Some Remarks Concerning k-Fibonacci-Like Numbers

S. Sambasiva Rao* and Maruthi Srinivas

***Abstract*—The k-Fibonacci sequence $\{F_{k,n}\}$ is defined as a linear recurrence relation depending on one real parameter $k > 0$ given by $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$ with initial conditions $F_{k,0} = 0, F_{k,1} = 1$. The k-Fibonacci-Like numbers (k-FLNs) are the terms of the sequence $\{S_{k,n}\}$ defined by preserving recurrence relation as in k-Fibonacci sequence and altering initial conditions with $S_{k,0} = S_{k,1} = 2$. In this paper we introduce Generalized k-Fibonacci-Like numbers and some interesting properties are obtained for these numbers, thereby we prove the valid Binet's formula for k-FLNs as a correction to the existing one derived by Yashwanth K.Panwar et al.(2014).**

***Index Terms*—k-Fibonacci-Like numbers, Binet's formula, Golden Ratio.**

***MSC 2010 Codes* – 11B39, 11B37, 11B99.**

I. INTRODUCTION

THE sequence $\{F_n\}$ defined recursively by adding two preceding terms with initial values $F_0 = 0$, $F_1 = 1$, is called Fibonacci sequence. These sequences are generalized in several directions in recent past, some by retaining the initial conditions and imposing a change in the recurrence relation, or others by retaining the recurrence relation and altering the initial conditions. For instance, S.Falcon and A.Plaza introduced the general k-Fibonacci sequence while studying the recursive application of two geometrical transformation used in the Four-Triangle Longest-Edge (4TLE) and many properties of these numbers are deduced directly from elementary matrix algebra (see [1]). Yashwanth K.Panwar et al.(see [2]) introduced the notion of k-Fibonacci-Like numbers (in short k-FLNs) aiming to generalize several identities involving classic Fibonacci numbers to k-FLNs with the aid of following Binet's formula proved by them

$$S_{k,n} = 2\{\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\} \tag{1}$$

for $n \geq 0$, where r_1 and r_2 are the roots of $q^2 - kq - 1 = 0$ with $r_1 > r_2$. Unfortunately, this formula (1) is incorrect as it yield $S_{k,1} = 2k$ instead of its actual value $S_{k,1} = 2$. The purpose of this study is to introduce the notion of Generalized k-Fibonacci-Like number and to prove some interesting properties of these numbers, consequently we deduce valid Binet's formula for k-FLNs.

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II. MAIN RESULTS

Throughout this section $\mathbb{N}$ denotes the set of non-negative integers and $\mathbb{R}^+$ the set of positive real numbers. For undefined terms and notations we refer to ([1], [2] and [3]). We first introduce a general k-Fibonacci-Like sequence that generalizes both k-Fibonacci and k-Fibonacci-Like sequences as follows.

Definition 2.1: Let $a, b \in \mathbb{N}$. Then for any $k \in \mathbb{R}^+$, the Generalized k-Fibonacci-Like sequence $\{G_{k,n}\}$ is defined recurrently by

$$G_{k,n} = kG_{k,n-1} + G_{k,n-2} \tag{2}$$

for $n \geq 2$ with $G_{k,0} = a, G_{k,1} = b$.
Each member of the Generalized k-Fibonacci-Like sequence $\{G_{k,n}\}$ will be called Generalized k-Fibonacci-Like Number (in short G-k-FLN). The first few of them are:
$a,\ b,\ bk + a,\ bk^2 + ak + b,\ bk^3 + ak^2 + 2bk + a,\ bk^4 + ak^3 + 3bk^2 + 2ak + b$.

Remark 2.2: It is evident from the Definition 2.1 that

1) if $a = 0$, $b = 1$ then $\{G_{k,n}\}$ is the k-Fibonacci sequence $\{F_{k,n}\}$= $\{0, 1, k, k^2 + 1, k^3 + 2k, \ldots.\}$,
2) if $a = 2$, $b = 2$ then $\{G_{k,n}\}$ is the k-Fibonacci-Like sequence $\{S_{k,n}\}$=$\{2, 2, 2k + 2, 2k^2 + 2k + 2, 2k^3 + 2k^2 + 4k + 2, \ldots.\}$,
3) if $k = 1$, $a = 0$, $b = 1$ then $\{G_{k,n}\}$ is the classic Fibonacci sequence $\{F_n\}$,
4) if $k = 2$, $a = 0$, $b = 1$ then $\{G_{k,n}\}$ is the Pell sequence $\{P_n\} = \{0, 1, 2, 5, \ldots.\}$,
5) if $k = 3$, $a = 0$, $b = 1$ then $\{G_{k,n}\}$ is the sequence $\{0, 1, 3, 10, 33, 109, \ldots.\}$,
6) if $k = 2, a = 2, b = 1$ then $\{G_{k,n}\} = \{H_n\} = \{2, 1, 4, 9, 22, 53, \ldots.\}$ (see [4]).

In what follows, $G_{k,n}$, $S_{k,n}$, $F_{k,n}$ and F_n represent n^{th} term of the Generalized k-Fibonacci-Like, k-Fibonacci-Like, k-Fibonacci and Fibonacci sequence respectively and $r_1 = \frac{k+\sqrt{k^2+4}}{2}$, $r_2 = \frac{k-\sqrt{k^2+4}}{2}$ are the roots of characteristic equation of linear recurrence relation (2).

Remark 2.3: It is clear that $r_1 > r_2$, $r_1r_2 = -1$ and $r_1 + r_2 = k$.

Theorem 2.4: (Binet's Formula) The n^{th} term of $\{G_{k,n}\}$ is given by

$$G_{k,n} = b\{\frac{r_1^n - r_2^n}{r_1 - r_2}\} + a\{\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}\}. \tag{3}$$

Proof: The general solution is $G_{k,n} = Ar_1^n + Br_2^n$ for all $n \geq 0$.
Solving the equations $G_{k,0} = A + B = a$, $G_{k,1} = Ar_1 + Br_2 = b$, we get $A = \frac{b-ar_2}{r_1-r_2}$ and $B = \frac{ar_1-b}{r_1-r_2}$.
Therefore,
$G_{k,n} = Ar_1^n + Br_2^n = \frac{b-ar_2}{r_1-r_2}r_1^n + \frac{ar_1-b}{r_1-r_2}r_2^n$
$= b\{\frac{r_1^n-r_2^n}{r_1-r_2}\} + a\{\frac{r_1r_2^n-r_2r_1^n}{r_1-r_2}\}$
$= b\{\frac{r_1^n-r_2^n}{r_1-r_2}\} + a\{\frac{r_1^{n-1}-r_2^{n-1}}{r_1-r_2}\}$ for all $n \geq 0$.
This completes the proof. □
As a consequence of Theorem 2.4 and Remark 2.2, we have

Corollary 2.5: The Binet's formula to find $F_{k,n}$ is

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \tag{4}$$

for all $n \geq 0$.

Corollary 2.6: The Binet's formula to find n^{th} term of $\{S_{k,n}\}$ is given by

$$S_{k,n} = 2\{\frac{r_1^n - r_2^n}{r_1 - r_2} + \frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}\}. \tag{5}$$

Remark 2.7: If we replace n by $-n$ in the formula (4) and using the fact $r_1r_2 = -1$, then for each $k \in \mathbb{R}^+$,
$F_{k,-n} = \frac{r_1^{-n}-r_2^{-n}}{r_1-r_2}$
$= \frac{1}{r_1^nr_2^n}\{\frac{r_2^n-r_1^n}{r_1-r_2}\}$
$= (-1)^{n+1}F_{k,n}$.
Thus, for each k in $\mathbb{R}^+$, we can define a sequence $\{F_{k,-n}\}_{n\in\mathbb{N}}$ by $F_{k,-n} = (-1)^{n+1}F_{k,n}$. In particular, $F_{k,0} = 0$, $F_{k,-1} = F_{k,1} = 1$, $F_{k,-2} = -F_{k,2} = -k$, $F_{k,-3} = F_{k,3} = k^2 + 1$ and so on.
In [5] Alvaro H. Salas introduced associated k-Fibonacci numbers and gave a combinatorial interpretation for them.

Definition 2.8: ([5]) The sequence $\{A_{k,n}\}$ associated to $\{F_{k,n}\}$ defined as

$$A_{k,n} = 1,\ A_{k,n} = F_{k,n} + F_{k,n-1} \tag{6}$$

for $n \geq 1$
In view of Corollaries 2.5, 2.6 and Remark 2.7, one can establish the relation

$$S_{k,n} = 2\{F_{k,n} + F_{k,n-1}\} = 2A_{k,n} \tag{7}$$

for all $n \geq 0$, where $A_{k,n}$ is the n^{th} term of associated k-Fibonacci sequence.
Now we prove some properties for the sum of the terms of $\{G_{k,n}\}$.

Theorem 2.9: Let $G_{k,i}$ be the i^{th} G-k-FLN. Then
1) $\sum_{i=0}^n G_{k,i} = \frac{1}{k}\{G_{k,n+1} + G_{k,n} + (k-1)a - b\}$,
2) $\sum_{i=0}^n G_{k,2i} = \frac{1}{k}\{G_{k,2n+1} + ka - b\}$,
3) $\sum_{i=0}^n G_{k,2i+1} = \frac{1}{k}\{G_{k,2n+2} - a\}$.

Proof: We prove (1): It follows from Definition 2.1 that

$$G_{k,i} = \frac{G_{k,i+1} - G_{k,i-1}}{k} \tag{8}$$

for all $i \geq 1$. If we add these equations, for $i = 1, i = 2, \ldots, i = n$, then we obtain
$\sum_{i=1}^n G_{k,i} = \frac{1}{k}\{G_{k,n+1} + G_{k,n} - G_{k,1} - G_{k,0}\}$
$= \frac{1}{k}\{G_{k,n+1} + G_{k,n} - b - a\}$
$\Rightarrow \sum_{i=0}^n G_{k,i} = \frac{1}{k}\{G_{k,n+1} + G_{k,n} + (k-1)a - b\}$.
Proof of (2) and (3) are similar. □
In view of Theorem 2.9 and Remark 2.2, we can deduce the following identities

1) $\sum_{i=0}^n S_{k,i} = \frac{1}{k}\{S_{k,n+1} + S_{k,n} + 2(k-2)\}$,
2) $\sum_{i=0}^n S_{k,2i} = \frac{1}{k}\{S_{k,2n+1} + 2(k-1)\}$,
3) $\sum_{i=0}^n S_{k,2i+1} = \frac{1}{k}\{S_{k,2n+2} - 2\}$,
4) $\sum_{i=0}^n F_{k,i} = \frac{1}{k}\{F_{k,n+1} + F_{k,n} - 1\}$,
5) $\sum_{i=0}^n F_{k,2i} = \frac{1}{k}\{F_{k,2n+1} - 1\}$,
6) $\sum_{i=0}^n F_{k,2i+1} = \frac{1}{k}F_{k,2n+2}$,
7) $\sum_{i=0}^n F_i = \{F_{n+1} + F_n - 1\} = F_{n+2} - 1$,
8) $\sum_{i=0}^n F_{2i} = \{F_{2n+1} - 1\}$,
9) $\sum_{i=0}^n F_{2i+1} = F_{2n+2}$,
10) $\sum_{i=0}^n P_i = \frac{1}{2}\{P_{n+1} + P_n - 1\}$,
11) $\sum_{i=0}^n P_{2i} = \frac{1}{2}\{P_{2n+1} - 1\}$,
12) $\sum_{i=0}^n P_{2i+1} = \frac{1}{2}P_{2n+2}$,

where P_n is the n^{th} term of Pell sequence.
Finally we prove a necessary and sufficient condition for a sequence to be a Generalized k-Fibonacci-Like sequence and we will discuss some of its consequences.

Theorem 2.10: A sequence $\{H_{k,n}\}$ defined recursively, depending on one real parameter $k > 0$, is a Generalized k-Fibonacci-Like if and only if there exist $a, b \in \mathbb{N}$ such that $H_{k,n} = bF_{k,n} + aF_{k,n-1}$ for all $n \geq 0$.

Proof: Suppose that $\{H_{k,n}\}$ is a Generalized k-Fibonacci-Like sequence. Then by Theorem 2.4 and Corollary 2.5, we can conclude that $H_{k,n} = bF_{k,n} + aF_{k,n-1}$ for all $n \geq 0$.
Conversely, suppose that there exist $a, b \in \mathbb{N}$ such that $H_{k,n} = bF_{k,n} + aF_{k,n-1}$ for all $n \geq 0$ Then
$H_{k,n} = bF_{k,n} + aF_{k,n-1}$
$= b\{kF_{k,n-1} + F_{k,n-2}\} + a\{kF_{k,n-2} + F_{k,n-3}\}$
$= k\{bF_{k,n-1} + aF_{k,n-2}\} + \{bF_{k,n-2} + aF_{k,n-3}\}$
$= kH_{k,n-1} + H_{k,n-2}$ for $n \geq 3$,
whereas $H_{k,0} = bF_{k,0} + aF_{k,-1} = a$, $H_{k,1} = bF_{k,1} + aF_{k,0} = b$ and $H_{k,2} = bF_{k,2} + aF_{k,1} = bk + a$. This shows that $H_{k,n}$ is the n^{th} G-k-FLN, and completes the proof. □

Corollary 2.11: If the initial conditions in the Defintion2.1 satisfy $G_{k,1} = kG_{k,0}$, then $G_{k,n} = G_{k,0}F_{k,n+1}$ for all $n \geq 0$.

Proof: This follows from Theorem 2.10.

Corollary 2.12: (Generalized Convolution Property) $G_{k,n+m} = F_{k,m}G_{k,n+1} + F_{k,m-1}G_{k,n}$.

Proof: In view of Theorem 2.10, there exist $a, b \in \mathbb{N}$ such that $G_{k,n} = bF_{k,n} + aF_{k,n-1}$ for all $n \geq 0$, and by Proposition 14 of [7], we have $F_{k,n+m} = F_{k,m}F_{k,n+1} + F_{k,m-1}F_{k,n}$.
$G_{k,n+m} = bF_{k,n+m} + aF_{k,n+m-1}$
$= b(F_{k,m}F_{k,n+1} + F_{k,m-1}F_{k,n}) + a(F_{k,m}F_{k,n} + F_{k,m-1}F_{k,n-1})$
$= F_{k,m}(bF_{k,n+1} + aF_{k,n}) + F_{k,m-1}(bF_{k,n} + aF_{k,n-1})$
$= F_{k,m}G_{k,n+1} + F_{k,m-1}G_{k,n}$.
This completes the proof. □
Particular cases are:

1) if $a = 2, b = 2$ then $S_{k,n+m} = F_{k,m}S_{k,n+1} + F_{k,m-1}S_{k,n}$,

2) if $a = 0$, $b = 1$, $k = 1$ then $F_{n+m} = F_m F_{n+1} + F_{m-1}F_n$,
3) if $a = 0$, $b = 1$, $k = 2$ then $P_{k,n+m} = P_{k,m}P_{k,n+1} + P_{k,m-1}P_{k,n}$ where P_n is the n^{th} term of Pell sequence.

It is a well-known fact that the ratio of consecutive Fibonacci numbers approaches $\phi = \frac{1+\sqrt{5}}{2}$ which is called the Golden ratio (see [6]). The following theorem is a natural generalization of this result to G-k-FLNs.

Corollary 2.13: Let$\{G_{k,n}\}$ be a nonzero Generalized k-Fibonacci-Like sequence. Then $\lim_{n\to\infty} \frac{G_{k,n+1}}{G_{k,n}} = r_1$.

Proof: By Theorem 2.10, there exist $a,\, b \in \mathbb{N}$ with at least one of them is nonzero such that $G_{k,n} = bF_{k,n} + aF_{k,n-1}$ for all $n \geq 0$, and by Proposition 6 of [7], we have $\lim_{n\to\infty} \frac{F_{k,n+1}}{F_{k,n}} = r_1$, therefore,

$$\lim_{n\to\infty} \frac{G_{k,n+1}}{G_{k,n}} = \lim_{n\to\infty} \frac{bF_{k,n+1}+aF_{k,n}}{bF_{k,n}+aF_{k,n-1}}$$

$$= \lim_{n\to\infty} \frac{b\frac{F_{k,n+1}}{F_{k,n}}+a}{b+a\frac{F_{k,n-1}}{F_{k,n}}} = \frac{br_1+a}{b+\frac{a}{r_1}} = r_1.$$

In particular, if $k = 1$, then $\lim_{n\to\infty} \frac{G_{k,n+1}}{G_{k,n}} = \frac{1+\sqrt{5}}{2}$ $\square$.

One can also provide a direct proof to the corollaries 2.12, 2.13 using the Binet's formulae (3) and (4) instead of deriving them as a consequence of Theorem 2.10 followed by corresponding results involving k-Fibonacci number.

III. Concluding Remarks

Several identities involving k-FLNs have been proved by Y.K.Panwar et.al.(see [2]) using the formula (1) are incomplete or false. Valid proofs for many of them can be offered with correct Binet's formula derived in this paper (see Corollary 2.6). Furthermore, several identities involving G-k-FLNs have been studied by us and will be communicated in the forthcoming paper.

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