Common Fixed Point Theorems of ψ - Weak Generalized Geraghty Contractions In Partially Ordered Partial b-Metric Spaces

K.P.R. Sastry, K.K.M. Sarma, Ch. Srinivasa Rao and Vedula Perraju

***Abstract*—In this paper we consider the concept of $\psi-$ weak generalized Geraghty contractive self mappings in a complete partially ordered partial b - metric space. We study the existence of fixed points for such self mappings in complete partially ordered partial b - metric spaces controlled by $\psi-$ weak generalized Geraghty contractive type condition and obtain some fixed point results of Babu.G.V.R. et.al [3] in complete partially ordered metric spaces as corollaries. Supporting example is also provided. An open problem is given at the end of the paper.**

***Index Terms*—Fixed point theorems, $\psi-$ contractive mappings, partial b - metric, ordered partial b - metric space, partially ordered partial b - metric space, Geraghty contraction.**

MSC 2010 Codes - 54H25, 47H10.

I. Introduction

MOST of the generalizations of fixed point theorems usually start from Banach [5] contraction principle. But all the generalizations may not be from this principle. In 1973, Geraghty [8] introduced an extension of the contraction in which the contraction constant was replaced by a function having some specified properties. In 1989, Bakktin [4] introduced the concept of a b - metric space as a generalization of a metric space. In 1993, Czerwik [7] extended many results related to the b - metric spaces. In 1994, Matthews [16] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, Neill.S.J.O' [21] generalized the concept of partial metric space by admitting negative distances. In 2013, Shukla [26] generalized both the concepts of b - metric and partial metric space by introducing the partial b - metric spaces. Many authors recently studied the existence of fixed points of self maps in different types of metric spaces [13,25,1,2,20,28]. Xian Zhang [29] proved a common fixed point theorem for two self maps on a metric space satisfying generalized contractive type conditions. Some authors studied some fixed point theorems in b - metric spaces [17,23,24,30]. After that some authors proved $\alpha-\psi$ versions of certain fixed point theorems in different type metric spaces [12,18,23,]. Mustafa [19] gave a generalization of Banach contraction principle in complete ordered partial b - metric space by introducing a generalized $\alpha-\psi$ weakly contractive mapping. Mukheimer.A [18] generalized the concept of Mustafa [19] by introducing the α-ψ-φ contractive mapping in a complete ordered partial b - metric space. In this paper we prove fixed point theorems for ψ - weak generalized Geraghty contractive self mappings in complete partially ordered partial b - metric spaces satisfying a contractive type condition by considering partial b - metric p as in definition 2.1(Shukla [26]) which is more general than that of any partial b - metric and obtained some fixed point results of Babu.G.V.R. et.al [3] in complete partially ordered metric space as corollaries. A supporting example is given and an open problem is also given at the end of the paper. Shukla [26] introduced the notation of a partial b - metric space as follows.

K.P.R. Sastry resides at 8-28-8/1, Tamil Street, Chinna Waltair Visakhapatnam-530 017, India. (E-mail: kprsastry@hotmail.co)

K.K.M. Sarma is a Professor in the Department of Mathematics at Andhra University, Visakhapatnam-530 003, India. (E-mail: sarmakmkandala@yaoo.in).

Ch. Srinivasa Rao is an Associate Professor in the Department of Mathematics at Mrs. A.V.N. College, Visakhapatnam - 530 001, India (E-mail: drcsr41@yahoo.com).

Vedula Perraju is an Associate Professor in the Department of Mathematics, Mrs. A.V.N. College, Visakhapatnam - 530 001, India. (E-mail: perrajuvedula2004@gmail.com), corresponding Author .

II. Some Preliminary Results

In this section, some definitions and preliminary results are given which we use in this paper.

Definition 2.1(Shukla.S[26]) Let X be a non empty set and let $s \geq 1$ be a given real number. A function
$p : X \times X \to [0, \infty)$ is called a partial
b - metric if for all $x, y, z \in X$ the following conditions are satisfied.
(i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$
(ii) $p(x, x) \leq p(x, y)$
(iii) $p(x, y) = p(y, x)$
(iv) $p(x, y) \leq s\{p(x, z) + p(z, y)\}$ - $p(z, z)$.
The pair (X, p) is called a partial b - metric space. The number $s \geq 1$ is called a coefficient of (X, p). □

Definition 2.2(Karapinar.E. et.al [12]) Let $(X, \leq)$ be a partially ordered set and $T : X \to X$ be a mapping. We say that T is non decreasing with respect to $\leq$ if $x, y \in X$, $x \leq y$ $\Rightarrow Tx \leq Ty$ □

Definition 2.3(Karapinar.E. et.al [12]) Let $(X, \leq)$ be a partially ordered set. A sequence $\{x_n\} \in X$ is said to be non decreasing with respect to $\leq$ if
$x_n \leq x_{n+1} \forall\, n \in \mathbb{N}$ □

Definition 2.4 (Mustafa.Z [19]) A triple $(X, \leq, p)$ is called an ordered partial b - metric space if $(X, \leq)$ is a partially ordered set and p is a partial b-metric on X. □

Definition 2.5(Mohammad Mursaleen. et. al.[18]) Define $\Psi = \{\psi/\psi : [0, \infty) \to [0, \infty)$ is non-decreasing and satisfies

$(2.5.1)\}$
ψ is continuous and $\psi(t) = 0 \Leftrightarrow t = 0$ (2.5.1) □

Definition 2.6(Geraghty [8]) A self map $f : X \to X$ is said to be a Geraghty contraction if there exists $\beta \in S$ such that
$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y)$
where $S = \{\beta : [0, \infty) \to [0, 1)/\beta(t_n) \to 1 \Rightarrow t_n \to 0\}$ □

Definition 2.7(Ciric. et.al [6]) Suppose $(X, \leq)$ is a partially ordered set and $f, g : X \to X$ are self maps. f is said to be g - non-decreasing if for $x, y \in X$, $gx \leq gy \Rightarrow fx \leq fy$ □

Definition 2.8 (Babu.G.V.R. et.al[3]) Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (X, d) is a metric space. Let f and g be two self mappings on X. Suppose there exists $\psi \in \Psi, \beta \in S$ and $L > 0$ such that
$\psi(d(f(x), f(y)) \leq \beta(M(x, y))M(x, y) + L.N(x, y)$ (2.8.1)
for all $x, y \in X$ with $gx \geq gy$, where
$M(x, y) =$
$max\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{1}{2}[d(gx, fy) + d(fx, gy)]\}$
and$N(x, y) = min\{d(gx, fy), d(gx, fy), d(gx, fy)\}$.
Then we say that (f, g) is a pair of ψ weak generalized Geraghty contraction maps. □

Definition 2.9(Jungck.G [9]) Two self maps f and g of a metric space (X, d) are said to be compatible if
$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ □

Definition 3.0(Jungck.G [10]) Two self maps f and g of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points, that is if $fu = gu$ for some $u \in X$, then $fgu = gfu$. □

Definition 3.1(Pant.R.P[22]) Two self maps f and g of a metric space (X, d) are said to be reciprocally continuous if $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$ whenever $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$. □

Babu.G.V.R. et.al [3] proved the following theorems :

Theorem 3.2 (Babu.G.V.R. et.al [3], Theorem 2.1) Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (X, d) is a complete metric space. Let f and g be two self maps on X such that f is g - non - decreasing. Suppose that (f, g) is a pair of generalized Geraghty contraction maps satisfying (2.8.1). Assume that
(i) $fX \subseteq gX$
(ii) there exists $x_0 \in X$ such that $gx_0 \leq fx_0$
(iii) $g(X)$ is a closed subset of X.
(iv) if any non - decreasing $\{x_n\}$ in X, converges to u, then $x_n \leq u \ \ \forall\ n \in \mathbb{N}$
Then f and g have a coincidence point in X. □

Theorem 3.3 (Babu.G.V.R. et.al [3], Theorem 2.2) In addition to the hypothesis of Theorem 3.2, if $gu < ggu$ where u is as in (iv) and f and g are weakly compatible then f and g have a common fixed point in X. □

Theorem 3.4 (Babu.G.V.R. et.al [3], Theorem 2.3) Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (X, d) is a complete metric space. Let f and g be two self maps on X, f is g- non - decreasing. Suppose that (f, g) is a pair of ψ - weak generalized Geraghty contraction maps. Assume that
(i) $fX \subseteq gX$
(ii) f and g are compatible.
(iii) there exists $x_0 \in X$ such that $gx_0 \leq fx_0$
(iv) f and g are reciprocally continuous.
Then f and g have a coincidence point in X. □

III. Main Result

In this section we prove coincident point and common fixed point theorems for two self maps on partially ordered partial b - metric spaces by using by partial b - metric p of definition 2.1 and obtain theorems 3.2,3.3 and 3.4 as corollaries. A supporting example is also given. An open problem is also given at the end.
We begin this section with the following definition

Definition 3.5(Mustafa.Z[19]) Suppose $(X, \leq)$ is a partially ordered set and p is a partial b - metric with $s \geq 1$ as the coefficient of (X, p). Then we say that the triplet $(X, \leq, p)$ is a partially ordered partial b - metric space.We observe that every ordered partial b - metric space is a partially ordered partial b - metric space.

Definition 3.6(Mustafa.Z [19])A sequence $\{x_n\}$ in a partial b - metric space (X, p) is said to be:
(i) convergent to a point $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$
(ii) a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite
(iii)a partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that
$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x)$. □

Now we introduce the notions of compatibility, weak compatibility and reciprocal continuity of two self maps on a partially ordered partial b - metric space.

Definition 3.7 Two self maps f and g of a partially ordered partial b - metric space $(X, \leq, p)$ are said to be compatible if $\lim_{n\to\infty} p(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that
$\lim_{m,n\to\infty} p(fx_m, fx_n) = \lim_{n\to\infty} p(fx_n, u) = p(u, u) = 0$
and
$\lim_{m,n\to\infty} p(gx_m, gx_n) = \lim_{n\to\infty} p(gx_n, u) = p(u, u) = 0$ for some $u \in X$. □

Definition 3.8 Two self maps f and g of a partially ordered partial b - metric space $(X, \leq, p)$ are said to be weakly compatible if they commute at their coincidence points. that is $fu = gu$ for some $u \in X$, then $fgu = gfu$. □

Definition 3.9 Two self maps f and g of a partially ordered partial b - metric space $(X, \leq, p)$ are said to be reciprocally continuous if
$\lim_{m,n\to\infty} p(fgx_m, fgx_n) = \lim_{n\to\infty} p(fgx_n, fz) = p(fz, fz) = 0$
and
$\lim_{m,n\to\infty} p(gfx_m, gfx_n) = \lim_{n\to\infty} p(gfx_n, fz) = p(gz, gz) = 0$
whenever $\{x_n\}$ is a sequence in X with
$\lim_{m,n\to\infty} p(fx_m, fx_n) = \lim_{n\to\infty} p(fx_n, z) = p(z, z) = 0$
and $\lim_{m,n\to\infty} p(gx_m, gx_n) = \lim_{n\to\infty} p(gx_n, z) = p(z, z) = 0$
for some $z \in X$. □

In the following definition we extend the notion of ψ weak generalized Geraghty contraction and weak generalized Geraghty contraction for two self maps on a partially ordered partial b - metric space.

Definition 4.0 Let $(X, \leq)$ be a partially ordered set and suppose that there exists a partial b - metric p such that (X,p) is a partial b - metric space. Let f and g be two self mappings on X. Suppose there exists $\psi \in \Psi, \beta \in S$ such that
$\psi(sp(f(x), f(y)) \leq \beta(\psi(M(x,y)))\psi(M(x,y))$ (4.0.1)
for all $x, y \in X$ whenever gx and gy are comparable, where
$M(x,y) =$
$max\{p(gx,gy), p(gx,fx), p(gy,fy), \frac{1}{2s}[p(gx,fy)+p(fx,gy)]\}$
Then we say that (f,g) is a pair of ψ weak generalized Geraghty contraction maps. We also say that (f,g) is a pair of weak generalized Geraghty contraction maps if $\psi(t) = t \; \forall \; t \in [0,\infty)$. □

Now we state the following useful lemmas 4.1 and 4.2, whose proofs can be found in Sastry. et. al [25].

Lemma 4.1 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in X such that

(i) $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} p(x_n, y_n) = 0 \Rightarrow x = y$. □

(ii) $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} x_n = y$
Then $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, y) = p(x,y)$ and hence $x = y$. □

Lemma 4.2 (i)$p(x,y) = 0 \Rightarrow x = y$ □

(ii) $\lim_{n\to\infty} p(x_n, x) = 0 \Rightarrow p(x,x) = 0$ and hence $x_n \to x$ as $n \to \infty$ □

Lemma 4.3 Let $(X, \leq, p)$ be a partially ordered partial b - metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.

Then (i) $\{x_n\}$ is a Cauchy sequence $\Rightarrow \lim_{m,n\to\infty} p(x_m, x_n) = 0$.

(ii) $\{x_n\}$ is not a Cauchy sequence $\Rightarrow \exists \; \epsilon > 0$ and sequences $\{m_k\}, \{n_k\} \ni m_k > n_k > k \in \mathbb{N}$; $p(x_{n_k}, x_{m_k}) > \epsilon$ and $p(x_{n_k}, x_{m_k - 1}) \leq \epsilon$

Proof : (i) Suppose $\{x_n\}$ is a Cauchy sequence then $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and finite. Therefore
$0 = \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{m,n\to\infty} p(x_m, x_n)$
Therefore $\lim_{m,n\to\infty} p(x_m, x_n) = 0$.

(ii) $\{x_n\}$ is not a Cauchy sequence $\Rightarrow \lim_{m,n\to\infty} p(x_m, x_n) \neq 0$ if it exists
$\Rightarrow \exists \; \epsilon > 0$ and for every N and $m, n > N \ni p(x_m, x_n) > \epsilon$
$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \Rightarrow \exists \; M \ni p(x_n, x_{n+1}) < \epsilon \; \forall \; n > M$.
Let $N_1 > M$ and n_1 be the smallest such that $m > n_1$ and $p(x_{n_1}, x_m) > \epsilon$ for at least one m. Let m_1 be the smallest such that $m_1 > n_1 > N_1 > 1$ and $p(x_{n_1}, x_{m_1}) > \epsilon$
so that $p(x_{n_1}, x_{m_1 - 1}) \leq \epsilon$. Let $N_2 > N_1$ and choose $m_2 > n_2 > N_2 > 2 \ni p(x_{n_2}, x_{m_2}) > \epsilon$ and $p(x_{n_2}, x_{m_2 - 1}) \leq \epsilon$.
Continuing this process we can get sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k > k$ and $p(x_{m_k}, x_{n_k}) > \epsilon$; $p(x_{n_k}, x_{m_k - 1}) \leq \epsilon$ □

Now we state our first main result

Theorem 4.4 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let f and g be two self maps on X, f is g- non - decreasing. Suppose that (f,g) is a pair of ψ - weak generalized Geraghty contraction maps, that is there exist $\psi \in \Psi$ and $\beta \in S$ such that
$\psi(sp(f(x), f(y)) \leq \beta(\psi(M(x,y)))\psi(M(x,y))$ for all $x, y \in X$ whenever gx and gy are comparable, where
$M(x,y) =$
$max\{p(gx,gy), p(gx,fx), p(gy,fy), \frac{1}{2s}[p(gx,fy)+p(fx,gy)]\}$ (4.4.1)

(i) $fX \subseteq gX$
(ii) there exists $x_0 \in X$ such that $gx_0 \leq fx_0$
(iii) $g(X)$ is is a closed subset of X.
(iv) if any non - decreasing sequence $\{x_n\}$ in X, converges to u, then $x_n \leq u \;\; \forall \; n \geq 0$

Then f and g have a coincidence point in X

Proof : let $x_0 \in X$ be as in (ii). If $gx_0 = fx_0$ then x_0 is a coincident point and there is nothing to prove. Now suppose $gx_0 < fx_0$. By $(i) \; \exists \; x_1 \in X$ such that $gx_1 = fx_0$.
Since $gx_0 < fx_0 = gx_1$ and f is $g-$ non decreasing, we have $fx_0 < fx_1$.
Since $f(X) \subseteq g(X)$ and $fx_1 \in fX \subseteq gX$, there exists $x_2 \in X$ such that $gx_2 = fx_1$ and $gx_1 \leq gx_2$. Continuing this process, we can find sequence $\{x_n\}$ with $fx_n = gx_{n+1}$ for $n = 1, 2, 3, ...$ Further, since $gx_1 \leq gx_2$ and f is $g-$ non decreasing, we have $fx_1 \leq fx_2$ so that $gx_2 \leq gx_3$.
$\therefore$ By induction, we get $gx_n \leq gx_{n+1} \; \forall \; n = 0, 1, 2, 3, ...$
Suppose $gx_{n+1} = gx_{n+2}$ for some $n \in \mathbb{N} \Rightarrow gx_{n+1} = fx_{n+1}$
$\Rightarrow x_{n+1}$ is a coincident point of f and g in X.
Hence we may assume that $gx_{n+1} \neq gx_{n+2} \; \forall \; n \in \mathbb{N}$.
Then we have $p(gx_{n+2}, gx_{n+1}) > 0$, therefore by $(4.4.1)$,
$\psi(sp(g(x_{n+2}), g(x_{n+1})))$
$= \psi(sp(f(x_{n+1}), f(x_n)))$
$\leq \beta(M(x_{n+1}, x_n)) \; M(x_{n+1}, x_n)$,
where $M(x_{n+1}, x_n)$
$= max\{p(gx_{n+1}, gx_n), p(gx_{n+1}, fx_{n+1}), p(gx_n, fx_n),$
$\frac{1}{2s}[p(gx_{n+1}, fx_n) + p(fx_{n+1}, gx_n)]\}$
$= max\{p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2}), p(gx_n, gx_{n+1}),$
$\frac{1}{2s}[p(gx_{n+1}, gx_{n+1}) + p(gx_{n+2}, gx_n)]\}$
$\leq max[p(gx_{n+1}, gx_n), \; p(gx_{n+1}, gx_{n+2}),$
$\frac{1}{2s}\{p(gx_{n+1}, gx_{n+1}) + s(p(gx_{n+2}, gx_{n+1}) + p(gx_{n+1}, gx_n)) - p(gx_{n+1}, gx_{n+1}) \;\}]$
$= max[\; p(gx_{n+1}, gx_n), \; p(gx_{n+1}, x_{n+2})]$
Suppose $p(gx_{n+1}, gx_n) \leq gp(x_{n+1}, gx_{n+2})$ (4.4.2)
Then $M(x_{n+1}, x_n) = p(gx_{n+1}, gx_{n+2})$
$\therefore \psi(sp(gx_{n+2}, gx_{n+1}))$
$\quad \leq \beta(\psi(p(gx_{n+2}, gx_{n+1})))\psi(p(gx_{n+2}, gx_{n+1}))$
$\quad < \psi(p(gx_{n+2}, gx_{n+1}))$
$\Rightarrow sp(gx_{n+2}, gx_{n+1}) < p(gx_{n+2}, gx_{n+1})$, a contradiction.
$\therefore M(x_{n+1}, x_n) = p(gx_{n+1}, gx_n)$ (4.4.3)
$\therefore \psi(p(gx_{n+2}, gx_{n+1})) \leq \psi(sp(gx_{n+2}, gx_{n+1}))$
$\quad \leq \beta(\psi(p(gx_{n+1}, gx_n)))\psi(p(gx_{n+1}, gx_n))$
$\quad < \psi(p(gx_{n+1}, gx_n))$
$\Rightarrow p(gx_{n+2}, gx_{n+1}) \leq sp(gx_{n+2}, gx_{n+1}) < p(gx_{n+1}, gx_n)$ (4.4.4)
$\therefore$ sequence $\{\psi(p(gx_{n+1}, gx_n))\}$ is strictly decreasing and converges to r (say).
Also sequence $p(gx_{n+1}, gx_n)$ is strictly decreasing and converges to λ (say). $\therefore r = \psi(\lambda)$ (4.4.5)
Suppose $r \neq 0$

$\therefore \frac{\psi(p(gx_{n+2},gx_{n+1}))}{\psi(p(gx_{n+1},gx_n))} \leq \beta(\psi(p(gx_{n+1},gx_n))) < 1$ (4.4.6)
taking limits as $n \to \infty$
$\therefore \lim_{n\to\infty} \beta(\psi(p(gx_{n+1},gx_n)))=1$
$\Rightarrow \lim_{n\to\infty} \psi(p(gx_{n+1},gx_n)) = 0$
$\therefore r = 0 \Rightarrow \psi(\lambda) = 0 \Rightarrow \lambda = 0$ (4.4.7)
Now we claim sequence $\{gx_n\}$ is a Cauchy sequence. Assume that $\{gx_n\}$ is not a Cauchy sequence. Then by lemma 4.3 $\exists\ \epsilon > 0$ and sequences $\{m_k\}$, $\{n_k\}$; $m_k > n_k > k$ such that $p(gx_{m_k}, gx_{n_k}) \geq \epsilon$ and $p(gx_{m_k-1}, gx_{n_k}) < \epsilon$.
$\therefore \psi(s\epsilon)$
$\leq\psi\{sp(gx_{m_k}, gx_{n_k})\}$
$=\psi\{sp(fx_{m_k-1}, fx_{n_k-1})\}$
$\leq\beta(\psi(M(x_{m_k-1}, x_{n_k-1})))\ \psi\{M(x_{m_k-1}, x_{n_k-1})\}$ (4.4.8)
where $M(x_{m_k-1}, x_{n_k-1})$
$= max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, fx_{n_k-1}),$
$p(gx_{m_k-1}, fx_{m_k-1}), \frac{1}{2s}[\{p(gx_{m_k-1}, fx_{n_k-1})+$
$p(fx_{m_k-1}, gx_{n_k-1})\}]$
$=max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, gx_{n_k}), p(gx_{m_k-1}, gx_{m_k}),$
$\frac{1}{2s}[\{p(gx_{m_k-1}, gx_{n_k})+p(gx_{m_k}, gx_{n_k-1})\}]$
$\leq max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, gx_{n_k}), p(gx_{m_k-1}, gx_{m_k}),$
$\frac{1}{2s}[\{sp(gx_{m_k-1}, gx_{n_k-1})+sp(gx_{n_k-1}, gx_{n_k})$
$-p(gx_{n_k-1}, gx_{n_k-1})+sp(gx_{m_k-1}, gx_{n_k-1})$
$+sp(gx_{m_k}, gx_{m_k-1})-p(gx_{m_k-1}, gx_{m_k-1})\}]$
$\leq max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, gx_{n_k}), p(gx_{m_k-1}, gx_{m_k}),$
$\frac{1}{2s}[\{2sp(gx_{m_k-1}, gx_{n_k-1})+ sp(gx_{n_k-1}, gx_{n_k})+$
$sp(gx_{m_k}, gx_{m_k-1})\}]$
$= p(gx_{m_k-1}, gx_{n_k-1})+\frac{1}{2}p(gx_{n_k-1}, gx_{n_k})+\frac{1}{2}p(gx_{m_k}, gx_{m_k-1})$
$\leq sp(gx_{m_k-1}, gx_{n_k})+sp(gx_{n_k}, gx_{n_k-1})-p(gx_{n_k}, gx_{n_k})+$
$\frac{1}{2}p(gx_{n_k-1}, gx_{n_k})+\frac{1}{2}p(gx_{m_k}, gx_{m_k-1})$
$\leq sp(gx_{m_k-1}, gx_{n_k})+sp(gx_{n_k}, gx_{n_k-1})+$
$\frac{1}{2}p(gx_{n_k-1}, gx_{n_k})+\frac{1}{2}p(gx_{m_k}, gx_{m_k-1})$
$\leq s\epsilon + s\eta+\frac{1}{2}\eta + \frac{1}{2}\eta$ where $\eta > 0$ for large k
$\therefore \psi(s\epsilon)$
$\leq\beta(\psi(M(x_{m_k-1}, x_{n_k-1}))\psi(s\epsilon + s\eta + \eta)$
$< \psi(s\epsilon + s\eta + \eta)$ (4.4.9)
(This is being for large k and true for every $\eta > 0$)
Since ψ is continuous , then we get for large k,
$\psi(s\epsilon) \leq \lim_{k\to\infty} \beta(\psi(M(x_{m_k-1}, x_{n_k-1})))\ \psi(s\epsilon) \leq \psi(s\epsilon)$
$\therefore \lim_{k\to\infty} \beta(\psi(M(x_{m_k-1}, x_{n_k-1}))) = 1$
$\therefore \lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}) = 0$
$\therefore$ By (4.4.8), $\psi(s\epsilon) = 0 \Rightarrow s\epsilon = 0$, a contradiction.
Consequently $\{gx_n\}$ is a Cauchy sequence.
$\therefore \{gx_n\} \to gy$ for some $y \in X$ by (iii)
Also
$0 = \lim_{n,m\to\infty} p(gx_n, gx_m)=\lim_{n\to\infty} p(gx_n, gy)=p(gy, gy)$ (4.4.10)
Now by (iv) of the hypothesis $gx_n \leq gy\ \ \forall\ n \in \mathbb{N}$
Therefore $gx_{n+1} \leq gy \Rightarrow fx_n \leq fy\ \ \forall\ n \in \mathbb{N}$(since f is $g-$ non decreasing)
$\psi\{sp(fx_n, fy)\}$
$\leq \beta(\psi(M(x_n, y))\ \psi\{M(x_n, y)\}$
where$M(x_n, y)$=$max\{p(gx_n, gy), p(gy, fy),\ p(gx_n, fx_n),$
$\frac{1}{2s}[p(gx_n, fy)+p(fx_n, gy)]\}$
$= max\{p(gx_n, gy),\ p(gy, fy), p(gx_n, gx_{n+1}),$
$\frac{1}{2s}[p(gx_n, fy)\ +\ p(gx_{n+1}, gy)]\}$
$= p(gy, fy)$ for large n. (4.4.11)
Now, $\lim_{n\to\infty}\beta(\psi(M(x_n, y))) = 1$
$\Rightarrow \lim_{n\to\infty} \psi(M(x_n, y)) = 0$
$\Rightarrow \psi(p(gy, fy)) = 0$ (by 4.4.11) (4.4.12)
$\Rightarrow p(gy, fy) = 0 \Rightarrow gy = fy$ (by lemma 4.2 (i)).
Therefore y is a coincident point of f and g.
Suppose $\exists\ \lambda$ such that $\beta(\psi(M(x_n, y))) = \lambda$, for infinitely many n (4.4.13)
$\therefore 0 \leq \lambda < 1$
$\psi(sp(fx_n, fy))$
$\leq \lambda\psi(M(x_n, y)) \leq \lambda\psi(p(gy, fy)) < \psi(p(gy, fy))$
$\Rightarrow sp(fx_n, fy) < p(gy, fy)$
$\Rightarrow \lim_{n\to\infty} \sup\ sp(fx_n, fy) \leq p(gy, fy)$ (4.4.14)
Now$p(gy, fy) \leq sp(gy, gx_{n+1})+sp(gx_{n+1}, fy)-p(gx_{n+1}, gx_{n+1})$
$\leq sp(gy, gx_{n+1})\ +\ sp(gx_{n+1}, fy)$
$\Rightarrow p(gy, fy)-sp(gy, gx_{n+1}) \leq sp(fx_n, fy)$
$\Rightarrow p(gy, fy) \leq \lim_{n\to\infty} \inf\ sp(fx_n, fy)$ (4.4.15)
$\therefore \lim_{n\to\infty} \sup\ sp(fx_n, fy) \leq p(gy, fy) \leq \lim_{n\to\infty} \inf\ sp(fx_n, fy)$
$\therefore \lim_{n\to\infty}\ sp(fx_n, fy) = p(gy, fy)$
$\therefore \psi(p(gy, fy)) = \psi(\lim_{n\to\infty}\ sp(fx_n, fy))$
$= \lim_{n\to\infty}\ \psi(sp(fx_n, fy))$ (since ψ is continuous)
$\leq \lambda\psi(p(gy, fy))$
$\Rightarrow\psi(p(gy, fy))= 0 \Rightarrow p(gy, fy) = 0 \Rightarrow gy = fy$ (4.4.16)
Hence y is a coincident point of f and g.
Now we state and prove our second main result.

Theorem 4.5 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let f and g be two self maps on X, f is g- non - decreasing. Suppose that (f, g) is a pair of ψ - weak generalized Geraghty contraction maps, that is there exist $\psi \in \Psi$ and $\beta \in S$ such that
$\psi(sp(f(x), f(y)) \leq \beta(\psi(M(x, y)))\psi(M(x, y))\ \ \forall\ \ x, y\ \in\ X$
whenever
gx and gy are comparable,
where $M(x, y)$=
$max\{p(gx, gy), p(gx, fx), p(gy, fy), \frac{1}{2s}[p(gx, fy)+p(fx, gy)]\}$
(4.5.1)
Assume that
(i) $fX \subseteq gX$
(ii) there exists $x_0 \in X$ such that $gx_0 \leq fx_0$
(iii) $g(X)$ is is a closed subset of X.
(iv) if any non - decreasing $\{x_n\}$ in X, converges to y, then that $x_n \leq y\ \ \forall\ n \geq 0$.
Further if f and g are weakly compatible f and g and if $gy \leq ggy\ \forall\ y \in X$, then f and g have a common fixed point in X

Proof We have by Theorem 4.4, $\{gx_n\}$ is a Cauchy sequence, which is non-decreasing that converges to gy and $gy = fy$.
$\therefore \lim_{n,m\to\infty}\ p(gx_n, gx_m)$ exists and is equal to 0
As sequence $\{gx_n\} \to gy$ implies
$0 = \lim_{n,m\to\infty}\ p(gx_n, gx_m) = \lim_{n\to\infty}\ p(gx_n, gy) = p(gy, gy)$
Since f and g are weakly compatible, we have $fgy = gfy$
Let gy= fy= u (say) (4.5.2)
$\therefore fu$=fgy=gfy= gu (4.5.3)
If $y = u$, then $u = fu = gu \Rightarrow u$ is a common fixed point of f and g in X.
Let$y \neq u \Rightarrow gy \neq gu \Rightarrow p(gy, gu) \neq 0$ (by lemma 4.2 (i))
We have from (4.5.1),

$\psi(sp(gy, gu)) = \psi(sp(fy, fu)) \leq \beta(M(y, u))M(y, u)$
where $M(y, u) =$
$max\{p(gy, gu), p(gy, fy), p(gu, fu), \frac{1}{2s}[p(gy, fu)+p(fy, gu)]\}$
$= p(gy, gu)$(by (4.5.2) and lemma 4.1 (i))
$\therefore \psi(sp(gy, gu)) \leq \beta(\psi(M(y, u)))\psi(M(y, u))$
$\Rightarrow \psi(sp(gy, gu)) \leq \beta(\psi(p(gy, gu)))\psi(p(gy, gu))$
$\Rightarrow \psi(p(gy, gu)) \leq \psi(sp(gy, gu)) < \psi(p(gy, gu))$
if$\psi(p(gy, gu)) > 0$, a contradiction
$\therefore \psi(p(gy, gu)) = 0 \Rightarrow p(gy, gu) = 0$
$\therefore gy = gu$
$\therefore$ By (4.5.2) and (4.5.3) $u = fu = gu$
$\therefore u$ is a common fixed point of f and g in X.
Now we state and prove our third main result.

Theorem 4.6 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let f and g be two self maps on X, f is g - non - decreasing. Suppose that (f, g) is a pair of ψ - weak generalized Geraghty contraction maps, that is there exist $\psi \in \Psi$ and $\beta \in S$ such that $\psi(sp(f(x), f(y)) \leq \beta(\psi(M(x, y))) \ \psi(M(x, y))$ for all $x, y \in X$ whenever gx and gy are comparable, where $M(x, y) =$
$max\{p(gx, gy), p(gx, fx), p(gy, fy), \frac{1}{2s}[p(gx, fy)+p(fx, gy)]\}$ (4.6.1)

Assume that
(i) $fX \subseteq gX$
(ii) f and g are compatible
(iii) there exists $x_0 \in X$ such that $gx_0 \leq fx_0$
(iv) f and g are reciprocally continuous.
Then f and g have a coincidence point in X

Proof: We have by Theorem 4.4, $\{gx_n\}$ is a Cauchy sequence, which is non-decreasing that converges to z (say).
$\therefore \lim_{n,m\to\infty} p(gx_n, gx_m)$ exists and is equal to 0
As sequence $\{gx_n\} \to z$ implies
$\lim_{n,m\to\infty} p(gx_n, gx_m) = \lim_{n\to\infty} p(gx_n, z) = p(z, z) = 0$
Now $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$
$\therefore \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = z$
Since f and g are reciprocally continuous,
$\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$
Also since f and g are compatible,
$\therefore \lim_{n\to\infty} p(fgx_n, gfx_n) = 0$
Then by lemma 4.2 (i), we get $fz = gz$
Hence z is a coincidence point of f and g in X.
The following corollaries can be established for the theorems 4.4, 4.5 and 4.6

Corollary 4.7 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $f : X \to X$ be a ψ weak generalized Geraghty contraction map. If there exists $x_0 \in X$ such that $x_0 \leq fx_0$ and f is non decreasing, if any non decreasing sequence $\{x_n\}$ in X converges to u, then we assume that $x_n \leq u \ \forall \ n \geq 0$. Then f has a fixed point.
Proof: follows from the theorem 4.4 by choosing $g = I_x$

Corollary 4.8 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $f : X \to X$ be a weak generalized Geraghty contraction map. If there exists $x_0 \in X$ such that $x_0 \leq fx_0$ and f is non decreasing, if any non decreasing sequence $\{x_n\}$ in X converges to u, then we assume that $x_n \leq u \ \forall \ n \geq 0$. Then f has a fixed point
Proof: follows from the theorem 4.4
by choosing $g = I_x$ and $\psi(t) = t$.

Corollary 4.9 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $f : X \to X$ be a ψ weak generalized Geraghty contraction map. If there exists $x_0 \in X$ such that $x_0 \leq fx_0$ and f is non decreasing and continuous. Then f has a fixed point.
Proof: follows from the theorem 4.6 by choosing $g = I_x$.

Corollary 4.10 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $f : X \to X$ be a weak generalized Geraghty contraction map. If there exists $x_0 \in X$ such that $x_0 \leq fx_0$ and f is non decreasing and continuous. Then f has a fixed point.
Proof: follows from the theorem 4.6 by choosing
$g = I_x$ and $\psi(t) = t$.
Now we give an example in support of theorem 4.4

Example: Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{10}\}$with usual ordering.
Define$p(x, y) = 0 \ if \ x = y$
$= 1 \ if \ x \neq y \in \{0, 1\}$
$= |x - y| \ if \ x, y \in \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\}$
$= 4$ otherwise
Clearly, $(X, \leq, p)$ is a partially ordered partial b - metric space with coefficient $s = \frac{8}{3}$ (P.Kumam et.al [15])
Define $f, g : X \to X$ by
$f1 = f\frac{1}{2} = f\frac{1}{3} = f\frac{1}{4} = f\frac{1}{5} = \frac{1}{2}$,
$f\frac{1}{6} = \frac{1}{4} = f\frac{1}{7} = f\frac{1}{8} = f\frac{1}{9} = f\frac{1}{10} = f0 \Rightarrow f(X) = \{\frac{1}{2}, \frac{1}{4}\}$
and
(i)For $x=0, y \in X \Rightarrow \psi(sp(fx, fy))=0$ or $\frac{1}{3} \leq \frac{2}{3} = \beta(\psi(M(x, y)))\psi(M(x, y))$
where $M(x, y)$
$=max\{p(gx, gy), p(gx, fx), p(gy, fy), \frac{1}{2s}[p(gx, fy)+p(fx, gy)]\}$
$= 4$
(ii)For $1 \leq m \leq 5$ and $6 \leq n \leq 10$
$\Rightarrow \psi(sp(fx, fy)) \leq \frac{1}{3} \leq \frac{2}{3} = \beta(\psi(M(x, y)))\psi(M(x, y))$
where $M(x, y) = 4$
(iii)For $6 \leq m \leq 10$ and $1 \leq n \leq 5 \Rightarrow \psi(sp(fx, fy)) \leq \frac{1}{3} \leq \frac{2}{3} = \beta(\psi(M(x, y)))\psi(M(x, y))$
where $M(x, y) = 4$
(iv)For $6 \leq m \leq 10$ and $6 \leq n \leq 10 \Rightarrow \psi(sp(fx, fy))=0 \leq \frac{2}{3} = \beta(\psi(M(x, y)))\psi(M(x, y))$
where $M(x, y) = 4$
$f\frac{1}{2} = \frac{1}{2} = g\frac{1}{2} \Rightarrow fg\frac{1}{2} = \frac{1}{2} = gf\frac{1}{2} \Rightarrow f$ and g are weakly compatible at $\frac{1}{2} \in X$
Clearly $g\frac{1}{10} = \frac{1}{9} < \frac{1}{4} = f\frac{1}{10}$.
Let $x_0 = \frac{1}{10} \Rightarrow gx_0 < fx_0 = \frac{1}{4} = g\frac{1}{3} = gx_1$
$\Rightarrow fx_1 = f\frac{1}{3} = \frac{1}{2} = g\frac{1}{2} = gx_2$
$\Rightarrow fx_2 = f\frac{1}{2} = \frac{1}{2} = g\frac{1}{2} = gx_2$
Therefore $\frac{1}{2} \in X$ is a fixed point.
The hypothesis and conclusions of of theorem 4.4 satisfied.

Open Problem

Are the theorems 4.4, 4.5, 4.6 and their corollaries true if continuity of ψ is dropped?

Acknowledgment

The authors are grateful to the referees for their valuable suggestions in rewriting the paper in the present form. The authors are also grateful to the Editor-in-Chief for his valuable comments to standardize it. Further the fourth author is grateful to management of Mrs.AVNCollege, Visakhapatnam and Mathematics Department of Andhra University for giving necessary permission and providing necessary facilities to carry on this research.

REFERENCES

[1] T. Abdeljawad, "Meir-Keeler α - Contractive fixed and common fixed point theorems", *Fixed Point Theory and Applications*, 2013:19, doi:10.1186/1687-1812-2013-19, 2013.

[2] H. Aydi, M. Bota, E. Karapinar and S. Mitrovic, "A fixed point theorem for set-valued quasi-contractions in b-metric spaces", *Fixed Point Theory and its Applications*, 2012:88, doi:86/1687-1812-2012-88, 2012.

[3] G.V.R. Babu, K.K.M. Sarma and V.A. Kumari, "Common fixed points of ψ-weak generalized Geraghty contractions in partially ordered metric spaces", *International Journal Of Mathematics And Scientific Computing*, vol. 4, no. 2, pp. 88-93, 2014.

[4] I.A. Bakhtin, "The contraction principle in quasi-metric spaces," *Funct. Anal. Unianowsk Gos. Ped.*, vol. 30, pp. 26-37, 1989.

[5] S. Banach, "Sur les operations dans les ensembles abstraits et leur application oux equations integrals", *Fundamenta Mathematicae*, vol. 3, pp. 133-181, 1922.

[6] L. Ciric, N. Cakic, M. Rajovic and J.S. Ume, "Monotone generalized nonlinear contractions in partially ordered metric spaces", *Fixed Point Theory and Applications*, 2008:131294, doi:10.1155/2008/131294, 2008.

[7] S. Czerwik, "Contraction mappings in b-metric spaces", *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, no. 1, pp. 5-11, 1993.

[8] M.A. Geraghty, "On contractive maps", *Proceedings of the American Mathematical Society*, vol. 40, no. 2, pp. 604-608, 1973.

[9] G. Jungck, "Compatible mappings and common fixed points", *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771-779, 1986.

[10] G. Jungck and B.E. Rhoades, "Fixed point for set valued function without continuity", *Indian Journal of Pure and Applied Mathematics*, vol. 29, no. 3, pp. 227-238, 1998.

[11] R. Kannan, "Some results on fixed points", *Bulletin of Calcutta Mathematical Society*, vol. 60, pp. 71-76, 1968.

[12] E. Karapinar and R.P. Agarwal, "A note on coupled fixed point theorems for $\alpha-\psi$ Contractive-type mappings in partially ordered metric spaces", *Fixed Point Theory and Applications*, 2013:154, doi:10.1186/1687-1812-2013-154, 2013.

[13] E. Karapinar and B. Samet, "Generalized $\alpha - \psi$ contractive type mappings and related fixed point theorems with applications", *Abstract and Applied Analysis*, Art. ID 793486, 2012.

[14] M. Khan, M. Swaleh and S. Sessa, "Fixed point theorems by altering distances between the points", *Bulletin of the Autralian Mathematical Society*, vol. 30, pp. 1-9, 1984.

[15] P. Kumam, N.V. Dung and V.T.L. Hang, "Some equivalences between cone b-metric spaces and b-metric spaces", *Abstract and Applied Analysis*, vol. 2013, Article ID 573740, 2013.

[16] S.G. Matthews, "Partal metric topology", *Ann. Acad. N.Y., Sci.*, vol. 728, pp. 183-197, 1994.

[17] K. Mehmet and H. Kiziltunc, "On some well-known fixed point theorems in b-metric spaces", *Turkish Journal of Analysis and Number Theory*, vol. 1, pp. 13-16, 2013.

[18] M. Murasaleen, S.A. Mohiuddine and R.P. Agarwal, "Coupled fixed point theorems for α - ψ contractive type mappings in partially ordered metric spaces", *Fixed Point Theory and Applications*, Article ID 228, 2012.

[19] Z. Mustafa, J.R. Roshan, V. Parveneh and Z. Kadelburg, "Some common fixed point result in ordered partal b-metric spaces", *Journal of Inequalities and Applications*, doi:10.1186/1029-242X-2013-562, 2013.

[20] H. Nashinea, M. Imdadb and M. Hesanc, "Common fixed point theorems under rational contractions in complex valued metric spaces", *Journal of Nonlinear Sciences and Applications*, vol. 7, pp. 42-50, 2014.

[21] S.J. O'Neill,"Partial metrics, valuations and domain theory", in: Proc. 11th Summer Conference on General Topology and Applications. Ann. New York Acad. Sci., vol. 806, pp. 304-315, 1996.

[22] R.P. Pant, "A common fixed point theorem under a new condition", *Indian Journal of Pure and Applied Mathematics*, vol. 30, no. 2, pp. 147-152, 1999.

[23] B. Samet, C. Vetro and P. Vetro, "Fixed Point theorems for $\alpha - \psi$ Contractive type mappings", *Nonlinear Analysis*, vol. 75, pp. 2154-2165, 2012.

[24] K.P.R. Sastry, K.K.M. Sarma, Ch. Srinivasarao and V. Perraju, "Coupled Fixed point theorems for α - ψ contractive type mappings", *International J. of Pure and Engg. Mathematics*, vol. 3, no. 1, pp. 245-262, 2015.

[25] K.P.R. Sastry, K.K.M. Sarma, Ch. Srinivasarao and V. Perraju, "α-ψ-φ contractive mappings in complete partially ordered partial b-metric spaces", *International J. of Math. Sci. and Engg. Applications*, vol. 9, pp. 129-146, 2015.

[26] W. Shatanawiam and H. Nashineb, "A generalization of Banach's contraction principle for nonlinear contraction in a partial metric space", *Journal of Nonlinear Sciences and Applications*, vol. 5, pp. 37-43, 2012.

[27] S. Shukla, "Partial b-metric spaces and fixed point theorems", *Mediterranean Journal of Mathematics*, doi:10.1007/s00009-013-0327-4, 2013.

[28] S. Singh and B. Chamola, "Quasi-contractions and approximate fixed points", *Natur. J., Phys. Sci.*, vol. 16, pp. 105-107, 2002.

[29] C. Vetroa and F. Vetrob, "Common fIxed points of mappings satisfying implicit relations in partial metric spaces", *Journal of Nonlinear Sciences and Applications*, vol. 6, pp. 152-161, 2013.

[30] X. Zhang, "Common fixed point theorems for some new generalised contractive type mappings", *Journal of Mathematical Analysis and Applications*, vol. 333, no. 2, pp. 780-786, 2007.

[31] H. Yingtaweesittikul, "Suzuki type fixed point for generalized multi-valued mappings in b-metric spaces", *Fixed Point Theory and Applications*, 2013:215, doi:10.1186/1687-1812-2013-215, 2013.