 3.1, suppose that the following condition holds:

"for each (x,y) and (u,v) in $X\times X$ there exists $(z,t) \in X\times X$ that is comparable with (x,y) and (u,v)". (3.2.1)

Then F has a unique coupled fixed point.

Proof: Condition (3.2.1) implies the condition (2.4.1) of Theorem 2.4. Hence the conclusion follows from Theorem 2.4. □

Corollary 3.3 Let $(X,\preceq)$ be a partially ordered set. Suppose that there is a metric d on X such that (X,d) is a complete metric space. Suppose that $F : X \times X \to X$ be a mapping. Assume that there exists $k \in [0,1)$ such that

$$d(F(x,y),F(u,v)) \leq k\max\{\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}]\}$$

for each $x,y,u,v \in X$ with $x \succeq u$ and $y \preceq v$. Further, assume that conditions (i), (ii) and (iii) of Theorem 3.1 hold. Then F has a coupled fixed point.

Proof: With the same argument as in the proof of Theorem 3.1, the conclusion of this corollary follows from Corollary 2.7.□

Remark 3.4 Theorem 1.8 can be obtained as a corollary to Corollary 3.3, which in turn Theorem 1.8 follows as a corollary to Theorem 3.1.

The following are examples in support of Theorem 3.1 and Theorem 3.2.

Example 3.5 Let $X = R^+$ with the usual metric and the usual ordering. We define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x-y}{4}, & if \quad x \geq y \\ 0, & otherwise. \end{cases}$$

Then F clearly satisfies mixed monotone property.

We define $\psi_1, \psi_2 : (R^+)^4 \to R^+$ by

$\psi_1(t_1,t_2,t_3,t_4) = \frac{t_1+t_2+t_3+t_4}{4}$ and

$\psi_2(t_1,t_2,t_3,t_4) = \frac{t_1+t_2+t_3+t_4}{32},\ t_1,t_2,t_3,t_4 \geq 0.$

We now verify the inequality (3.1.1) in the following cases:

Let $(x,y),(,v) \in X \times X$ such that $x \geq u$ and $y \leq v$.

<u>Case</u> (i): $x \geq y$ and $u \geq v$.

In this case, $F(x,y) = \frac{x-y}{4}, F(u,v) = \frac{u-v}{4}, F(y,x) = 0$ and $F(v,u) = 0$.

Hence

$$\varphi_1(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \tfrac{1}{8}[(x-u)+(v-y)].$$

$$\begin{aligned}&\psi_1(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}])\\ &-\psi_2(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}])\\ &= \begin{cases} \frac{1}{32}[\frac{95}{16}x - \frac{17}{16}u + \frac{121}{16}y + \frac{9}{16}v], & if \quad u \geq \frac{x-y}{4} \\ \frac{1}{32}[\frac{109}{16}x - \frac{73}{16}u + \frac{107}{16}y + \frac{9}{16}v], & if \quad u < \frac{x-y}{4}. \end{cases}\end{aligned}$$

In any case it is now easy to verify inequality (3.1.1).

<u>Case</u> (ii): $x \geq y$ and $u < v$.

In this case, $F(x,y) = \frac{x-y}{4}, F(u,v) = 0, F(y,x) = 0$ and $F(v,u) = \frac{v-u}{4}$.

<u>Subcase</u> (a): $y \geq \frac{v-u}{4}$.

$$\varphi_1(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \tfrac{1}{8}[(x-u)+(v-y)].$$

$$\begin{aligned}&\psi_1(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}])\\ &-\psi_2(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}])\\ &= \begin{cases} \frac{1}{32}[\frac{95}{16}x + \frac{23}{16}u + \frac{9}{16}y + \frac{95}{16}v], & if \quad u \geq \frac{x-y}{4} \\ \frac{1}{32}[\frac{109}{16}x + \frac{49}{16}u - \frac{5}{16}y + \frac{95}{16}v], & if \quad u < \frac{x-y}{4}. \end{cases}\end{aligned}$$

so that the inequality (3.1.1) holds clearly.

<u>Subcase</u> (b): $y < \frac{v-u}{4}$.

$$\varphi_1(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \tfrac{1}{8}[(x-u)+(v-y)].$$

$$\begin{aligned}&\psi_1(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}])\\ &-\psi_2(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}])\end{aligned}$$

$$= \begin{cases} \frac{1}{32}[\frac{95}{16}x - \frac{5}{16}u - \frac{47}{16}y + \frac{109}{16}v], & if \quad u \geq \frac{x-y}{4} \\ \frac{1}{32}[\frac{109}{16}x - \frac{61}{16}u - \frac{61}{16}y + \frac{109}{16}v], & if \quad u < \frac{x-y}{4}. \end{cases}$$

Now it is easy to see that the inequality (3.1.1) holds.

Case (iii): $x < y$ and $u \geq v$.

This case does not arise, for, we have $x \geq u$ and $y \leq v$.

Therefore $x \geq u \geq v \geq y > x$, a contradiction.

Case (iv): $x < y$ and $u < v$.

In this case, $F(x,y) = 0, F(u,v) = 0, F(y,x) = \frac{y-x}{4}$ and $F(v,u) = \frac{v-u}{4}$.

The verification of the inequality in this case is similar as in Case (ii).

From all the above cases, we conclude that F satisfies the inequality (3.1.1). Also there exists a point $(x_0, y_0) = (0,1) \in X \times X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Hence all the hypotheses of Theorem 3.1, Theorem 3.2 hold good and $(0,0)$ is the unique coupled fixed point of F. .

The following example is in support of Theorem 3.1. Further, if we relax condition (3.2.1) of Theorem 3.2, then the uniqueness of coupled fixed point fails to hold.

Example 3.6 Let $X = [0,1] \bigcup \{2,3,4,\ldots.\}$ with the usual metric. We define partial ordering $\preceq$ on X by

$$\preceq := \{(x,y) \in X \times X \ / \ x,y \in [0,1], x \leq y\} \cup \ \{(x,y) \in X \times X \ / \ x \in \{2,3,4...\}, x \leq y\}.$$

We define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} 1, & if \quad x,y \in [0,1] \\ x, & if \ x \in [0,1] \text{ and } y \in \{2,3,4\ldots.\} \\ 0, & if \ x \in \{2,3,4\ldots.\} \text{ and } y \in [0,1] \\ 2, & if \ x,y \in \{2,3,4\ldots.\}. \end{cases}$$

We define $\psi_1, \psi_2 : (R^+)^4 \to R^+$ by

$\psi_1(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$ and

$\psi_2(t_1,t_2,t_3,t_4) = \frac{t_1+t_2+t_3+t_4}{16}, \ t_1,t_2,t_3,t_4 \geq 0.$

Clearly F satisfies mixed monotone property. The elements (x,y) and (u,v) in $X \times X$ such that $x \succeq u$ and $y \preceq v$ are:

(i) $(x,y),(u,v)$ such that $x,y,u,v \in [0,1]$ with $x \succeq u$ and $y \preceq v$.

(ii) $(x,y),(u,v)$ such that $x,u \in [0,1], \ y,v \in \{2,3,4\ldots.\}$ with $x \succeq u$ and $y \preceq v$.

(iii) $(x,y),(u,v)$ such that $x,u \in \{2,3,4\ldots.\}, \ y,v \in [0,1]$ with $x \succeq u$ and $y \preceq v$.

(iv) $(x,y),(u,v)$ such that $x,y,u,v \in \{2,3,4\ldots.\}$ with $x \succeq u$ and $y \preceq v$.

So we now verify the inequality (2.3.1) in the following for the cases:

Case (i): $(x,y), \ (u,v)$ in $X \times X$, $x,y,u,v \in [0,1]$ with $x \succeq u$ and $y \preceq v$.

In this case $F(x,y) = 1, F(u,v)) = 1, F(y,x) = 1$ and $F(v,u) = 1$ so that the inequality (2.3.1) holds trivially.

Case (ii): $(x,y), \ (u,v)$ in $X \times X$, with $x,u \in [0,1], \ y,v \in \{2,3,4\ldots.\}$ and $x \succeq u$ and $y \preceq v$.

In this case $F(x,y) = x, \ F(u,v) = u, \ F(y,x) = 0$ and $F(v,u) = 0$.

$$\varphi_1(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) = \tfrac{1}{2}(x-u),$$

$$\begin{aligned}&\psi_1(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\ &\quad \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \\ &\quad \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}]) \\ &-\psi_2(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\ &\quad \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \\ &\quad \tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}]) \\ &= \max\{\tfrac{(x-u)+(v-y)}{2}, \tfrac{y}{4}, \tfrac{v}{4}, \tfrac{1}{2}[(x-u) + \tfrac{y+v}{2}]\} \\ &\qquad - \tfrac{1}{16}[(x-u) + \tfrac{5}{4}v + \tfrac{y}{4}].\end{aligned}$$

Now,

$$\begin{aligned}&\varphi_1(\tfrac{d(F(x,y),F(u,v))+d(F(y,x),F(u,v))}{2}) \\ &= \tfrac{1}{2}(x-u) \\ &\leq \tfrac{1}{2}[(x-u) + \tfrac{y+v}{2}] - \tfrac{1}{16}[(x-u) + \tfrac{5}{4}v + \tfrac{y}{4}] \\ &\psi_1(\tfrac{d(x,u)+d(y,v)}{2}, \tfrac{d(x,F(x,y))+d(y,F(y,x))}{2}, \\ &\quad \tfrac{d(u,F(u,v))+d(v,F(v,u))}{2}, \\ &\tfrac{1}{2}[\tfrac{d(x,F(u,v))+d(y,F(v,u))}{2} + \tfrac{d(u,F(x,y))+d(v,F(y,x))}{2}])\end{aligned}$$

Hence the inequality (3.1.1) holds.

Case (iii): $(x,y), \ (u,v)$ in $X \times X$, with $x,u \in \{2,3,4\ldots.\}, \ y,v \in [0,1]$ and $x \succeq u$ and $y \preceq v$.

In this case $F(x,y) = 0, \ F(u,v) = 0, \ F(y,x) = y$ and $F(v,u) = v$.

The verification of the inequality (3.1.1) in this case is similar as in Case (ii).

Case (iv): $(x,y), \ (u,v)$ in $X \times X$, with $x,y,u,v \in \{2,3,4\ldots.\}$ and $x \succeq u$ and $y \preceq v$.

In this case $F(x,y) = 2, F(u,v)) = 2, F(y,x) = 2$ and $F(v,u) = 2$ so that the inequality (3.1.1) holds trivially.

Therefore from all the above cases, it is clear that the inequality (3.1.1) holds. Also there is a point $(x_0,y_0) = (0,2)$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. Hence all the hypotheses of Theorem 3.1 are satisfied and $(1,1)$ and $(2,2)$ are two coupled fixed points of F.

Here we observe that $(1,1)$ and $(2,2)$ are neither comparable with each other nor there exist $(z,t) \in X \times X$ which is comparable with (1,1) and (2,2). Hence condition (3.2.1) fails to hold, and F has more than one coupled fixed point.

Example 3.7 Let $X = R^+$ with the usual metric and the usual ordering. We define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x-y}{2}, & if \quad x \geq y \\ \\ 0, & otherwise. \end{cases}$$

Then F clearly satisfies mixed monotone property.

We define $\psi_1, \psi_2 : (R^+)^4 \to R^+$ by

$\psi_1(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$ and

$\psi_2(t_1,t_2,t_3,t_4) = \frac{t_1+t_2+t_3+t_4}{32}, \ t_1,t_2,t_3,t_4 \geq 0.$

Then F satisfies all the hypotheses of Theorem 3.1 with $(x_0, y_0) = (1, 0)$ and $(0, 0)$ is the unique coupled fixed point of F. But Theorem 1.8 can not be applied; for, at $x = 1, \ u = v = y = 0$, we have $d(F(x,y), F(u,v)) = \frac{1}{2}$ and $\frac{d(x,u)+d(y,v)}{2} = \frac{1}{2}$, so that inequality (1.8.1) fails to hold.

Remark 3.8 Example 3.7 and Remark 3.4 suggest that Theorem 3.1 is a generalization of Theorem 1.8.

IV. Conclusion

In this paper, we proved the existence of coupled fixed points of F using generalized altering distance functions in four variables under F - invariant set without using mixed monotone property. We deduced the existence of coupled fixed points of F in partially ordered metric spaces from our main results. Several examples are provided in support of our results. One of our results (Theorem 3.1) generalized the results of Bhaskar and Lakshmikantham [8].

References

[1] M. S. Khan, M. Swaleh and S. Sessa, “Fixed point theorems by altering distances between the points“, *Bull. Austral. Math. Soc.*, vol. 30, pp. 1-9, 1984.

[2] B. S. Choudhury, “A common unique fixed point result in metric space involving generalized altering distances“, *Mathematical Communications*, vol. 10, pp. 105-110, 2005.

[3] B. S. Choudhury and P. N. Dutta, “A unique fixed point result in metric space involving a two variable function“, *Filomat*, vol. 14, pp. 43-48, 2000.

[4] D. Doric, “Common fixed point for generalized (ψ, φ)-weak contractions“, *Applied Mathematics Letters*, vol. 22, pp. 1896-1900, 2009.

[5] K. P. R. Sastry, G. V. R. Babu, “Some fixed point theorems by altering diatances between the points“, *Indian, J. Pure. Appl. Math.*, vol. 30, no. 6, pp. 641-647, 1999.

[6] G. V. R. Babu, G. Lalitha and M. L. Sandhya, “Common fixed point theorems involving two generalized altering distance function in four variables“, *Proceedings of Jangjeon Mathematical Society*, vol. 10, pp. 83-93, 2007.

[7] D. Guo and V. Lakshmikantham, “Coupled fixed points of nonlinear operators with applications“, *Nonlinear Analysis*, TMA, vol. 11, no. 5, pp. 623-632, 1987.

[8] T. Gnanabhaskar, V. Lakshmikantham, “Fixed point theorems on partially ordered sapces and applications“, *Nonlinear Analysis*, TMA, vol. 65, pp. 1379-1393, 2006.

[9] M. Abbas, T. Nazir, S. Radenovic, “Common fixed points of four maps in partially orderd metric spaces“, *Applied Mathematics Letters*, vol. 24, pp. 1520-1526, 2011.

[10] A. Amini-Harindi, H. Emami, “A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, Nonlinear Analysis“, vol. 72, no. 5, pp. 2238-2242, 2010.

[11] B. S. Choudhury, Amaresh Kundu, “A coupled coincidence point result in partially ordered metric spaces for compatible mappings“, *Nonlinear Analysis*, vol. 73, pp. 2524-2531, 2010.

[12] J. Harjani, K. Sadarangani, “Fixed point theorems for weakly contarctive mappings in partially ordered sets“, *Nonlinear Analysis*, vol. 71, pp. 3403-3410, 2009.

[13] J. Harjani, K. Sadarangani, “Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations“, *Nonlinear Analysis*, vol. 72, pp. 1188-1197, 2010.

[14] J. Harjani, B. Lopez and K. Sadarangani, “Fixed point theorems for weakly contractive mappings in ordered metric spaces“, *Comp. and Math. with Appl.*, vol. 61, pp. 790-796, 2011.

[15] V. Lakshmikantham, Lj. B. Ciric, “Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces“, *Nonlinear Analysis*, vol. 70, pp. 4341-4349, 2009 .

[16] Nguyen Van Luong, Nguyen Vuan Thuan, “Coupled fixed points in partially ordered metric spaces and application“, *Nonlinear Analysis*, vol. 74, pp. 983-992, 2011.

[17] H. K. Nashine, B. Samet, “Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces“, *Nonlinear Analysis*, 2010, doi: 10.1016/j.na.2010.11.024.

[18] J. J. Nieto, R. R. Lopez, “Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations“, *Order*, vol. 22, pp. 223-239, 2005.

[19] S. Radenovic, Z. Kadelburg, “Generalized weak contractions in partially ordered metric spaces“, *Comput. Math. Appl.*, vol. 60, pp. 1776-1783, 2010.

[20] A. C. M. Ran, M. C. B. Reuings, “A fixed point theorem in partially orderd sets and some application to matrix equations“, *Proc. Amer. Math. Soc.*, vol. 132, pp. 1435-1443, 2004.

[21] B. Samet and C. Vetro, “Coupled fixed point, F-invariant set and fixed points of N order“, *Ann. funct. Analysis*, vol. 7, no. 1, pp. 46-56, 2010.

[22] D. Doric, Z. Kadelburg, S. Radenovic, “Coupled fixed point results for mappings without mixed monotone property“, *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1803-1808, 2012.

[23] W. Sintunavarat, Y. J. Cho, P. Kumam, “Coupled fixed point theorem for weak contraction mappings under F-invariant set“, *Abstract and Applied Analysis*, vol. 2012, 2012, 15 pages.

[24] W. Sintunavarat, A. Petrusel, P. Kumam, “Common coupled fixed point theorems for w^*-compatible mappings without mixed monoone property“, *Rend. Circ. Mat. Palermo*, vol. 61, pp. 361-383, 2012.

[25] W. Sintunavarat, S. Radenovic, Z. Golubovic and P. Kumam, “Coupled fixed point theorems for F-invariant set“, *Applied Math.Inf. Science*, vol. 7, no. 1, pp. 247-255, 2013.