More on Diophantine Equations of Type $p^x + q^y = z^2$

Julius Fergy T. Rabago

***Abstract*—In this note we study some Diophantine equations of type $p^x + q^y = z^2$, where p and q are primes.**

***Index Terms*—Exponential Diophantine equation, integer solutions.**
***MSC 2010 Codes* – 11D61.**

I. Introduction

CASES of the Diophantine equation of the form

$$a^x + b^y = c^z$$

have been studied (*see* for instance $[3, 4, 5, 6, 7, 8]$). In 2005, Acu [1] studied Diophantine equations of type $a^x + b^y = c^z$ for primes a and b. In 2007, Acu [2] gave all solutions to the Diophantine equation $2^x + 5^y = z^2$ in non-negative integers. The only solutions are $(3, 0, 3)$ and $(2, 1, 3)$. On the other hand, Suvarnamani, Singta, and Chotchaisthit [3] showed that the two Diophantine Equations $4^x+7^y = z^2$ and $4^x+11^y = z^2$ have no solution in non-negative integers.

In [9], Rabago showed that the Diophantine equation $3^x + 13^y = z^2$ has exactly three solutions in positive integers. The solutions are $(1, 1, 4), (3, 2, 14)$ and $(5, 1, 16)$. In addition, Rabago showed that the exact solutions to $5^x + 11^y = z^2$ in positive integers are $(1, 1, 4), (2, 1, 6)$ and $(5, 1, 56)$.

In [10], all solutions to the Diophantine equation $2^x + 17^y = z^2$ were found. The given Diophantine equation has exactly five solutions in positive integers. The solutions are $(3, 1, 5), (5, 1, 7), (6, 1, 9), (7, 3, 71)$ and $(9, 1, 23)$.

In [11], Rabago showed that the only solution to the Diophantine equation $17^x+19^y = z^2$ in non-negative integers is $(1, 1, 6)$. Here we see that $6^2 = 36 = 17^1+19^1$. In fact, the integer 36 can be partition into two prime numbers different from 17 and 19. More precisely, 36 can also be expressed as $36 = 5+31 = 7+29 = 13+23$. Furthermore, Rabago showed that the Diophantine equation $71^x + 73^y = z^2$ has a unique solution in non-negative integers. The solution is $(1, 1, 12)$. Here we see that $12^2 = 144 = 71^1 + 73^1$. We can write 144 as a sum of two primes other than 71 and 73. In particular, $144 = 47 + 97 = 61 + 83$.

These give us a motivation to study the following Diophantine equations:

$$5^x + 31^y = z^2, \tag{1}$$

$$7^x + 29^y = z^2, \tag{2}$$

$$13^x + 23^y = z^2, \tag{3}$$

$$47^x + 97^y = z^2, \tag{4}$$

$$61^x + 83^y = z^2. \tag{5}$$

Hence, in the present paper, we consider the Diophantine equations (1), (2), and (3) and show that there is no possible solution to these equations in non-negative integers except $(x, y, z) = (1, 1, 6)$. On the other hand, we also show that the Diophantine equations (4) and (5) have exactly one solution in non-negative integers. The solution is $(1, 1, 12)$.

II. Main Results

We begin this section by the following theorem.

Theorem 2.1: [11, Theorem 2.2] The Diophantine equation $p^y + 1 = z^2$ has no positive integer solution for prime $p > 3$. □

We now present our new theorems.

Theorem 2.2: The only solution (x, y, z) to the Diophantine equations (1), (2), and (3) in non-negative integers is $(1, 1, 6)$.

Proof: Let x, y and z be non-negative integers and, consider the case when one of the three unknowns x, y, and z in (1), (2), and (3) is zero. If x or y is zero then we can use Theorem 2.1 to conclude that none of these equations is true. On the other hand, the case $z = 0$ is obviously not possible. We proceed on the case when $\min\{x, y, z\} > 0$.

1) Consider equation (1). Because $31 = 4(7)+3$ then (1) is possible only when y is odd. So, we let $y = 2n + 1$ for some non-negative integer n and suppose (1) has a solution in non-negative integers. We have $5^x+31^{2n+1} = z^2$. We divide x into two cases.
Case 1.1 x is even. Suppose $x = 2k$ for some non-negative integer k. Hence $(z + 5^k)(z - 5^k) = 31^y$. Let $\alpha+\beta = y$ and $\beta > \alpha$, then $2\cdot5^k = (z+5^k)-(z-5^k) = 31^\beta - 31^\alpha$. So, $\alpha = 0$ and $2 \cdot 5^k = 31^{2n+1} - 1$. This is impossible since $2\cdot5^k \equiv 1, 2 \pmod 3$ but $31^{2n+1} - 1 \equiv 0 \pmod 3$. Thus, (1) is impossible for positive even integer x.
Case 1.2 x is odd, say $x = 2k + 1$, then $5^{2k+1} + 31^{2n+1} = z^2$. We note that $5^{2k+1}+31^{2n+1} \equiv 0 \pmod 4$ so $z = 2m$ for some natural number m. Hence, $5^{2k+1} + 15 \cdot 31^{2n} = 4m^2 - 16 \cdot 31^{2n}$ or equivalently $5(5^{2k} + 3 \cdot 31^{2n}) = 4(m + 2 \cdot 31^n)(m - 2 \cdot 31^n)$. Since $m+2\cdot31^n = 1$ is never true for any m and n, it follows

Julius Fergy T. Rabago is a lecturer in the Department of Mathematics and Physics, Central Luzon State University, Science City of Muñoz, Nueva Ecija Philippines (e-mail: julius_fergy.rabago@up.edu.ph).

that $m+2\cdot 31^n = 5$, $5^{2k}+3\cdot 31^{2n} = 4$ and $m-2\cdot 31^n = 1$. This gives us the values $n = 0$, $k = 0$ and $m = 3$. Thus, we obtain the only solution $(x, y, z) = (1, 1, 6)$ to the Diophantine equation $5^x + 31^y = z^2$.

2) For equation (2), we only consider the case when x is odd since $7 = 4(1)+3$. We let $x = 2k+1$ and consider two possibilities for y.
Case 2.1 First suppose that y is even, say $y = 2n$. We have, from (2), $(z + 29^n)(z - 29^n) = 7^x$. Letting $\alpha + \beta = x$ and $\beta > \alpha$, we obtain $2 \cdot 29^n = (z + 29^n) - (z - 29^n) = 7^\beta - 7^\alpha$. Clearly, we see that $\alpha = 0$ and $2 \cdot 29^n = 7^x - 1$. This is a contradiction because $2 \cdot 29^n \equiv 1, 2 \pmod 3$ but $7^{2n+1} - 1 \equiv 0 \pmod 3$. So, (2) is never possible for positive even integer y.
Case 2.2 y is odd. Again assume that $z = 2m$ and, suppose that $y = 2n + 1$ then $7^{2k+1} + 29^{2n+1} = z^2$. Hence, $7^{2k+1} + 28 \cdot 29^{2n} = (2m^2 + 29^n)(2m^2 - 29^n)$. If $2m - 29^n = 7$ and $2m + 29^n = 7^{2k} + 4 \cdot 29^{2n}$ then we have $7^{2k} - 7 + 4 \cdot 29^{2n} = 2 \cdot 29^n$. So, $2 \cdot 29^n(2 \cdot 29^n - 1) = 7 - 7^{2k} > 0$. Clearly, we see that there is no possible values for n and k for the equality to hold. On the other hand, if $2m + 29^n = 7$ and $2m - 29^n = 7^{2k} + 4 \cdot 29^{2n}$ then $2 \cdot 29^n(2 \cdot 29^n + 1) = 7 - 7^{2k}$. It follows that k must not be greater than zero. This implies that $n = 0$ and $2(2 \cdot 29^n + 1) = 7 - 7^{2k}$ or $k = 0$. This gives us one and only solution to $7^x + 29^y = z^2$. That is, we have $(x, y, z) = (1, 1, 6)$.

3) Now, consider equation (3). Since $23 = 4(5) + 3$ then (3) is only possible whenever y is odd. Let $y = 2n + 1$. We divide x into two cases.
Case 3.1 x is even. Suppose $x = 2k$, then $2 \cdot 13^k = (z + 13^k) - (z - 13^k) = 23^\beta - 23^\alpha$ where $\alpha + \beta = y$ and $\beta < \alpha$. Hence, $\alpha = 0$ and $2 \cdot 13^k = 23^y - 1$. This is impossible because $2 \cdot 13^k \equiv 2 \pmod 3$ but $23^{2n+1} \equiv 1 \pmod 3$. Thus, $13^x + 23^y = z^2$ for x an even positive integer is not possible.
Case 3.2 x is odd, say $x = 2k + 1$. We have $13^{2k+1} + 23^{2n+1} = z^2$. Assume that $z = 2m$ then, $16 \cdot 13^{2k} - 3 \cdot 13^{2k} + 23^{2n+1} = 4m^2$. Hence, $23^{2n+1} - 3 \cdot 13^{2n} = 4(m+2\cdot 13^n)(m+2\cdot 13^n)$. It follows that $m-2\cdot 13^n = 1$ and $4(m + 2 \cdot 13^n) = 23^{2n+1} - 3 \cdot 13^{2n}$. This implies that $13^n(3 \cdot 13^n + 16) = 23^{2n+1} - 4$ in which follows that $n = 0$. So, $m = 3$ and $k = 0$. Thus, we obtain the only solution $(x, y, z) = (1, 1, 6)$ to the Diophantine equation $13^x + 23^y = z^2$.

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Theorem 2.3: The only solution (x, y, z) to the Diophantine equations (4) and (5) in non-negative integers is $(1, 1, 12)$.

Proof: Let x, y and z be non-negative integers. If $z = 0$ then (4) and (5) are obviously impossible. If x or y is zero then by Theorem 2.1, equation (4) and (5) is impossible. We consider the remaining cases in which $\min\{x, y, z\} > 0$.

4) Suppose equation (4) has a solution in positive integers. Since $47 = 11(4) + 3$ then we only consider the case when x is odd. We have two possibilities for y.
Case 4.1 y is even. If $y = 2n$ for some natural number n then $(z + 97^n)(z - 97^n) = 47^x$. This implies that $2 \cdot 97^n = 47^x - 1$. This is impossible because $2 \cdot 97^n \equiv 2 \pmod 3$ and $47^x - 1 \equiv 1 \pmod 3$. Thus, (4) is impossible for positive even integer y.
Case 4.2 y is odd. If $y = 2n + 1$ we have $47^{2k+1} + 97^{2n+1} = z^2$. Note that $47^x + 97^y \equiv 0 \pmod 2$ so $z = 2m$ for some natural number m. It follows that $47(47^{2k} + 97^{2n}) = 47^{2k+1} + 47 \cdot 97^{2n} = 2(2m^2 - 25 \cdot 97^{2n})$. Hence, $47^{2k}+97^{2n} = 2$ and this is only true when k and n is zero. Also, we see that $2m^2 - 25 \cdot 97^{2n} = 47$ or $m = 6$. Thus, we obtain a unique solution (x, y, z) to the Diophantine equation $47^x + 97^y = z^2$. That is, we have $(x, y, z) = (1, 1, 12)$.

5) Suppose equation (5) has a solution in positive integers. We only consider the case when y is odd. That is, we let $y = 2n + 1$. We have the following two cases for x.
Case 5.1 x is even. If $x = 2n$ then $(z+61^n)(z-61^n) = 83^y$. So, $2 \cdot 61^n = 83^y - 1$. This is impossible because $2 \cdot 61^n \equiv 2 \pmod 3$ and $83^y - 1 \equiv 1 \pmod 3$. Thus, (5) is impossible for positive even integer x.
Case 5.2 x is odd. If $x = 2n+1$ then we have $61^{2k+1} + 83^{2n+1} = z^2$. We let $z = 2m$. It follows that $61^{2k+1} - 61 \cdot 83^{2n} = 61(61^{2k} + 83^{2n}) = 2(2m^2 - 11 \cdot 83^{2n})$. Hence, $61^{2k}+83^{2n} = 2$ and $2m^2-11\cdot 83^{2n} = 61$. These equations implies that $k = 0$ and $n = 0$, and $m = 6$, respectively. Thus, we have a unique solution $(x, y, z) = (1, 1, 12)$ to the Diophantine equation $61^x + 83^y = z^2$ in non-negative integers.

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III. Conclusion

In the paper, we have shown that the only solution (x, y, z) in non-negative integers to the Diophantine equations $5^x + 31^y = z^2$, $7^x + 39^y = z^2$, and $13^x + 23^y = z^2$ is $(1, 1, 6)$. Also, we have shown that the two Diophantine equations $47^x + 97^y = z^2$ and $61^x + 83^y = z^2$ have a unique solution (x, y, z) in non-negative integers. The solution is $(1, 1, 12)$.

References

[1] D. Acu, On a Diophantine equations of type $a^x + b^y = c^z$, *Gen. Math.*, **13**, 2005, no. 1, 67-72.
[2] D. Acu, On a Diophantine equation $2^x + 5^y = z^2$, *Gen. Math.*, **15**, 2007, no. 4, 145-148.
[3] A. Suvarnamani, A. Singta, S. Chotchaisthit, On two Diophantine Equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, *Sci. Technol. RMUTT J.*, **1**, 2011, 25-28.
[4] B. Sroysang, More on the Diophantine Equation $8^x + 19^y = z^2$, *International Journal of Pure and Applied Mathematics,* **81**, 2012, no. 4, 601-604.
[5] B. Sroysang, On the Diophantine Equation $3^x + 5^y = z^2$, *International Journal of Pure and Applied Mathematics,* **81**, 2012, no. 4, 605-608.
[6] B. Sroysang, On the Diophantine Equation $31^x+32^y = z^2$, *International Journal of Pure and Applied Mathematics,* **81**, 2012, no. 4, 609-612.
[7] J. F. T. Rabago, On an Open Problem by B. Sroysang, *Konuralp Journal of Mathematics*, (2013), to appear.
[8] J. F. T. Rabago, A Note on an Open Problem by B. Sroysang, *Sci. Technol. RMUTT J.*, (2013), submitted.
[9] J. F. T. Rabago, On Two Diophantine Equations, *Chiang Mai Journal of Science*, (2013), submitted.
[10] J. F. T. Rabago, On the Diophantine Equation $2^x + 17^y = z^2$, *General Mathematics*, (2013), submitted.
[11] J. F. T. Rabago, A Note on Two Diophantine Equations $17^x+19^y = z^2$ and $71^x + 73^y = z^2$, *Mathematics Journal of Interdisciplinary Sciences*, (2013), submitted.