On Isolated and Non-Isolated Subsemigroups of Full Transformation Semigroup

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***Abstract*—Some properties of isolated and non-isolated subsemigroups are studied in this work.**

***Index Terms*—Ideals, Isolated and Non-Isolated subsemigroups.**

***MSC 2010 Codes* – 20M15, 20M20**

I. Introduction

A SUBSEMIGROUP T of a semigroup S as a completely isolated subsemigroup if $x.y \in T$ implies that $x \in T$ or $y \in T$ for all $x, y \in S$. Also a subsemigroup T of a semigroup S is called isolated provided that $x^n \in T$ implies that $x \in T$ for all $x \in S$ and $n \in N$ [1].

A completely non - isolated subsemigroup T' is defined for all $x, y \in T'$, $x.y \in S$. In the same vein, a non - isolated subsemigroup T' is defined if for $a \in T'$, $a^n \in S$ and $n \in N$ [2].

A subsemigroup I of a semigroup S forms a right ideal if whenever $i \in I$ and $s \in S$, $is \in I$ and analogously, I is a left ideal if $si \in I$. If I is both a left and a right ideal, then it is an ideal [1].

Let $X = \{1, 2, 3.\dots\dots n\}$ be a natural ordering of numbers and $\alpha : Dom(\alpha) \subseteq X \mapsto Im(\alpha) \subset X$. The right[left] waist of α is $W^+(\alpha)$=$max(Im\alpha)[W^-(\alpha) = min(Im\alpha)]$ as defined by Umar[3].

Hence identity difference transformation is defined given that

$\mid W^+(\alpha) - W^-(\alpha) \mid \leq 1$, $\mid . \mid$ stands for modulus.

Generally speaking, identity difference full transformation semigroup (IDT_n) is non - invertible in nature for $n \geq 3$. Lemma 2.1 is necessary to show that T is non - commutative while Lemma 2.2 and Theorem 2.3 showed that $T$is isolated and completely isolated respectively. T is a left-sided ideal by Proposition 2.4. The cardinality of $T^{'}$ is derived in Theorem 2.6 while by lemma 2.5, $T^{'}$ is non-commutative and non-isolated. For the purpose of this study, $T = IDT_n, S = T_n$ and $T^{'} = \widetilde{IDT_n}$.

T and $T^{'}$ are both disjoint subsets of T_n where T is isolated and identity free for $n \leq 2$ while $T^{'}$ contains the identity element, symmetric group, S_n for $n \geq 3$ and is non - isolated. The symmetric group satisfies the property that $a^k = i$ where $a \in S_n$, i is the identity element $k \in N$. It is also known that the largest value a can attain is n, that is $\max(k) = n$.

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II. Identity Difference Full Transformation Semigroup and It's Complement

Lemma 2.1: T is a non - commutative semigroup.

Proof: Associativity of composition of mappings necessarily leads to the idea of semigroup as follow:

Let $\alpha : W \mapsto X, \beta : X \mapsto Y$ and $\eta : Y \mapsto Z$ for all $w \in W, x \in X, y \in Y$ and $z \in Z$. Taking that $\alpha(w) = x, \beta(x) = y$ and $\eta(y) = z$.

$\Rightarrow \alpha\beta = y$ and $\beta\eta = z \dots (i)$.

Since,

$\beta\alpha = x$ and $\eta\beta = y \dots (ii)$

then from (i) and $(ii), T$ is non - commutative.

But $\alpha(\beta\eta) = (\beta\alpha)\eta$.

$\Rightarrow$ Both $\alpha(\beta\eta)$ and $(\beta\alpha)\eta$ are defined equally.

Also, the closure property is defined by choosing arbitrary elements

$a, b, c, d \in T$ for composition of mappings as:

$$\begin{pmatrix} a & b & c & d \\ a & b & a & a \end{pmatrix} \begin{pmatrix} a & b & c & d \\ b & a & b & a \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ b & a & b & b \end{pmatrix}.$$

Hence T is a semigroup noting that $Im(\alpha) = \{j, j+1\}, j = 0, 1, \dots n-1$ for each $\alpha \in T\bullet$

Lemma 2.2: T is an isolated subsemigroup.

Proof: From the definition of an isolated subsemigroup, let $x^n \in T$ implies that $x \in T$ for all $x \in S$ and $n \in N$.

For instance, take $x = \begin{pmatrix} a & b & c & d & e \\ b & a & a & b & b \end{pmatrix} \in T$, then

$$x^2 = \begin{pmatrix} a & b & c & d & e \\ b & a & a & b & b \end{pmatrix} \begin{pmatrix} a & b & c & d & e \\ b & a & a & b & b \end{pmatrix} = \begin{pmatrix} a & b & c & d & e \\ a & b & b & a & a \end{pmatrix} \in T;$$

$$x^3 = \begin{pmatrix} a & b & c & d & e \\ a & b & b & a & a \end{pmatrix} \begin{pmatrix} a & b & c & d & e \\ b & a & a & b & b \end{pmatrix} = \begin{pmatrix} a & b & c & d & e \\ b & a & a & b & b \end{pmatrix} \in T.$$

If $x = \begin{pmatrix} a & b & c & d & e \\ c & a & b & a & d \end{pmatrix} \in S$,

then $$x^2 = \begin{pmatrix} a & b & c & d & e \\ c & a & b & a & d \end{pmatrix} \begin{pmatrix} a & b & c & d & e \\ c & a & b & a & d \end{pmatrix} = \begin{pmatrix} a & b & c & d & e \\ b & c & a & b & a \end{pmatrix} \in S;$$

$$x^3 = \begin{pmatrix} a & b & c & d & e \\ b & c & a & b & a \end{pmatrix} \begin{pmatrix} a & b & c & d & e \\ c & a & b & a & d \end{pmatrix} = \begin{pmatrix} a & b & c & d & e \\ a & b & c & b & c \end{pmatrix} \in S.$$

Thus continuing by iteration, $x^n \in T$ implies $x \in T \forall x \in S\bullet$

The fact that T is an isolated subsemigroup of S makes it a completely isolated subsemigroup [2]. In the light of this, the following theorem is essential.

Theorem 2.3: T is a completely isolated subsemigroup of S.

Proof: T is shown to be non - commutative in lemma 2.1. From the definition of a completely isolated subsemigroup, it is seen that

$x.y \in T$ does not automatically imply that $y.x \in T$. That is, if $x.y \in T$ is true, $y.x \in T$ may or may not be true, $\forall x, y \in S$.

This leads to three possibilities:

- $x \in T, y \in T \Rightarrow x.y \in T$ and $y.x \in T \ldots (i)$;
- $x \in T, y \in S \Rightarrow x.y \in S$ and $y.x \in T \ldots (ii)$;
- $y \in T, x \in S \Rightarrow x.y \in T$ and $y.x \in S \ldots (iii)$.

From (i) and $(ii), x.y \in T$ implies that $x \in T$ or $y \in T$. Also, $y.x \in T$ is true if only $x \in T$ from (i) and (ii). It is observed that in all, $x.y \neq y.x$•

Proposition 2.4: Let $T \subset S$, then $TS \subset S$ and $ST \subset T$.

Proof: From the definition of ideals, let $x_i \in T, i = 1, 2, 3$ and $x_j \in S, j = 1, 2, \ldots 10$.

$\Rightarrow x_i \subset x_j$. By composition of mappings, $x_i x_j \subset x_j$ and $x_j x_i \subset x_i$ which means that $TS \subset S$ and $ST \subset T$. Conclusively, T is a left sided ideal•.

a) : It is vital to state that since $T \subset S$ is an isolated subsemigroup, then $\sqrt[n]{x} \in T$ for all $x \in T$. This is further explained, in that, if T is an isolated subsemigroup, then $x^n \in T$ on the condition that $x \in T$ for all $x \in S$. If $y \in T$ is such that $y^n = x, \Rightarrow y = \sqrt[n]{x} \in T, n \in N$.

b) : Let $T^{'} = \widetilde{IDT_n} = S \backslash T$ denote the compliment of the completely isolated subsemigroup T. The following results emerged from the study of $T^{'}$.

Lemma 2.5: $T^{'}$ is a non - commutative and non - isolated semigroup.

Proof: $T^{'}$ satisfies the condition $W^{+}(\alpha) - W^{-}(\alpha) > 1$. Following the proof in lemma 2.1, $T^{'}$ is a non - commutative semigroup.

From the definition of non - isolation, consider the following examples for arbitrary elements in $T^{'}$. - $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} \notin T^{'}$. - $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \notin T^{'}$.

Hence the result •

Lemma 2.5 established the fact that the compliment of a completely isolated subsemigroup may not be completely isolated.

Theorem 2.6: $T^{'}$ contains $n^n + n - 2^n(n-1) - 2$ elements.

Proof: T_n consists of bijection and singular mappings of which $T^{'}$ is a subset. In T_n, each map has the whole set $n \in N$ of natural numbers as the domain while the image, $Im(\alpha) \subseteq N$ and each is dintinctly defined.

Following the rule of full transformation where each element in the domain maps $i = 1, \ \forall n$, which makes the length of image of the first element to be 1. Starting with the image of the right-most nth element in the domain i changes in this manner, $1 \mapsto 2 \mapsto 3 \mapsto 4 \ldots \mapsto n$ until the last element is $\begin{pmatrix} 1 & 2 & 3 & . & . & . & n \\ n & n & n & . & . & . & n \end{pmatrix}$.

The cardinality of T_n was obtained to be n^n and the cardinality of T is $n + (n-1)(2^n - 2)$ [1]. Since $T^{'} = S \backslash T$, $\Rightarrow n^n - [(n + (n-1)(2^n - 2))] = n^n + n - 2^n(n-1) - 2$•

Lemma 2.7: T is an identity - free semigroup

Proof: Unlike $T^{'}$, T is identity - free for $n \geq 2$ since $Im(\alpha) = \{j, j+1\}$, for $j = 0, 1, 2, \ldots n-1$•

III. Conclusion

It was shown in this work, that the compliment of a completely isolated subsemigroup may not be completely isolated. Also, the cardinality of

$T^{'} = n^n + n - 2^n(n-1) - 2$.

References

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