Line Gracefulness of Some Path Related Graphs

S. K. Vaidya and N. J. Kothari

***Abstract*—We investigate line graceful labeling for the graphs arising from different graph operations on path P_n. We prove that the spliting graph, shadow graph, total graph and middle graph of path admit line graceful labeling. Moreover we prove that the graph obtained by duplication of each edge of path by a vertex and the alternate triangular snake are line graceful graphs.**

***Index Terms*—Line graceful, edge graceful, total graph, middle graph.**

***MSC 2010 Codes* – 05C78, 05C38, 05C76.**

I. INTRODUCTION

HERE, by a graph $G = (V(G),\ E(G))$, we mean finite, simple, connected and undirected graph. For standard terminology and notations, we refer to West [1].

Definition 1.1 A *graph labeling* is an assignment of numbers to the vertices or edges or both subject to certain condition(s).

Many diversified applications of graph labeling are reported by Yegnanaryanan and Vaidhyanathan [2]. An extensive survey on graph labeling can be found in Gallian [3].

Definition 1.2 A function f is called *graceful labeling* of graph if $f : V(G) \rightarrow \{0, 1, 2, 3, ..., q\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 2, ..., q\}$ defined as $f^*(e = uv) = |f(x) - f(y)|$ is bijective. A graph which admits graceful labeling is called a graceful graph.

Most of the graph labeling problems trace their origin with graceful labeling which was introduced independently by Rosa [4] and Golomb [5]. A variant of graceful labeling termed as edge graceful labeling is introduced by Lo [6].

Definition 1.3 A graph $G = (V(G), E(G))$ is said to be *edge graceful* if there exists a bijection $f : E(G) \rightarrow \{1, 2, 3, ..., q\}$ such that the induced mapping $f^* : V(G) \rightarrow \{0, 1, ..., p - 1\}$ defined by $f^*(v) = \sum_{vv_i \in E(G)} f(vv_i) \pmod p$ is bijection.

Lo [6] discussed edge gracefulness of many graph families and also derived a necessary condition for a graph to be edge graceful. Lee [7] has conjectured that all trees of odd order are edge-graceful. Lee *et al.* [8], [9], [10] have proved many results on edge graceful labeling of graphs. Gnanajothi[11] introduced and studied line graceful labeling in her Ph. D. thesis which is little weaker than edge graceful labeling.

Definition 1.4 A mapping $f : E(G) \rightarrow \{0, 1, 2, ..., p\}$ is called *line graceful* of graph with p vertices, if induced function $f^* : V(G) \rightarrow \{0, 1, 2, ..., p - 1\}$ defined by $f^*(v) = \sum_{vv_i \in E(G)} f(vv_i)\,(mod\,p)$ is bijective.

Vaidya and Kothari [12] have investigated many results on line gracefulness of graphs.

Definition 1.5 For every vertex $v \in V(G)$, the *open neighbourhood set* $N(v)$ is the set of all vertices adjacent to v in G.

Definition 1.6 For a graph G the *splitting graph* $S'(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.

Definition 1.7 The *shadow graph* $D_2(G)$ of a connected graph G is obtained by taking two copies of G, say G' and G''. Join each vertex u' in G' to the neighbours of corresponding vertex u'' in G''.

Definition 1.8 The *total graph* $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

Definition 1.9 The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

Definition 1.10 Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.

Definition 1.11 An alternate triangular snake $A(Tn)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternately) to a new vertex v_i. That is every alternate edge of a path is replaced by C_3.

II. MAIN RESULT

Proposition 2.1[11] If the graph is line graceful then its order is not congruent to $2\ (mod\,4)$.

Theorem 2.2 $S'(P_n)$ is line graceful for $n \equiv 0, 2\,(mod\,4)$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n and $u_1, u_2, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, \ldots, v_n$ which are added to obtain $S'(P_n)$. We note that $|V(S'(P_n))| = 2n$ and $|E(S'(P_n))| = 3(n - 1)$. Define edge labeling $f : E(S'(P_n)) \rightarrow \{0, 1, \ldots, 2n\}$ as follows.

Case 1: $n \equiv 0, 2\,(mod\,4)$

For $1 \leq i \leq n - 1$

$$f(v_iv_{i+1}) = \begin{cases} 1 & \text{for odd } i \\ 0 & \text{for even } i \end{cases}$$

$$f(u_iv_{i+1}) = 0$$
$$f(v_iu_{i+1}) = 2i.$$

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Above defined edge labeling function satisfies the condition for line graceful labeling.

Case 2: $n \equiv 1, 3\,(mod\,4)$

In this case $|V(S'(P_n))| = 2n \equiv 2\,(mod\,4)$.

Then according to Proposition 2.1, $S'(P_n)$ is not line graceful. Thus we proved that $S'(P_n)$ admits line graceful labeling for $n \equiv 0, 2 \pmod 4$.

Example 2.3 Line graceful labeling of $S'(P_6)$ is shown in Figure 1.

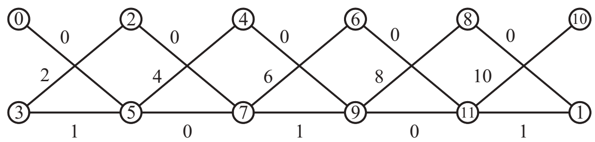

Figure 1

Theorem 2.4 $D_2(P_n)$ is line graceful for $n \equiv 0, 2\,(mod\,4)$.

Proof. Consider two copies of path P_n. Let $v_1, v_2, \ldots, v_n$ be the vertices of first copy of path P_n and $u_1, u_2, \ldots, u_n$ be the vertices of second copy of path P_n. Hence $V(D_2(P_n)) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and

$E(D_2(P_n)) = \{v_iv_{i+1}, u_iu_{i+1}, v_iu_{i+1}, u_iv_{i+1}; 1 \le i \le n-1\}$

We note that $|V(D_2(P_n))| = 2n$ and $|E(D_2(P_n))| = 4(n-1)$.

Define edge labeling $f : E(D_2(P_n)) \to \{0, 1, \ldots, 2n\}$ as follows.

Case 1: $n \equiv 0, 2\,(mod\,4)$

For $1 \le i \le n-1$

$$f(v_iv_{i+1}) = \begin{cases} 1 & \text{for odd } i \\ 0 & \text{for even } i \end{cases}$$

$$f(u_iv_{i+1}) = 0$$
$$f(v_iu_{i+1}) = 2i$$
$$f(u_iu_{i+1}) = 0.$$

Above defined edge labeling function satisfies the condition for line graceful labeling.

Case 2: $n \equiv 1, 3\,(mod\,4)$

In this case $|V(D_2(P_n))| = 2n \equiv 2\,(mod\,4)$.

Then according to Proposition 2.1 $D_2(P_n)$ is not line graceful.

Thus we proved that $D_2(P_n)$ admits line graceful labeling for $n \equiv 0, 2$ (mod 4).

Example 2.5 Line graceful labeling of $D_2(P_8)$ is shown in Figure 2.

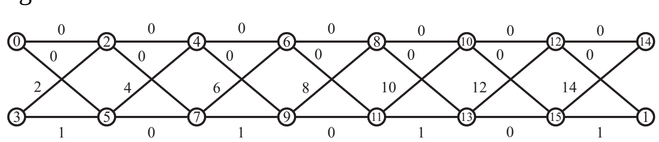

Figure 2

Theorem 2.6 $T(P_n)$ is line graceful for all n.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. Then $V(T(P_n)) = V(P_n) \cup E(P_n)$ and $E(T(P_n)) = \{v_iv_{i+1}; 1 \le i \le n-1, v_ie_i; 1 \le i \le n-1, e_ie_{i+1}; 1 \le i \le n-2, v_ie_{i-1}; 2 \le i \le n\}$. We note that $|V(T(P_n))| = 2n-1$ and $|E(P_n)| = 4n-5$.

Define edge labeling $f : E(T(P_n)) \to \{0, 1, \ldots, 2n-1\}$ as follows.

$$\begin{array}{lll} f(v_iv_{i+1}) & = 0 & for \quad 1 \le i \le n-1 \\ f(e_ie_{i+1}) & = 0 & for \quad 1 \le i \le n-2 \\ f(v_ie_i) & = 2i-1 & for \quad 1 \le i \le n-1 \\ f(v_ie_{i-1}) & = 2i-2 & for \quad 2 \le i \le n \end{array}$$

Above defined edge labeling function satisfies the condition for line graceful labeling.

Hence $T(P_n)$ admits line graceful labeling for all n.

Example 2.7 A line graceful labeling of $T(P_7)$ is shown in Figure 3.

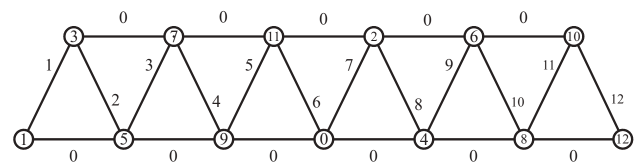

Figure 3

Theorem 2.8 $M(P_n)$ is line graceful for all n.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. Then $V(M(P_n)) = V(P_n) \cup E(P_n)$ and $E(M(P_n)) = \{v_ie_i; 1 \le i \le n-1, v_ie_{i-1}; 2 \le i \le n, e_ie_{i+1}; 1 \le i \le n-2\}$. Here $|V(M(P_n))| = 2n-1$ and $|E(P_n)| = 3n-4$.

Define edge labeling $f : E(M(P_n)) \to \{0, 1, \ldots, 2n-1\}$ as follows.

$$\begin{array}{lll} f(v_ie_i) & = 2i-1 & for \quad 1 \le i \le n-1 \\ f(v_ie_{i-1}) & = 2i-2 & for \quad 2 \le i \le n \\ f(e_ie_{i+1}) & = 0 & for \quad 1 \le i \le n-2 \end{array}$$

Above defined edge labeling function satisfies the condition for line graceful labeling.

Hence $M(P_n)$ admits line graceful labeling for all n.

Example 2.9 Line graceful labeling of $M(P_9)$ is shown in Figure 4.

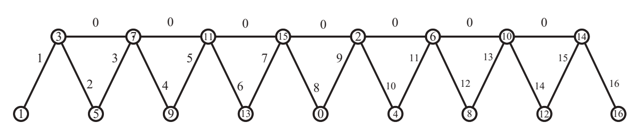

Figure 4

Theorem 2.10 The graph obtained by duplication of each edge of P_n by a vertex admits line graceful labeling.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n and G be the graph obtained by duplication of each edge v_iv_{i+1} of path P_n by vertex $u_i\,(1 \le i < n)$. Then $V(G) = V(P_n) \cup \{u_1, u_2, \ldots, u_{n-1}\}$ and

$E(G) = \{v_iv_{i+1}; 1 \le i \le n-1\} \cup \{v_iu_i; 1 \le i \le n-1\} \cup \{v_{i+1}u_i; 1 \le i \le n-1\}$.

We observe that $|V(G)| = 2n-1$ and $|E(G)| = 3n-3$.

Define edge labeling $f : E(G) \to \{0, 1, \dots, 2n-1\}$ as follows.

$$\begin{array}{lll} f(v_i v_{i+1}) & = 0 & \text{for} \quad 1 \le i \le n-1 \\ f(v_i u_i) & = 2i-1 & \text{for} \quad 1 \le i \le n-1 \\ f(v_{i+1} u_i) & = 2i & \text{for} \quad 1 \le i \le n-1 \end{array}$$

Above defined edge labeling function satisfies the condition for line graceful labeling.

Hence, graph G admits line graceful labeling.

Example 2.11 Graph obtained duplication of each edge of P_7 by a vertex and its line graceful labeling is shown in Figure 5.

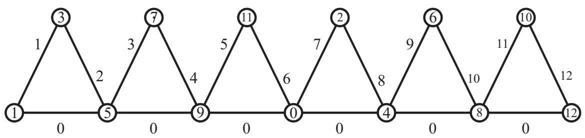

Figure 5

Theorem 2.12 Alternate triangular snake $A(T_n)$ is line graceful except $3n \equiv 4, 5 \pmod 8$.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of path P_n. The graph $A(T_n)$ is obtained by joining the vertices $v_i v_{i+1}$ (alternately) to new vertex u_i, $1 \le i \le n-1$ for even n and $1 \le i \le n-2$ for odd n. Therefore $V(A(T_n)) = V(P_n) \cup \left\{u_i / 1 \le i \le \lfloor \frac{n}{2} \rfloor\right\}$ and $E(A(T_n)) = E(P_n) \cup \left\{v_{2i-1}u_i; 1 \le i \le \lfloor \frac{n}{2} \rfloor, u_i v_{2i}; 1 \le i \le \left\lfloor \dfrac{n}{2} \right\rfloor\right\}$. We note that

$$|V(A(T_n))| = \begin{cases} \frac{3n-1}{2} & \text{for odd } n \\ \frac{3n}{2} & \text{for even } n \end{cases}$$

and

$$|E(A(T_n))| = \begin{cases} 2n-2 & \text{for odd } n \\ 2n-1 & \text{for even } n \end{cases}$$

Define edge labeling $f : E(A(T_n)) \to \{0, 1, \dots, |E(A(T_n))|\}$ as follows.

Case 1: $3n \equiv 0 \pmod 8$

<u>For $1 \le i \le \lfloor \frac{n}{2} \rfloor$</u>

$$f\left(v_{(2i-1)} v_{(2i)}\right) = 0$$

<u>For $1 \le i \le \lfloor \frac{n}{4} \rfloor$</u>

$$f\left(v_{(2i)} v_{(2i+1)}\right) = 3i$$

$$f\left(v_{\left(\frac{n}{2}+2\right)} v_{\left(\frac{n}{2}+3\right)}\right) = 4$$

<u>For $1 \le i \le \frac{n-8}{8}$, $n \ne 8$</u>

$$f\left(v_{\left(\frac{n}{2}+2+4i\right)} v_{\left(\frac{n}{2}+4+4i\right)}\right) = 4 + 6i$$

$$f\left(v_{(n-4i)} v_{(n-4i+1)}\right) = 6\left(\frac{n-4i}{4}\right)$$

$$f\left(v_{\left(\frac{n}{2}+1\right)} u_{\left(\frac{n}{4}+1\right)}\right) = 2$$

$$f\left(v_{\left(\frac{n}{2}+1+4i\right)} u_{\left(\frac{n}{4}+1+2i\right)}\right) = f\left(v_{\left(\frac{n}{2}+1\right)} u_{\left(\frac{n}{4}+1\right)}\right) + 6i$$

<u>For $0 \le i < \frac{n}{4}$</u>

$$f\left(u_{\left(\frac{n}{4}+1+i\right)} v_{\left(\frac{n}{2}+2i+2\right)}\right) = f\left(v_{\left(\frac{n}{2}+2i\right)} v_{\left(\frac{n}{2}+1+2i\right)}\right) + 2$$

<u>For $0 \le i < \frac{n}{8}$</u>

$$f\left(v_{\left(\frac{n}{2}+3+4i\right)} u_{\left(\frac{n}{4}+2+2i\right)}\right) = f\left(u_{\left(\frac{n}{4}+1+2i\right)} v_{\left(\frac{n}{2}+2+4i\right)}\right) + 2$$

Above defined edge labeling function satisfies the condition for line graceful labeling for $3n \equiv 0 \pmod 8$.

Case 2: $3n \equiv 1 \pmod 8$

<u>For $1 \le i \le \lfloor \frac{n}{2} \rfloor$</u>

$$f\left(v_{(2i-1)} v_{(2i)}\right) = 0$$

<u>For $1 \le i \le \lfloor \frac{n}{4} \rfloor$</u>

$$f\left(v_{(2i)} v_{(2i+1)}\right) = 3i$$

<u>For $0 \le i \le \lfloor \frac{n}{8} \rfloor$</u>

$$f\left(v_{\left(\frac{n+1}{2}+4i\right)} v_{\left(\frac{n+1}{2}+1+4i\right)}\right) = 2 + 6i$$

<u>For $0 \le i < \lfloor \frac{n}{8} \rfloor$</u>

$$f\left(v_{\left(\lceil \frac{n}{2} \rceil+2+4i\right)} v_{\left(\lceil \frac{n}{2} \rceil+3+4i\right)}\right) = 6\left\lceil \frac{n}{8} \right\rceil + 6i$$

$$f\left(u_{\left(\lceil \frac{n}{4} \rceil+1+2i\right)} v_{2\left(\lceil \frac{n}{4} \rceil+1\right)+4i}\right) = 4 + 6i$$

$$f\left(u_{\left(\lceil \frac{n}{4} \rceil+2+2i\right)} v_{2\left(\frac{n}{4}+2\right)+4i}\right) = 3\left\lceil \frac{n}{4} \right\rceil + 5 + 6i$$

$$f\left(v_{\left(\lceil \frac{n}{2} \rceil+1+4i\right)} u_{\left(\lceil \frac{n}{4} \rceil+1+2i\right)}\right) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + 1 + 6i$$

$$f\left(v_{\left(\lceil \frac{n}{2} \rceil+3+4i\right)} u_{\left(\lceil \frac{n}{4} \rceil+2+2i\right)}\right) = 6 + 6i$$

Above defined edge labeling function satisfies the condition for line graceful labeling for $3n \equiv 1 \pmod 8$.

Case 3: $3n \equiv 2, 3, 6, 7 \pmod 8$

<u>For $1 \le i \le \lfloor \frac{n}{2} \rfloor$</u>

$$f(v_{2i-1} v_{2i}) = 0$$

$$f(v_{2i} v_{2i+1}) = 3i$$

$$f(v_{2i-1} u_i) = 3i - 2$$

$$f(u_i v_{2i}) = 3i - 1$$

Above defined edge labeling function satisfies the condition for line graceful labeling for $3n \equiv 2, 3, 6, 7 \pmod 8$.

Case 4: $3n \equiv 4 \pmod 8$

$$\begin{array}{lll} \Rightarrow 3n & \equiv 0 & \pmod 2 \\ \Rightarrow n & \equiv 0 & \pmod 2 \\ \Rightarrow |V(A(T_n))| & = \frac{3n}{2} & \\ \Rightarrow |V(A(T_n))| & = \frac{3n}{2} \equiv 2 & \pmod 8 \\ \Rightarrow \frac{3n}{2} & = 8k + 2 & \\ \Rightarrow \frac{3n}{2} & \equiv 2 & \pmod 4 \end{array}$$

Then according to a Proposition 2.1, $A(T_n)$ is not line graceful.

Case 5: $3n \equiv 5 \pmod 8$

$$\begin{array}{lll}
\Rightarrow 3n & \equiv 1 & \pmod 2 \\
\Rightarrow n & \equiv 1 & \pmod 2 \\
\Rightarrow |V(A(T_n))| & = \frac{3n-1}{2} &
\end{array}$$

but

$$\begin{array}{lll}
\Rightarrow 3n & \equiv 5 & \pmod 8 \\
\Rightarrow 3n & = 8k+5 & \\
\Rightarrow 3n-1 & = 8k+4 & \\
\Rightarrow \frac{3n-1}{2} & = 4k+2 & \\
\Rightarrow |V(A(T_n))| & = \frac{3n-1}{2} \equiv 2 & \pmod 4
\end{array}$$

Then according to a Proposition 2.1, $A(T_n)$ is not line graceful.

Thus we have shown that $A(T_n)$ is line graceful except $3n \equiv 4, 5 \pmod 8$.

Example 2.13 $A(T_7)$ and its line graceful labeling is shown in Figure 6.

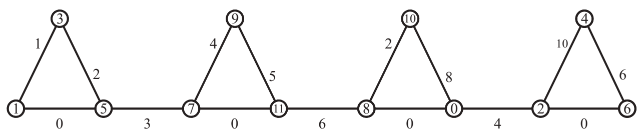

Figure 6

III. Concluding Remarks

We investigate line graceful labeling for the lager graphs obtained by means of graph operations on path P_n.

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