k-cordiality of Wheel, Path Related and Cycle Related Graphs

M. V. Modha and K. K. Kanani*

***Abstract*—We prove that the wheels W_n are k-cordial for all odd k and for all $n = mk + j$, $m \geq 0$, $1 \leq j \leq k-1$ except for $j = \frac{k-1}{2}$. We discuss here k-cordial labeling of total graph of the path for any k and square graph of cycle for odd k for $n \geq k$. In addition to this we prove that the path union of n copies of cycle C_k is k-cordial for odd k. It is also hereby evidenced that pan graph C_n^{+1} is k-cordial for odd k and $n \geq k$.**

***Index Terms*—Abelian Group, k-Cordial Labeling, Square Graph,Total Graph, Path union of Cycles.**

***MSC 2010 Codes* – 05C78**

I. INTRODUCTION

IN this work, by a graph we mean finite, connected, undirected, simple graph $G = (V(G), E(G))$ of order $|V(G)|$ and size $|E(G)|$.

Definition 1.1 A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). The most recent findings on various graph labeling techniques can be found in Gallian[1].

Definition 1.2 Let $< A, * >$ be any Abelian group. A graph $G = (V(G), E(G))$ is said to be *A-cordial* if there is a mapping $f : V(G) \rightarrow A$ which satisfies the following two conditions when the edge $e = uv$ is labeled as $f(u) * f(v)$

(i) $|v_f(a) - v_f(b)| \leq 1$; for all $a, b \in A$,

(ii) $|e_f(a) - e_f(b)| \leq 1$; for all $a, b \in A$.

Where

$v_f(a)$=the number of vertices with label a;

$v_f(b)$=the number of vertices with label b;

$e_f(a)$=the number of edges with label a;

$e_f(b)$=the number of edges with label b.

We note that if $A =< Z_k, +_k >$, that is additive group of modulo k then the labeling is known as k-cordial labeling. The concept of A-cordial labeling was introduced by Hovey[2] and proved the following results.

- All the connected graphs are 3-cordial.
- All the trees are $3, 4, 5$-cordial.
- Cycles are k-cordial for all odd k.

In [3,4] Kanani and Modha proved various results related to 5-cordial and 7-cordial labeling.

M. V. Modha is a Research Scholar in the School of Science, R. K. University, Tramba, Rajkot and Associate Professor in Mathematics M. D. Science College Porbandar-360575 Gujarat, India. (E-mail:mitavmodh@gmail.com)

K. K. Kanani is Assistant Professor in Mathematics, Department of Humanities and Science, Government Engineering College Rajkot-360005 Gujarat, India. (E-mail: kananikkk@yahoo.co.in).

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In [5,6] Modha and Kanani proved fans f_n are k-cordial for all k, The Bistar $B(m, n)$ is k-cordial graph for all k. The restricted square graph $B^2_{n,n}$ of Bistar $B_{n,n}$ is k-cordial for all odd k, The Comb Graph $P_n \bigodot K_1$ is k-cordial for all k. Here we consider the following definitions of standard graphs.

- The *wheel* W_n is defined as the join $C_n + K_1$.
- The *total graph* $T(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.
- Let G be a simple connected graph. The *square of graph* G denoted by G^2 is defined to be the graph with the same vertex set as G and in which two vertices u and v are joined by an edge $\Leftrightarrow$ in G we have $1 \leq d(u, v) \leq 2$.
- Let G be a graph and $G_1, G_2, \ldots, G_n$, be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} (for $i = 1, 2, \ldots, n-1$) is called path union of G.
- The *n-pan graph* is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge. The n-Pan graph is also defined as n-cycle with one pendant edge.

For any undefined term we rely upon Gross and Yellen[7]

II. MAIN RESULTS

Theorem 2.1 The Wheels W_n are k-cordial for all odd k and for all $n = mk+j$, $m \geq 0$, $1 \leq j \leq k-1$ except for $j = \frac{k-1}{2}$.

Proof: Let W_n be the wheel.
Let $n = mk + j$ where $m \geq 0$ and $1 \leq j \leq k-1$, $j \neq \frac{k-1}{2}$.

Let v_0 be the apex vertex and $v_1, v_2, .., v_{mk}, v_1^{'}, v_2^{'} .., v_j^{'}$ be the n rim vertices of the wheel W_n. We note that $|V(G)| = n+1$ and $|E(G)| = 2n$.

To define k-cordial labeling we consider the following cases.

<u>Case 1</u>: $\frac{k+1}{2}$ is odd.

<u>Subcase I</u>: $m \geq 0$, $j = 1$.

$$\begin{aligned} &f(v_0) = 0; \\ &f(v_i) = p_i; \qquad (2i-1) \equiv p_i (mod\ k),\ 1 \leq i \leq mk. \\ &f(v_1^{'}) = 1. \end{aligned}$$

Subcase II: $m \geq 0,\ j = 2$.

$$\begin{aligned}
&f(v_0) = 0;\\
&f(v_i) = p_i; \quad (2i-1) \equiv p_i (mod\ k),\ 1 \leq i \leq mk.\\
&f(v_1^{'}) = k-1;\\
&f(v_2^{'}) = 1.
\end{aligned}$$

Subcase III: $m \geq 0,\ 3 \leq j \leq \frac{k-3}{2}$.

$$\begin{aligned}
&f(v_0) = 0;\\
&f(v_i) = k - p_i; \quad 2i-1 \equiv p_i (mod\ k),\\
&\qquad 1 \leq i \leq mk, \text{ if } m \geq 1.\\
&f(v_1^{'}) = k-1;\\
&f(v_i^{'}) = \tfrac{k+1}{2} + \lceil \tfrac{k-5}{8} \rceil + \tfrac{i}{2}; \quad i \text{ is even},\\
&f(v_i^{'}) = \tfrac{k+3}{4} + \lfloor \tfrac{k-5}{8} \rfloor + \tfrac{i-1}{2}; \quad i \text{ is odd},\ 2 \leq i \leq (j-1).\\
&f(v_j^{'}) = 1.
\end{aligned}$$

Subcase IV: $m \geq 0,\ j = \frac{k+1}{2}$.

$$\begin{aligned}
&f(v_0) = 0;\\
&f(v_i) = p_i; \quad \tfrac{k+i}{2} \equiv p_i (mod\ k),\ i \text{ is odd},\\
&\qquad = p_i; \quad \tfrac{i}{2} \equiv p_i (mod\ k),\ i \text{ is even},\\
&\qquad 1 \leq i \leq mk.\\
&f(v_i^{'}) = \tfrac{k+i}{2}; \quad i \text{ is odd},\\
&\qquad = \tfrac{k-1}{4} + \tfrac{i}{2}; \ i \text{ is even},\ 1 \leq i \leq j.
\end{aligned}$$

Subcase V: $m \geq 0,\ \frac{k+3}{2} \leq j \leq k-1$.

$$\begin{aligned}
&f(v_0) = 0;\\
&f(v_i) = p_i; \quad \tfrac{k+i}{2} \equiv p_i (mod\ k),\ i \text{ is odd},\\
&\qquad = p_i; \quad \tfrac{i}{2} \equiv p_i (mod\ k),\ i \text{ is even},\\
&\qquad 1 \leq i \leq mk.\\
&f(v_i^{'}) = \tfrac{k+i}{2}; \quad i \text{ is odd},\\
&\qquad = \tfrac{k-l}{4} + \tfrac{i}{2}; \ i \text{ is even},\ 1 \leq i \leq j,\\
&\qquad \text{where } \tfrac{k+l-2}{2} \leq j \leq \tfrac{k+l}{2} \text{ and } l = 5, 9, ..., k-4.
\end{aligned}$$

Case 2: $\frac{k+1}{2}$ is even.

Subcase I: $m \geq 0,\ j = 1$.

$$\begin{aligned}
&f(v_0) = 0;\\
&f(v_i) = p_i; \quad (2i-1) \equiv p_i (mod\ k),\ 1 \leq i \leq mk.\\
&f(v_1^{'}) = 1.
\end{aligned}$$

Subcase II: $m \geq 0,\ j = 2$.

$$\begin{aligned}
&f(v_0) = 0;\\
&f(v_i) = p_i; \quad (2i-1) \equiv p_i (mod\ k),\ 1 \leq i \leq mk.\\
&f(v_1^{'}) = k-1;\\
&f(v_2^{'}) = 1.
\end{aligned}$$

Subcase III: $m \geq 0,\ 3 \leq j \leq \frac{k-3}{2}$.

$$\begin{aligned}
&f(v_0) = 0;\\
&f(v_i) = k - p_i; \quad 2i-1 \equiv p_i (mod\ k),\\
&\qquad 1 \leq i \leq mk, \text{ if } m \geq 1.\\
&f(v_1^{'}) = k-1;\\
&f(v_i^{'}) = \tfrac{k+1}{2} + \lceil \tfrac{k-3}{8} \rceil + \tfrac{i}{2}; \quad i \text{ is even},\\
&f(v_i^{'}) = \tfrac{k+1}{4} + \lfloor \tfrac{k-3}{8} \rfloor + \tfrac{i-1}{2}; \quad i \text{ is odd},\ 2 \leq i \leq (j-1).\\
&f(v_j^{'}) = 1.
\end{aligned}$$

Subcase IV: $m \geq 0,\ \frac{k+1}{2} \leq j \leq k-1$.

$$\begin{aligned}
&f(v_0) = 0;\\
&f(v_i) = p_i; \quad \tfrac{k+i}{2} \equiv p_i (mod\ k),\ i \text{ is odd},\\
&\qquad = p_i; \quad \tfrac{i}{2} \equiv p_i (mod\ k),\ i \text{ is even},\ 1 \leq i \leq mk.\\
&f(v_i^{'}) = \tfrac{k+i}{2}; \quad i \text{ is odd},\\
&\qquad = \tfrac{k-l}{4} + \tfrac{i}{2}; \ i \text{ is even},\ 1 \leq i \leq j,\\
&\qquad \text{where } \tfrac{k+l-2}{2} \leq j \leq \tfrac{k+l}{2} \text{ and } l = 3, 7, 11, ..., k-4
\end{aligned}$$

The labeling pattern defined above covers all possible arrangement of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for k-cordial labeling. Hence, the Wheels W_n are k-cordial for all odd k and for all $n = mk + j$, $m \geq 0$, $1 \leq j \leq k-1$ except for $j = \frac{k-1}{2}$.

Illustration 2.2(a) The wheel W_{11} and its 25-cordial labeling is shown in *figure 1*.

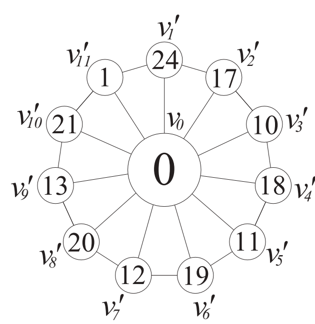

Figure 1:25-cordial labeling of wheel W_{11}

Illustration 2.2(b) The wheel W_{27} and its 17-cordial labeling is shown in *figure 2*.

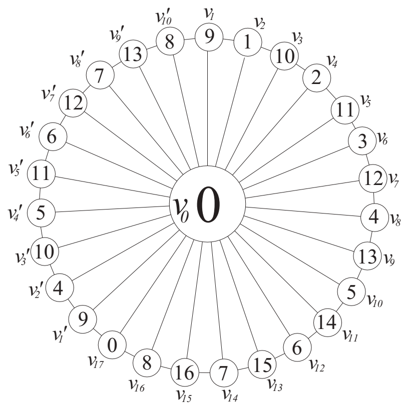

Figure 2:17-cordial labeling of wheel W_{27}.

Theorem 2.3 The Total graph $T(P_n)$ of path P_n is k-cordial for all k.

Proof: Let G=$T(P_n)$ be the total graph of the path P_n. Let $v_1, v_2, v_3, ..., v_n$ be the vertices of path P_n and $e_1, e_2, ..., e_{n-1}$ be the $n-1$ edges. Let $v_1^{'}, v_2^{'}, ..., v_{n-1}^{'}$ be the newly added vertices corresponding to edges $e_1, e_2, ..., e_{n-1}$ to form G. We note that $|V(G)|$=$2n-1$ and $|E(G)|$=$4n-5$.

To define k- cordial labeling $f : V(G) \rightarrow Z_k$ we consider the following cases.

Case 1: k is odd.
$f(v_i) = p_i$; where $(4i-3) \equiv p_i (mod\ k)$, $1 \leq i \leq n$.
$f(v_i^{'}) = p_i^{'}$; where $(4i-1) \equiv p_i^{'} (mod\ k)$, $1 \leq i \leq n-1$.

Case 2: k is even.
$f(v_i) = p_i - 1$; where $(2i-1) \equiv p_i (mod\ k)$, $1 \leq i \leq n$.
$f(v_i^{'}) = p_i^{'}$; where $(2i-1) \equiv p_i^{'} (mod\ k)$, $1 \leq i \leq n-1$.

The labeling pattern defined above covers all possible arrangement of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for k-cordial labeling. Hence the Total graph $T(P_n)$ of path P_n is k-cordial for all k.

Illustration 2.4(a) The total graph $T(P_{11})$ and its 13-cordial labeling is shown in *Figure 3*.

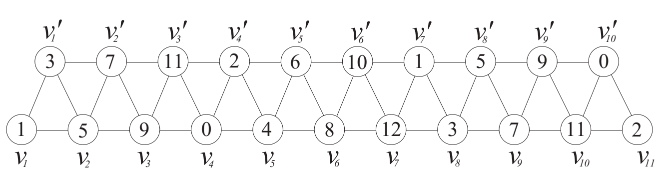

Figure 3:13-cordial labeling of total graph $T(P_{11})$.

Illustration 2.4(b) The total graph $T(P_8)$ and its 10-cordial labeling is shown in *Figure 4*.

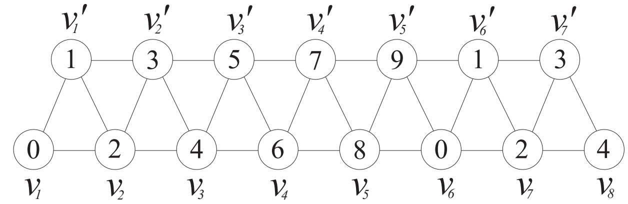

Figure 4: 10-cordial labeling of total graph $T(P_8)$.

Theorem 2.5 The Square graph C_n^2 of cycle C_n is k-cordial for all odd k and $n \geq k$.

Proof: Let G=C_n^2 be the square of the cycle C_n with vertices $v_1, v_2, v_3, ..., v_n$. We note that $|V(G)|$=n and $|E(G)|$=$2n$.

Let n=$mk+j$ where $m \geq 1$, $0 \leq j \leq k-1$. First we divide the mk vertices of the cycle into k blocks of m vertices. Now add one vertex to each odd block of the above mentioned k blocks for $j \leq \frac{k-1}{2}$ and then add one vertex to each even block of the above mentioned k blocks respectively for remaining vertices $j > \frac{k+1}{2}$ of the cycle C_n. Label every vertex of ith block with $i-1$.

The labeling pattern defined above covers all possible arrangement of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for k-cordial labeling. Hence the square graph C_n^2 of cycle C_n is k-cordial for all odd k and $n \geq k$.

Illustration 2.6 The square graph C_{13}^2 and its 9-cordial labeling is shown in *Figure 5*.

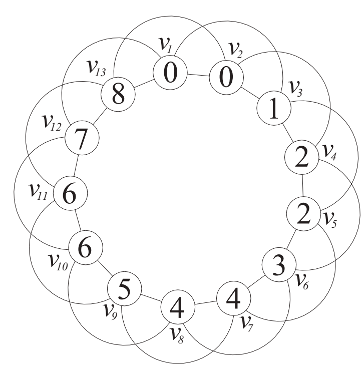

Figure 5: 9-cordial labeling of square graph C_{13}^2.

Theorem 2.7 The path union of n copies of cycle C_k is k-cordial graph for odd k.

Proof: Let $G_1, G_2, ..., G_n$ be n copies of the cycle C_k and G be the path union of cycle C_k. Let us denote the successive vertices of the i^{th} copy G_i by $v_{i1}, v_{i2}, ..., v_{ik}$. Let e_i=$v_{i1}v_{(i+1)1}$ be the edge joining G_i and G_{i+1} for i=$1, 2, ..., n-1$ We note that $|V(G)|$=nk and $|E(G)|$=$nk+(k-1)$.

We define k-cordial labeling $f : V(G) \rightarrow Z_k$ as follows.

$f(v_{ij}) = p_i + j - 2$; $i \equiv p_i (mod\ k)$, $1 \leq i \leq n$,$1 \leq j \leq k$.

The labeling pattern defined above covers all possible arrangement of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of k-cordial labeling. Hence, the path union of n copies of cycle C_k is k-cordial graph for odd k.

Illustration 2.8 The graph G obtained by a path union of 4 copies of cycle C_7 and its 7-cordial labeling is shown in *Figure 6*.

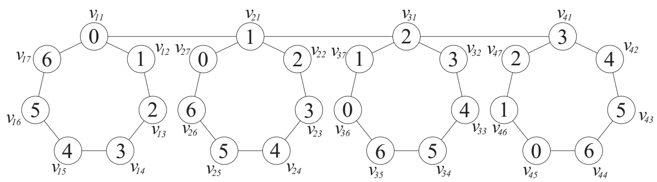

Figure 6: 7-cordial labeling of 4 copies of cycle C_7.

Theorem 2.9 The Pan graph C_n^{+1} is k-cordial for odd k and $n \geq k$.

Proof: Let C_n^{+1} be the Pan graph.We note that $|V(G)| = n+1$ and $|E(G)| = n+1$.

Let C_n be cycle, Consider n=$mk + j$, where $m \geq 1$, $0 \leq j \leq k-1$ and $v^{'}$ be the pendent vertex. Divide the n vertices of the cycle C_n into j blocks of $m+1$ vertices and $k-j$ blocks of m vertices. Label every vertex of i^{th} block with $i-1$. Now attach the pendent vertex $v^{'}$ to the $((m+1)j+1)^{th}$ vertex of the cycle and label the vertex $v^{'}$ with j.

The labeling pattern defined above covers all possible arrangement of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for k-cordial labeling. Hence the Pan graph C_n^{+1} is k-cordial for all odd k and $n \geq k$.

Illustration 2.10 The pan graph C_{13}^{+1} and its 5-cordial labeling is shown in *Figure 7*.

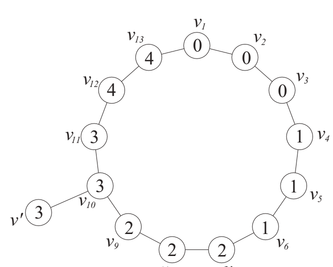

Figure 7: 5-cordial labeling of pan graph C_{13}^{+1}.

III. Conclusion

Here we have contributed general results to the theory of k-cordial labeling. Research is a never ending process and other graph families related to k-cordial labeling can be another site of exploration that will open fresh avenues and raise new questions, thereby stimulating the researcher and expanding our horizons.

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References

[1] J. A. Gallian, "A dynamic survey of graph labeling", *The Electronics Journal of Combinatorics*, vol.18, 2015.
[2] M. Hovey, "A-cordial graphs", *Discrete Math.*, vol.93, pp. 183-194, 1991.
[3] K. K. Kanani and M. V. Modha, "7-cordial labeling of standard graphs", *Internat. J. Appl. Math. Res.*, vol. 3, no. 4, pp. 547–560, 2014.
[4] K. K. Kanani and M. V. Modha, "Some new families of 5-cordial graphs", *Int. J. Math. Soft Comp.*, vol. 4, no. 1, pp. 129–141, 2015.
[5] M. V. Modha and K. K. Kanani, "k-cordial labeling of fan and double fan", *Internat. J. Appl. Math. Res.*, vol. 4, no. 2, pp. 362–369, 2015.
[6] M. V. Modha and K. K. Kanani, "On k-cordial labeling of some graphs", *British J. of Math. and Comp. Sci.*, vol. 13, no. 3, pp. 1–7, 2016.
[7] J. Gross and J. Yellen, *Handbook of Graph Theory*, CRC Press, 2004.
[8] K. K. Kanani and N. B. Rathod, "Some new 4-cordial graphs", *J. Math. Comput. Sci.*, vol. 4, no. 5, pp. 834–848, 2014.
[9] R. Tao, "On k-cordiality of cycles, crowns and wheels", *Systems Sci. Math. Sci.*, vol. 11, pp. 227–229, 1998.
[10] M. Z. Youssef, "On k-cordial labeling", *Australas. J. Combin.*, vol. 43, pp. 31–37, 2009.