Finite Dimensional Fuzzy Cone Normed Linear Spaces

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Abstract—**In this paper, an idea of fuzzy cone normed linear space is introduced. Some basic definitions viz. convergence of sequence, Cauchy sequence, closedness, completeness etc are given. One lemma is established and with the help of this lemma some results on finite dimensional fuzzy cone normed linear spaces are established.**

Index Terms—**Fuzzy real number, Fuzzy cone metric, Fuzzy Cone normed linear space.**

MSC 2010 Codes **– 46S40, 03E72**

I. Introduction

The idea of fuzzy set theory was introduced by L.A.Zadeh [15] in 1965 and fuzzy logic has become an important area of research in various branches of mathematics such as metric and topological spaces, automata theory, optimization, control theory etc. Fuzzy set theory also found applications for modeling, uncertainty and vagueness in various fields of science and engineering.

Fuzzy functional analysis is a recent development and it is based on fuzzy metric space theory and fuzzy normed linear space theory. Many authors have made important contributions [1], [5], [7], [10] in fuzzy functional analysis.

On the other hand, a number of generalizations of metric spaces as well as normed linear spaces have been done. In metric space theory, one is D-metric space initiated by Dhage [4] in 1992 and its corresponding generalize form in fuzzy setting developed by Sedghi et al. [13], [14], Bag [2]. Recently the idea of cone metric space is relatively new which is introduced by H.Long-Guang et al. [8] and it is a generalization of classical metric space. Its corresponding generalize form in fuzzy setting called fuzzy cone metric space is introduced by Bag [3].

The idea of cone normed linear space which is a generalization of classical normed linear space is established by T.K.Samanta et al.[12]. In such space, authors have considered a real Banach space as the range set of the cone norm.

In this paper, idea of fuzzy cone normed linear space is introduced and some basic definitions are given.

Here the range of fuzzy cone norm is considered as $E^*(I)$ where E is a given real Banach space and $E^*(I)$ denotes the set of all non-negative fuzzy real numbers defined on E.

It is shown that fuzzy cone normed linear space is a generalization of Felbin's [5] type fuzzy normed linear space (when L = min and U = max). Finally some results in finite dimensional fuzzy cone normed linear space are established.

The organization of the paper is as follows: Section II comprises some preliminary results which are used in this paper. Definition of fuzzy cone normed linear space and some basic properties are discussed in Section III. In Section IV, some results in finite dimensional fuzzy cone normed linear space are established.

II. Some preliminary results.

A fuzzy number is a mapping $x: R \to [0, 1]$ over the set R of all reals.

A fuzzy number x is convex if $x(t) \geq \min(x(s), x(r))$ where $s \leq t \leq r$.

The α-level set of a fuzzy real number η is denoted by $[\eta]_\alpha$ and defined by $[\eta]_\alpha = \{t \in R: \eta(t) \geq \alpha\}$.

If there exists a $t_0 \in R$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy number (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha, b_\alpha]$ means the interval $(-\infty, b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.

A fuzzy number x is called non-negative if

$$x(t) = 0, \forall t < 0$$

Kaleva (Felbin) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by $E(R(I))$ and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$.

A partial ordering " $\preceq$ " in E is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0, 1]$ where $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0, 1]$.

According to Mizumoto and Tanaka [10] , the arithmetic operations $\oplus$, $\ominus$, $\odot$ on $E \times E$ are defined by

$$(x \oplus y)(t) = \sup_{s \in R} \min\{x(s), y(t-s)\}, \quad t \in R$$
$$(x \ominus y)(t) = \sup_{s \in R} \min\{x(s), y(s-t)\}, \quad t \in R$$
$$(x \odot y)(t) = \sup_{s \in R, s \neq 0} min \ \{x(s) , y(\tfrac{t}{s})\}, \quad t \in R$$

Propositon 2.1[10]. *Let* $\eta , \delta \in E(R(I))$ *and* $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$, $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha]$, $\alpha \in (0, 1]$. *Then*

$$[\eta \oplus \delta]_\alpha = [a^1_\alpha + a^2_\alpha , b^1_\alpha + b^2_\alpha]$$
$$[\eta \ominus \delta]_\alpha = [a^1_\alpha - b^2_\alpha , b^1_\alpha - a^2_\alpha]$$
$$[\eta \odot \delta]_\alpha = [a^1_\alpha a^2_\alpha , b^1_\alpha b^2_\alpha]$$

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* The present work is partially supported by Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F.510/4/DRS/2009 (SAP-I)].

Definition 2.1[7]. A sequence $\{\eta_n\}$ in E is said to be convergent and converges to η denoted by $\lim_{n\to\infty} \eta_n = \eta$ if $\lim_{n\to\infty} a_\alpha^n = a_\alpha$ and $\lim_{n\to\infty} b_\alpha^n = b_\alpha$ where $[\eta_n]_\alpha = [a_\alpha^n, b_\alpha^n]$ and $[\eta]_\alpha = [a_\alpha, b_\alpha]$ $\forall \alpha \in (0, 1]$.

Note 2.1[7]. If $\eta,\ \delta \in G(R^*(I))$ then $\eta \oplus \delta \in G(R^*(I))$.

Note 2.2[7]. For any scalar t, the fuzzy real number $t\eta$ is defined as $t\eta(s) = 0$ if t=0 otherwise $t\eta(s) = \eta(\frac{s}{t})$.

Definition of fuzzy norm on a linear space as introduced by C. Felbin is given below:

Definition 2.2[5]. Let X be a vector space over R.
Let $||\ ||\ : X \to R^*(I)$ and let the mappings
$L, U\ : [0\ ,\ 1]\times[0\ ,\ 1] \to [0\ ,\ 1]$ be symmetric, nondecreasing in both arguments and satisfy
$L(0\ ,\ 0) =\ 0$ and $U(1\ ,\ 1) =\ 1$.
Write
$[||x||]_\alpha\ =\ [||x||_\alpha^1\ ,\ ||x||_\alpha^2]$ for $x \in X,\ 0\ < \alpha\ \le 1$ and suppose for all $x \in X,\ x \ne \underline{0}$, there exists $\alpha_0\ \in (0\ ,\ 1]$ independent of x such that for all $\alpha\ \le\ \alpha_0$,

(A) $||x||_\alpha^2\ <\ \infty$

(B) $\inf||x||_\alpha^1\ >\ 0$.

The quadruple $(X\ ,\ ||\ ||,\ L\ ,\ U)$ is called a fuzzy normed linear space and $||\ ||$ is a fuzzy norm if

(i) $||x||\ =\ \bar{0}$ if and only if $x\ =\ \underline{0}$;

(ii)$||rx||\ =\ |r|||x||,\ x \in X,\ r \in R$;

(iii) for all $x, y\ \in X$,

(a) whenever $s\ \le\ ||x||_1^1,\ t \le ||y||_1^1$ and $s+t\ \le ||x+y||_1^1$, $||x+y||(s+t)\ \ge\ L(||x||(s)\ ,\ ||y||(t))$,

(b) whenever $s\ \ge\ ||x||_1^1,\ t \ge ||y||_1^1$ and $s+t\ \ge ||x+y||_1^1$, $||x+y||(s+t)\ \le\ U(||x||(s)\ ,\ ||y||(t))$

Remark 2.1[5]. Felbin proved that,
if $L\ = \bigwedge$(Min) and $U = \bigvee$(Max) then the triangle inequality (iii) in the Definition 1.1 is equivalent to
$||x+y||\ \preceq ||x||\ \bigoplus ||y||$.

Further $||\ ||_\alpha^i;\ i = 1, 2$ are crisp norms on X for each $\alpha \in (0\ ,\ 1]$.

Definition 2.3[8]. Let E be a real Banach space and P be a subset of E. P is called a cone if

(i) P is closed, nonempty and $P \ne \{\underline{0}\}$;

(ii) $a, b \in R,\ a, b \ge 0, x, y \in P \Rightarrow ax + by \in P$.

(iii) $x \in P$ and $-x \in P \Rightarrow x = \bar{0}$.

Given a cone $P \subset E$, we define a partial ordering $\le$ with respect to P by $x \le y$ iff $y - x \in P$. On the other hand $x < y$ indicates that $x \le y$ but $x \ne y$ while $x << y$ will stand for $y - x \in$IntP where IntP denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,
with $0 \le x \le y$ implies $||x|| \le K||y||$.
The least positive number satisfying above is called the normal constant of P.
The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence such that $x_1 \le x_2 \le .... \le x_n \le .... \le y$ for some $y \in E$, then there is $x \in E$ such that $||x_n - x|| \to 0$ as $n \to \infty$.
Equivalently, the cone P is regular if every decreasing sequence which is bounded below is convergent. It is clear that a regular cone is a normal cone.

In the following we always assume that E is a real Banach space, P is a cone in E with IntP $\ne \phi$ and $\le$ is a partial ordering with respect to P.

Definition 2.4[12]. Let V be a vector space over the field R. The mapping $||\ ||_c :\ V \to E$ is said to be a cone norm if it satisfies the following conditions:

(i) $||x||_c \ge \theta\ \forall x \in V$;

(ii) $||x||_c = \theta$ iff $x = \theta_V$;

(iii) $||\alpha x||_c = |\alpha|||x||_c\ \forall x \in V,\ \alpha \in R$;

(iv) $||x+y||_c \le ||x||_c+||y||_c\ \forall x, y \in V$. Then $||\ ||_c$ is called a cone norm on V and $(V\ ,\ ||\ ||_c)$ is called a cone normed linear space.

Definition 2.5[3]. Let $(E, ||\ ||)$ be a fuzzy real Banach space where $||\ || : E \to R^*(I)$.
Denote the range of $||\ ||$ by $E^*(I)$. Thus $E^*(I) \subset R^*(I)$.

Definition 2.6[3]. A member $\eta \in E^*(I)$ is said to be an interior point if $\exists r > 0$ such that
$S(\eta, r) = \{\delta \in E^*(I)\ : \eta \ominus \delta \prec \bar{r}\} \subset E^*(I)$.
Set of all interior points of $E^*(I)$ is called interior of $E^*(I)$.

Definition 2.7[3]. A subset of F of $E^*(I)$ is said to be fuzzy closed if for any sequence $\{\eta_n\}$ in F such that $\lim_{n\to\infty} \eta_n = \eta$ implies $\eta \in F$.

Definition 2.8[3]. A subset P of $E^*(I)$ is called a fuzzy cone if

(i) P is fuzzy closed, nonempty and $P \ne \{\bar{0}\}$;

(ii) $a, b \in R,\ a, b \ge 0,\ \eta, \delta \in P \Rightarrow a\eta \oplus b\delta \in P$.

Given a fuzzy cone $P \subset E^*(I)$, define a partial ordering $\le$ with respect to P by $\eta \le \delta$ iff $\delta \ominus \eta \in P$ and $\eta < \delta$ indicates that $\eta \le \delta$ but $\eta \ne \delta$ while $\eta << \delta$ will stand for $\delta \ominus \eta \in$IntP where IntP denotes the interior of P.

The fuzzy cone P is called normal if there is a number $K > 0$ such that for all $\eta, \delta \in E^*(I)$,
with $\bar{0} \le \eta \le \delta$ implies $\eta \preceq K\delta$. The least positive number satisfying above is called the normal constant of P.
The fuzzy cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{\eta_n\}$ is a sequence such that $\eta_1 \le \eta_2 \le .... \le \eta_n \le .... \le \eta$ for some $\eta \in E^*(I)$, then there is $\delta \in E^*(I)$ such that $\eta_n \to \delta$ as $n \to \infty$.
Equivalently, the fuzzy cone P is regular if every decreasing sequence which is bounded below is convergent. It is clear that a regular fuzzy cone is a normal fuzzy cone.

III. Fuzzy Cone Normed Linear Spaces

In this section an idea of fuzzy cone normed linear space is introduced and prove some properties. In the following we always assume that E is a real Banach space, P is a fuzzy cone in E with IntP $\ne \phi$ and $\le$ is a partial ordering with respect to P.

Definition 3.1. Let V be a vector space over the field R. The mapping $||\ ||_P :\ V \to E^*(I)$ is said to be a fuzzy cone norm if it satisfies the following conditions:

(CN1) $||x||_P \ge \theta\ \forall x \in V$;

(CN2) $||x||_P = \theta$ iff $x = \theta_V$;;

(CN3) $||\alpha x||_P = |\alpha|||x||_P\ \forall x \in V,\ \alpha \in R$;

(CN4) $||x+y||_P \leq ||x||_P \oplus ||y||_P \; \forall x,y \in V$. Then $||\;||_P$ is called a cone norm on V and $(V \;,\; ||\;||_P)$ is called a cone normed linear space.

Note 3.1. Fuzzy cone normed linear space is a generalized fuzzy normed linear space.
For, choose $E = R$ and $P = \{\eta \in E^*(I) : \; \eta \succeq \bar{0}\}$ and partial ordering $\leq$ as $\preceq$ then $(X, ||\;||)$ is a Felbin's type fuzzy normed linear space when L= min and U=Max.

Example 3.1 Let $(E, ||\;||')$ be a Banach space. Define $||\;|| : \; E \to R^*(I)$ by

$$||x||(t) = \begin{cases} 1 & \text{if } t > ||x||' \\ 0 & \text{if } t \leq ||x||' \end{cases}$$

Then $[||x||]_\alpha = [||x||' \;,\; ||x||'] \;\; \forall \alpha \in (0,1]$.
It is easy to verify that,
(i) $||x|| = \bar{0}$ iff $x = \underline{0}$ (ii) $||rx|| = |r| ||x||$ (iii) $||x+y|| \preceq ||x|| \oplus ||y||$.
Thus $(E, ||\;||)$ is a fuzzy normed linear space. Let $\{x_n\}$ be a Cauchy sequence in $(E, ||\;||)$
So, $\lim_{m,n\to\infty} ||x_n - x_m|| = \bar{0}$.
$\Rightarrow \lim_{m,n\to\infty} ||x_n - x_m|| = 0 \Rightarrow \{x_n\}$ be a Cauchy sequence in $(E, ||\;||')$.
Since $(E, ||\;||')$ is complete, $\exists x \in E$ such that $\lim_{m,n\to\infty} ||x_n - x||' = 0$.
i.e. $\lim_{n\to\infty} ||x_n - x|| = \bar{0}$.
Thus $(E, ||\;||)$ is a real fuzzy Banach space.
Define $P = \{\eta \in E^*(I) \; : \eta \succeq \bar{0}\}$.
(i) P is fuzzy closed.
For, consider a sequence $\{\delta_n\}$ in P such that $\lim_{n\to\infty} \delta_n \to \delta$.
i.e. $\lim_{n\to} \delta^1_{n,\alpha} = \delta^1_\alpha$ and $\lim_{n\to} \delta^2_{n,\alpha} = \delta^2_\alpha$ where $[\delta_n]_\alpha = [\delta^1_{n,\alpha} \;,\; \delta^2_{n,\alpha}]$ and $[\delta]_\alpha = [\delta^1_\alpha \;,\; \delta^2_\alpha] \;\; \forall \alpha \in (0,1]$.
Now $\delta_n \succeq \bar{0} \;\; \forall n$.
So, $\delta^1_{n,\alpha} \geq 0$ and $\delta^2_{n,\alpha} \geq 0 \;\; \forall \alpha \in (0,1]$.
$\Rightarrow \lim_{n\to} \delta^1_{n,\alpha} \geq 0$ and $\lim_{n\to} \delta^2_{n,\alpha} \geq 0 \;\; \forall \alpha \in (0,1]$
$\Rightarrow \delta^1_\alpha \geq 0$ and $\delta^2_\alpha \geq 0 \;\; \forall \alpha \in (0,1]$
$\Rightarrow \delta \succeq \bar{0}$.
So $\delta \in P$. Hence P is fuzzy closed.
(ii) It is obvious that, $a, b \in R, \; a, b \geq 0 \; \eta, \delta \in P \Rightarrow a\eta \oplus b\delta \in P$.
Thus P is a fuzzy cone in E.
Now choose the ordering $\leq$ as $\preceq$ and define $||\;||_P \;: E \to E^*(I)$ by $||x||_P = ||x||$.
Then it is easy to verify that $||\;||_P$ satisfies the conditions (CN1) to (CN4). Hence $(E \;,\; ||\;||_P)$ is a fuzzy cone normed linear space.

Definition 3.2. Let $(V, ||\;||_P)$ be a fuzzy cone normed linear space. Let$\{x_n\}$ be a sequence in V and $x \in V$. If for every $c \in E$ with $\bar{0} << ||c||$ there is a positive integer N such that for all $n > N, \; ||x_n - x||_P << ||c||$, then $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. We denote it by $\lim\limits_{n\to\infty} x_n = x$.

Lemma 3.1. *Let $(V \;,\; ||\;||_P)$ be a fuzzy cone normed linear space space and P be a normal fuzzy cone with normal constant K. Let$\{x_n\}$ be a sequence in V. Then $\{x_n\}$ converges to x iff $||x_n - x||_P \to \bar{0}$ as $n \to \infty$.*

Proof. First we suppose that $\{x_n\}$ converges to x. For every real number $\epsilon > 0$, choose $c \in E$ with $\bar{0} << ||c||$ and $K||c|| \prec \bar{\epsilon}$.
Then $\exists$ a natural number N, such that $\forall n > N, \; ||x_n - x||_P << ||c||$.
So that when $n > N, \;\; ||x_n - x||_P \preceq K||c|| \prec \bar{\epsilon}$ (since P is normal).
i.e. $||x_n - x||^1_{P,\alpha} < \epsilon$ and $||x_n - x||^2_{P,\alpha} < \epsilon \; \forall n \geq N, \; \forall \alpha \in (0,1]$.
i.e. $\lim\limits_{n\to\infty} ||x_n - x||^1_{P,\alpha} = 0$ and $\lim\limits_{n\to\infty} ||x_n - x||^2_{P,\alpha} = 0 \; \forall \alpha \in (0,1]$.
i.e. $\lim\limits_{n\to\infty} ||x_n - x||_P = \bar{0}$.
Conversely, suppose that $\lim\limits_{n\to\infty} ||x_n - x||_P = \bar{0}$.
For, $c \in E$ with $\bar{0} << ||c||$, there is $\delta > 0$ such that $||x|| \prec \bar{\delta}$.
This implies that $||c|| \ominus ||x|| \in$ IntP.
For this δ there is a positive integer N such that $\forall n > N, \; ||x_n - x||_P \prec \bar{\delta}$.
Let $||x_n - x||_P = ||y_n||$ where $y_n \in E$ for each n. So $||y_n|| \prec \bar{\delta} \;\; \forall n > N$.
i.e. $||c|| \ominus ||y_n|| \in$ IntP $\;\; \forall n > N$
$\Rightarrow ||y_n|| << ||c|| \;\; \forall n > N$
$\Rightarrow ||x_n - x||_P << ||c|| \;\; \forall n > N$
$\Rightarrow x_n \to x$ as $n \to \infty$.

Lemma 3.2. *Let $(V \;,\; ||\;||_P)$ be a fuzzy cone normed linear space and P be a normal fuzzy cone with normal constant K. Let$\{x_n\}$ be a sequence in V. If $\{x_n\}$ is convergent then its limit is unique.*

Proof. If possible suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$. Thus for any $c \in E$ with $\bar{0} << ||c||$, there exists a natural number N such that $\forall n > N, \; ||x_n - x||_P << ||c||$ and $||y_n - y||_P << ||c||$.
We have $||x - y||_P \leq ||x - x_n||_P \oplus ||x_n - y||_P \;\; \leq 2||c||$.
Hence $||x - y||_P \preceq 2K||c||$.
Since c is arbitrary, we have $||x - y||_P = \bar{0}$. i.e. $x = y$.

Definition 3.3. Let $(V \;,\; ||\;||_P)$ be a fuzzy cone normed linear space and $\{x_n\}$ be a sequence in V. If for any $c \in E$ with $\bar{0} << ||c||$, there exists a natural number N such that $\forall m, n > N, \; ||x_n - x_m||_P << ||c||$, then $\{x_n\}$ is called a Cauchy sequence in V.

Definition 3.4. Let $(V \;,\; ||\;||_P)$ be a fuzzy cone normed linear space. If every Cauchy sequence is convergent in V, then V is called a complete fuzzy cone normed linear space.

Lemma 3.3. *Let $(V \;,\; ||\;||_P)$ be a fuzzy cone normed linear space and $\{x_n\}$ be a sequence in V. If $\{x_n\}$ is convergent then it is a Cauchy sequence.*

Proof. Let $\{x_n\}$ converges to x. So for any $c \in E$ with $\bar{0} << ||c||$ there exists a natural number N such that $\forall m, n > N, \; ||x_n - x||_P << ||\frac{c}{2}||$ and $||x_m - x||_P << ||\frac{c}{2}||$.
Hence $||x_n - x_m||_P \leq ||x_n - x||_P \oplus ||x - x_m||_P << ||c|| \;\; \forall m, n > N$.
Thus $\{x_n\}$ is a Cauchy sequence.

IV. Finite Dimensional Fuzzy Cone Normed Linear Spaces

In this section some results on finite dimensional fuzzy cone normed linear spaces are established.

Lemma 4.1. *Let $\{x_1, \; x_2, \; ...., \; x_n\}$ be a linearly independent set of vectors in a fuzzy cone (P) normed linear space $(X \;,\; ||\;||_P)$ with normal constant K. Then there exists a member $c \in$ IntP such that for every set of scalars*

α_1 , α_2 , ... , α_n *we have*
$||\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n||_P \geq \sum_{i=1}^{n} |\alpha_i| ||c||$..........*(1).*

Proof. Let $s = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$. If $s = 0$ then each α_i's is zero and hence (1) is true.
So we assume that $s > 0$. Then (1) becomes
$||\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n||_P \geq ||c||$.........(2) where $\beta_i = \frac{\alpha_i}{s}$ and $\sum_{i=1}^{n} |\beta_i| = 1$.
Thus it is sufficient to prove that there exists an element $c \in$ IntP such that (2) holds for any set of scalars $\beta_1, \ \beta_2, \ .... \ \beta_n$ with $\sum_{i=1}^{n} |\beta_i| = 1$.
If possible suppose that this is not true. Then there exists a sequence $\{y_m\}$ in X where
$y_m = \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + ... + \beta_n^{(m)} x_n$ with $\sum_{i=1}^{n} |\beta_i^{(m)}| = 1, \ m = 1, 2, ...$
such that $||y_m||_P \to \bar{0}$ as $m \to \infty$.
Since $\sum_{i=1}^{n} |\beta_i^{(m)}| = 1$ for $m = 1, 2, ...$ we have
$|\beta_i^{(m)}| \leq 1$ for $i = 1, 2, ..., n$ and $m = 1, 2, ...$
Hence for a fixed $i = 1, 2, ..., n$ the sequence $\beta_i^{(m)}$ is bounded. Therefore by Bolzano-Weierstrass theorem $\{\beta_1^{(m)}\}$ has a subsequence converging to β_1 (say) and suppose $\{y_{1,m}\}$ denotes the corresponding subsequence of $\{y_m\}$. By the same argument the sequence $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ (say) for which the corresponding subsequence of real scalars $\{\beta_2^{(m)}\}$ converges to β_2 (say). Continuing this procress, after n-step, we obtain a sequence $\{y_{n,m}\}$ of $\{y_m\}$ of the form
$y_{n,m} = \sum_{i=1}^{n} \delta_i^{(m)} x_i$ with $\sum_{i=1}^{n} |\delta_i^{(m)}| = 1 \ m = 1, 2, ...$
where $\delta_i^{(m)} \to \beta_i$ as $m \to \infty$ and for each $i = 1, 2, ..., n$. So $\sum_{i=1}^{n} |\beta_i| = 1$.
Since $\sum_{i=1}^{n} |\beta_i| = 1$, thus not all β's are zero. Since $\{x_1, \ x_2, \ ...., \ x_n\}$ is a linearly independent set of vectors, so $y \neq \underline{0}$.
We have $||\delta_i^{(m)} x_i - \beta_i x_i||_P = |\delta_i^{(m)} - \beta_i| ||x_i||_P$.
So $\lim_{m \to \infty} ||\delta_i^{(m)} x_i - \beta_i x_i||_P = \bar{0_X}$ $(\delta_i^{(m)} \to \beta_i)$.
Thus $\delta_i^{(m)} x_i \to \beta_i x_i$ as $m \to \infty$ for each $i = 1, \ 2, ..., n$.
For every $\epsilon > 0$, choose $c \in E$ with $\bar{0} << ||c||$ and $K||c|| \prec \bar{\epsilon}$. Then for each i, $\exists$ a natural number N_i such that $||\delta_i^{(m)} x_i - \beta_i x_i||_P << ||c|| \quad \forall m \geq N_i$ and for each $i = 1, \ 2, ..., n$.
Let $N = \max_{1 \leq i \leq n} N_i$.
Then $||\delta_i^{(m)} x_i - \beta_i x_i||_P << ||c|| \quad \forall m \geq N$ and for each $i = 1, \ 2, ..., n$.
$\Rightarrow ||\delta_i^{(m)} x_i - \beta_i x_i||_P \leq ||c|| \quad \forall m \geq N$ and for each $i = 1, \ 2, ..., n$.
Let $||\delta_i^{(m)} x_i - \beta_i x_i||_P = ||z_i^{(m)}||$ where $z_i^{(m)} \in E$ for each $i = 1, \ 2, ..., n$.
From above we have, $||z_i^{(m)}|| \leq ||c|| \quad \forall m \geq N$ and for each $i = 1, \ 2, ..., n$.
$\Rightarrow ||z_i^{(m)}|| \preceq K||c|| \prec \bar{\epsilon} \quad \forall m \geq N$ and for each $i = 1, \ 2, ..., n$.
$\Rightarrow \lim_{n \to \infty} ||z_i^{(m)}|| = \bar{0_E}$.
$\Rightarrow \lim_{n \to \infty} ||y_{n,m} - y||_P = \bar{0_E}$.
$\Rightarrow \lim_{m \to \infty} y_{n,m} = y$.
Now $||y_{n,m}||_P = ||y_{n,m} - y + y||_P \leq ||y_{n,m} - y||_P \oplus ||y||_P$.
Since $||y_{n,m} - y||_P \oplus ||y||_P \in E^*(I)$ we may choose $||y_{n,m} - y||_P \oplus ||y||_P = ||z'_{n,m}||$ where $z'_{n,m} \in E$.
Thus $||z_{n,m}|| \leq ||z'_{n,m}||$ where $||z_{n,m}|| = ||y_{n,m}||_P, \ z_{n,m} \in E$
$\Rightarrow ||z_{n,m}|| \preceq K||z'_{n,m}||$
$\Rightarrow ||z_{n,m}||_\alpha^1 \leq K||z'_{n,m}||_\alpha^1$ and $||z_{n,m}||_\alpha^2 \leq K||z'_{n,m}||_\alpha^2 \quad \forall \alpha \in (0 \ , \ 1]$
$\Rightarrow ||y_{n,m}||_{P,\alpha}^1 \leq K\{||y_{n,m} - y||_{P,\alpha}^1 + ||y||_{P,\alpha}^1\}$ and $||y_{n,m}||_{P,\alpha}^2 \leq K\{||y_{n,m} - y||_{P,\alpha}^2 + ||y||_{P,\alpha}^2\} \quad \forall \alpha \in (0, 1]$
$\Rightarrow ||y_{n,m}||_{P,\alpha}^1 - K||y||_{P,\alpha}^1 \leq K\{||y_{n,m} - y||_{P,\alpha}^1$ and $||y_{n,m}||_{P,\alpha}^2 - ||y||_{P,\alpha}^2 \leq K\{||y_{n,m} - y||_{P,\alpha}^2 \quad \forall \alpha \in (0, 1]$
$\Rightarrow \lim_{m \to \infty} ||y_{n,m}||_{P,\alpha}^1 = ||Ky||_{P,\alpha}^1$ and $\lim_{m \to \infty} ||y_{n,m}||_{P,\alpha}^2 = ||Ky||_{P,\alpha}^2 \quad \forall \alpha \in (0, 1]$
$\Rightarrow ||y_{n,m}||_P = ||Ky||_P$ as $m \to \infty$.
Since $||y_m||_P \to \bar{0}$ as $m \to \infty$, it follows that $||y_{n,m}||_P \to \bar{0}$ as $m \to \infty$.
So $||Ky||_P = \bar{0}$. i.e. $y = \underline{0}$ which is a contradiction.
Hence the lemma is proved.

Theorem 4.1. *Every finite dimensional fuzzy cone normed linear space with normal constant K is complete.*

Proof. Let $(X \ , \ || \ ||_P)$ be a fuzzy cone normed linear space with normal constant K. Let $\{x_n\}$ be a Cauchy sequence in X.
Let dim X=m and $\{e_1, \ e_2, \ ......, e_m\}$ be a basis for X. Then each x_n has a unique representation as
$x_n = \beta_1^{(n)} e_1 + \beta_2^{(n)} e_2 + ... + \beta_m^{(n)} e_m$ where $\beta_1^{(n)}, \ \beta_2^{(n)} \ , ..., \ \beta_m^{(n)}$ are scalars for each $n = 1, 2, ...$
Since $\{x_n\}$ is a Cauchy sequence, for every $e \in E$ with $||e|| >> \bar{0}$ there exists a positive integer N such that $||x_n - x_k|| << ||e|| \quad \forall n, k \geq N$.
Now from Lemma 4.1, it follows that $\exists c \in IntE$ with $||c|| \succ \bar{0}$ such that $||\epsilon|| >> ||x_n - x_k|| \geq ||c|| \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(}k)| \quad \forall n, k \geq N$.
$\Rightarrow ||c|| \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(}k)| \leq ||e|| \quad \forall n, k \geq N$.
$\Rightarrow ||c|| \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(}k)| \preceq K||e|| \quad \forall n, k \geq N$.
$\Rightarrow ||c||_\alpha^1 \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(}k)| \leq K||e||_\alpha^1$ and $||c||_\alpha^2 \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(}k)| \geq K||e||_\alpha^2 \quad \forall n, k \leq N \quad \forall \alpha \in (0 \ , \ 1]$
$\Rightarrow ||c||_\alpha^1 |\beta_i^{(n)} - \beta_i^{(}k)| \leq K||e||_\alpha^1$ and $||c||_\alpha^2 |\beta_i^{(n)} - \beta_i^{(}k)| \geq K||e||_\alpha^2 \quad \forall n, k \leq N \quad \forall \alpha \in (0 \ , \ 1]$, for each $i = 1, 2, ...$
Since $e \in E$ is arbitrary, from above it follows that each of the m sequences $\{\beta_i^{(n)}\}$ is Cauchy in R. Since R is complete, thus each $\{\beta_i^{(n)}\}$ converges and denote by β_i are their limits for each $i = 1, 2, ..., m$.

We define $x = \beta_1 e_1 + \beta_2 e_2 + ... + \beta_m e_m$. Clearly $x \in X$. We have,

$||x_n - x||_P = ||\sum_{i=1}^{m}(\beta_i^{(n)} - \beta)e_i||_P \leq |\beta_1^{(n)} - \beta_1|||e_1||_P \oplus |\beta_2^{(n)} - \beta_2|||e_2||_P \oplus ...... \oplus |\beta_m^{(n)} - \beta_m|||e_m||_P$.

Since $|\beta_i^{(n)} - \beta_i| \to 0$ as $n \to \infty$ for each $i = 1, 2, ..., m$ we get $||x_n - x||_P \to \bar{0}$ as $n \to \infty$.

Thus the Cauchy sequence $\{x_n\}$ converges to $x \in X$. Since $\{x_n\}$ is arbitrary it follows that X is complete.

Definition 4.1. Let $(X\ ,\ ||\ ||_P)$ be a fuzzy cone normed linear space.

(i) Let $c \in E$ with $\bar{0} << ||c||$ and $b \in X$.

Define $B_c(b)\ = \{x \in X :\ ||x - b||_P << ||c||\}$.

(ii) A subset B of X is said to be closed if any sequence $\{x_n\}$ in B converges to some point $x \in B$.

(iii) A subset F of X is said to be the closure of B if for any $x \in F$, there exists a sequence $\{x_n\}$ in B such that $x_n \to x$ as $n \to \infty$ with respect to the cone norm $||\ ||_P$.

(iv) A subset C of X is said to be bounded if $C \subset B_c(b)$ for some $b \in X$ and $c \in E$ with $\bar{0} << ||c||$.

(v) A subset F of X is said to be compact if for any sequence $\{x_n\}$ in F, there exists a subsequence of $\{x_n\}$ which converges to some point in F.

Theorem 4.2. *In a finite dimensional fuzzy cone normed linear space with normal constant K, a subset M of X is compact if and only if M is closed and bounded.*

Proof. Let M be a compact subset of X. Then from definition it is easy to verify that M is closed.

Next we show that M is bounded. If possible suppose that M is not bounded. Let x_0 be a fixed element in X. Then there exists a point $x_1 \in M$ such that $||x_1 - x_0||_P > ||c||$ for some $c \in E$ with $||c|| >> \bar{0}$.

By the same reason there exists a point $x_2 \in M$ such that $||x_2 - x_0||_P > ||x_1 - x_0||_P \oplus ||c||$.

Continuing this process we obtain a sequence $x_1,\ x_2,\ ......$ of the set M such that

$||x_n - x_0||_P\ > ||x_1 - x_0||_P \oplus ||x_2 - x_0||_P \oplus ...... \oplus ||x_{n-1} - x_0||_P \oplus ||c||$.

i.e. $||x_n - x_0||_P\ >\ ||x_m - x_0||_P \oplus ||c||\ \ \forall m < n$.

i.e. $||x_n - x_0||_P\ \ominus\ ||x_m - x_0||_P\ > ||c||\ \ \forall m < n...........(i)$.

Now $||x_n - x_0||_P \leq ||x_n - x_m||_P \oplus ||x_m - x_0||_P$

i.e $||x_n - x_0||_P \ominus ||x_m - x_0||_P \leq ||x_n - x_m||_P...........(ii)$.

From (i) and (ii) we have, $||c||\ < ||x_n - x_m||_P\ \ \forall m < n...........(iii)$.

From (iii), it follows that, neither the sequence nor any subsequence of $\{x_n\}$ can converge. This contradicts the fact that M is compact. Hence M is bounded.

Conversely suppose that M is closed and bounded and we have to show that M is compact.

Let dim X =n. Let $\{e_1,\ e_2,\ ....,\ e_n\}$ be a basis for X. Chose $\{x_m\}$ in M. Since M is bounded, there exists $p \in E$ such that $x_m \in B_p(b)$ for some $b \in X$ and $\forall m$.

i.e. $||x_m - b||_P << ||p||\ \ \forall m$.

Now $||x_m||_P\ = ||x_m - b + b||_P \leq ||x_m - b||_P \oplus ||b||_P << ||p|| \oplus ||b||_P\ \ \forall m$.

i.e. $||x_m||_P\ << ||p|| \oplus ||b||_P\ \ \forall m$.

Let $x_m = \beta_1^{(m)} e_1 + \beta_2^{(m)} e_2 + ... + \beta_n^{(m)} e_n$ where $\beta_1^{(m)},\ \beta_2^{(m)}\ , ...,\ \beta_n^{(m)}$ are scalars for each m=1,2,.......

Thus by Lemma 4.1, $\exists c \in IntP$ such that

$||p|| \oplus ||b||_P >> ||x_m||_P = ||\sum_{j=1}^{n}(\beta_j^{(m)} e_j||_P \geq ||c|||\sum_{j=1}^{n}|\beta_i^{(m)}|$.

i.e. $||p|| \oplus ||b||_P \geq ||c|||\sum_{j=1}^{n}|\beta_i^{(m)}|$..........(i)

Let $||b||_P\ = ||b'||$ for some $b' \in E$.

So from above we get $||p|| \oplus ||b'|| \geq ||c|||\sum_{j=1}^{n}|\beta_i^{(m)}|$

i.e. $||c||\sum_{j=1}^{n}|\beta_i^{(m)}| \preceq K(||p|| \oplus ||b'||)$

i.e $||c||_\alpha^1 \sum_{j=1}^{n}|\beta_i^{(m)}| \leq K(||p||_\alpha^1 + ||b'||_\alpha^1)$ and

$||c||_\alpha^2 \sum_{j=1}^{n}|\beta_i^{(m)}| \leq K(||p||_\alpha^2 + ||b'||_\alpha^2)\ \ \forall \alpha \in (0\ ,\ 1]$.

From above two relations it follows that each sequence $\{\beta_j^{(m)}\}$ $(j = 1, 2, ..., n)$ is bounded. By Bolzano-Weiestrass theorem it follows that each of the sequence $\{\beta_j^{(m)}\}$ has a convergent subsequence say $\{\beta_j^{(m_k)}\}$ for each $j = 1, 2, ..., n$.

Let $x_{m_k} = \beta_1^{(m_k)} e_1 + \beta_2^{(m_k)} e_2 + ... + \beta_n^{(m_k)} e_n$ where $\beta_1^{(m_k)},\ \beta_2^{(m_k)}\ , ...,\ \beta_n^{(m_k)}$ are convergent sequences of scalars

Let $\beta_j = \lim_{k \to \infty} \beta_j^{(m_k)}$ for $j = 1, 2, ..., n$. and $x\ = \beta_1 e_1 + \beta_2 e_2 + ... + \beta_n e_n$.

Now $||x_{m_k} - x||_P\ = ||\sum_{j=1}^{n}(\beta_j^{(m_k)} - \beta_j)e_j||_P \leq \sum_{j=1}^{n}|\beta_j^{(m_k)} - \beta_j|||e_j||_P$.

Since $|\beta_j^{(m_k)} - \beta_j| \to 0$ as $k \to \infty$, from the proof of the Lemma 4.1, it follows that $\lim_{k \to \infty}||x_{m_k} - x||_P\ =\ \bar{0}$.

i.e. $\{x_{m_k}\}$ is a convergent subsequence of $\{x_m\}$ and converges to x. Since M is closed, it follows that $x \in M$. Thus every sequence in M has a convergent subsequence and converges to an element of M. Hence M is compact.

CONCLUSION

In this paper, an idea of fuzzy cone normed linear space is introduced which is a generalization of fuzzy normed linear space. In fuzzy cone normed linear space, range of fuzzy cone norm is considered as ordering fuzzy real numbers defined on a real fuzzy Banach space. It is seen that Felbin's type (max, min) fuzzy normed linear space is a particular case of fuzzy cone normed linear space. I think that there is a large scope of developing more results of fuzzy functional analysis in this context.

ACKNOWLEDGMENT

The author is grateful to the referees for their valuable suggestions in rewriting the paper in the present form. The author is also grateful to the Editor-in-Chief for his valuable comments to standardize it.

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