On The Solutions of Two Sum of Divisor Equations

John Rafael M. Antalan

***Abstract*—In this paper, we find all the solutions of the sum of divisor equation $\sigma(n_1) + \sigma(n_2) = 2(n_1 + n_2)$ and show that the sum of divisor equation $\sigma(n_1)+\sigma(n_2) = 2n_1n_2$ has a solution by using the concept of abundant, deficient and perfect numbers.We then characterize those that are clearly not solutions and those that are possible solutions of the second sum of divisor equation. Lastly, we end this paper by posting some problems related to the topic.**

***Index Terms*—Sum of divisor function, Deficient Numbers, Abundant Numbers, Perfect Numbers, Almost Perfect Numbers, Quasiperfect Numbers .**

***MSC 2010 Codes* – 11A25**

I. INTRODUCTION

IN [1,pp.119,no.16], it is a problem to show that Goldbach's Conjecture implies that for each even integer $2n$ there exist integers n_1 and n_2 with $\sigma(n_1) + \sigma(n_2) = 2n$. The power of Goldbach's Conjecture in this certain type of problem can be seen by considering few even integers.

Let $E = \{2, 4, 6, 8, 10, ...\}$, instead of finding for n_1 and n_2 that will satisfy the equation above in each of the elements of E, (i.e. if $2n = 2$ then $n_1 = 1$ and $n_2 = 1$ or if $2n = 8$ then $n_1 = 1$ and $n_2 = 4$) the existence of such integers for all elements of E follow at once from the Goldbach Conjecture. However, Goldbach's Conjecture is not yet proven as stated in [2].

More specific than equation $\sigma(n_1) + \sigma(n_2) = 2n$, we consider in this paper the equation

$$\sigma(n_1) + \sigma(n_2) = 2(n_1 + n_2) \ (\mathbf{1})$$

and we give all its solutions in the set of counting numbers. The result is so simple and the method in finding its solutions is so easy but the attack in getting the solutions is more important than the easiness of finding it. Also in this paper, we show that a solution to the Sum of Divisor Equation

$$\sigma(n_1) + \sigma(n_2) = 2n_1n_2 \ (\mathbf{2})$$

exists.

The motivation in finding the solution of the two given sum of divisor equation was due to the fact that no one tried to solve it as a quick search in the web reveals. Finally, we end this paper by stating a problem related to the topic.

II. PRELIMINARIES

Before going to the main results, we consider a definition.

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Definition 2.1 A positive integer n is perfect whenever $\sigma(n) = 2n$. It is abundant whenever $\sigma(n) > 2n$ while deficient whenever $\sigma(n) < 2n$. □

We note that a positive integer n maybe either abundant, deficient or perfect.[3]

As a matter of natation we denote n to be $k - deficient$ or $k - abundant$ if n is deficient or abundant by a positive integer k respectively. In other words n is $k - deficient$ if $\sigma(n) = 2n - k$ and $k - abundant$ if $\sigma(n) = 2n + k$.

With this definitions and notations we are now in the position to show our main results.

III. RESULTS

We now present our results in this section.

Theorem 3.1 The solutions of the sum of divisor equation $\sigma(n_1) + \sigma(n_2) = 2(n_1 + n_2)$ are given by the following sets:

1) $S_1 = \{(n_1, n_2)|n_1, n_2 \in P\}$ where P is the set of perfect numbers.
2) $S_2 = \{(n_1, n_2)|n_1 \in D, n_2 \in A\}$ where D and A are respectively the set of all deficient and abundant numbers.
3) $S_3 = \{(n_1, n_2)|n_1 \in A, n_2 \in D\}$.

Proof: Let $n_1 \in Z^+$. Note that n_1 can be either perfect, abundant or deficient.

Case 1: If n_1 is perfect then $\sigma(n_1) = 2n_1$. We assume that it is a solution to the given sum of divisor equation. So the sum of divisor equation $\sigma(n_1)+\sigma(n_2) = 2(n_1 + n_2)$ becomes $2n_1+\sigma(n_2) = 2n_1+2n_2$ which implies $\sigma(n_2) = 2n_2$. Thus, n_2 is also a perfect number solution to the given sum of divisor equation.

Case 2: If n_1 is deficient by k then $\sigma(n_1) = 2n_1 - k$. We assume that it is a solution to the given sum of divisor equation. So the sum of divisor equation $\sigma(n_1) + \sigma(n_2) = 2(n_1 + n_2)$ becomes $2n_1 - k + \sigma(n_2) = 2n_1 + 2n_2$ which implies $\sigma(n_2) = 2n_2 + k$ is a $k-abundant$ solution to the given sum of divisor equation.

Case 3: If n_1 is a $k - abundant$ number solution to the given sum of divisor equation is proved similarly.

This completes the proof of theorem 1. □

Before presenting the second theorem, we need a preliminary proposition and a definition.

Proposition 3.1 The inequality $n_1n_2 > n_1 + n_2$ takes all positive integers n_1 and n_2 as integral solutions, except for a finite number of cases. In particular, the given inequality will not hold for (n_1, n_2) with $n_1 \in Z^+$ and $n_2 = 1$; with $n_1 = 1$ and $n_2 \in Z^+$ and $n_1 = n_2 = 2$.

Proof: Clearly, when $n_1 = 1$ and $n_2 \in Z^+$ or when $n_1 \in Z^+$ and $n_2 = 1$ the given inequality is not satisfied.Also for $n_1 = n_2 = 2$ the given inequality is not satisfied. For the remaining cases, consider the given inequality $n_1n_2 > n_1 + n_2$. This implies that $n_1 > \frac{n_2}{n_2-1}$ where $n_2 \neq \{1, 2\}$. If $n_2 = 3$, then $n_1 > \frac{3}{2} = 1.5$. Thus $\{2, 3\}$ is a solution to the given inequality, in fact it is the "minimal solution". For when $n_2 > 3$, $\frac{n_2}{n_2-1} \longrightarrow 1$ so $n_1 \geq 2$. Since the given inequality also implies $n_2 > \frac{n_1}{n_1-1}$ for $n_1 \neq \{1, 2\}$ it follows that if $(n_1 = k, n_2 = l)$ is a solution, then $(n_1 = l, n_2 = k)$ is also a solution. And we are done. □

Definition 3.1 A positive integer n is called an almost perfect number whenever $\sigma(n) = 2n - 1$. □

Note that integer powers of 2 are almost perfect number. [4]

With these, we have...

Theorem 3.2 There exists a solution to the sum of divisor equation $\sigma(n_1) + \sigma(n_2) = 2(n_1n_2)$ in the set of deficient numbers. In particular the set $P_1 = \{(n_1, n_2)|n_1 = 1, n_2 \in APN\}$ where APN is the set of almost perfect number and the set $P_2 = \{(n_1, n_2)|n_1 \in APN, n_2 = 1\}$ are its solution.

Proof: Let n_1 and n_2 be deficient numbers with $\sigma(n_1) = 2n_1 - j$ and $\sigma(n_2) = 2n_2 - k$. So $\sigma(n_1) + \sigma(n_2) = 2n_1 + 2n_2 - j - k = 2(n_1 + n_2) - j - k$. But we want that $\sigma(n_1) + \sigma(n_2) = 2(n_1n_2)$. Thus, it must be the case that $2(n_1 + n_2) - j - k = 2(n_1n_2)$. Using proposition 1, we see that n_1 or n_2 must be 1 and the other one is an integer in order for the last displayed equation to hold. If $n_1 = 1$ then $\sigma(n_1) = 1$ and in return $\sigma(n_1) + \sigma(n_2) = 2(n_2) - k + 1$. So $\sigma(n_1) + \sigma(n_2) = 1 + 2(n_2) - k = 2(1n_2)$ which means $k = 1$. So the set P_1 defined above is a solution. Similarly, setting $n_2 = 1$ will yield the solution P_2.□

Example 3.1 It is easy to verify that 1 and 2 (an almost perfect number) is a solution to the sum of divisor equation (2). □

The next proposition gives us those set that are not solutions of (2).

Proposition 3.2 Any pair (n_1, n_2) with $n_1, n_2 \neq 1$ coming from the set of (i) Perfect Numbers and (ii) Deficient Numbers is not a solution to $\sigma(n_1) + \sigma(n_2) = 2(n_1n_2)$.

Proof: If n_1 and n_2 is perfect. Then $\sigma(n_1) = 2n_1$ and $\sigma(n_2) = 2n_2$. This means that $\sigma(n_1) + \sigma(n_2) = 2(n_1 + n_2)$. In order to satisfy the given equation, it must be that $2n_1n_2 = 2(n_1 + n_2)$. By proposition1 the only solution of the latter equality is $n_1 = n_2 = 2$ but with this value of n_1 and n_2 the main equality is not satisfied.

The case when n_1 and n_2 is deficient follows from theorem 3.2.

Lastly, we consider the case when one of the solution is perfect and the other is deficient. Without laws of generality, let n_2 be a perfect number and n_1 be a $k - deficient$ number. So, $\sigma(n_2) = 2n_2$ and $\sigma(n_1) = 2(n_1) - k$. In effect $\sigma(n_1) + \sigma(n_2) = 2(n_1 + n_2) - k$. Using proposition 1 again we have $2(n_1n_2) > 2(n_1 + n_2) - k$ for all positive integers except for $n_1 = 1$ while n_2 is a positive integer and the other way around. If $n_1 = 1$ this implies $\sigma(n_1) = 1$ and in return $\sigma(n_1) + \sigma(n_2) = 1 + 2(n_2) \neq 2(n_2)$. Also, if $n_2 = 1$ this implies $\sigma(n_2) = 1$ and in return $\sigma(n_1) + \sigma(n_2) = 2(n_1) + 1 \neq 2(n_1 1)$.□

At this point, we arrived at the realization that if other solutions to the sum of divisor equation $\sigma(n_1) + \sigma(n_2) = 2(n_1n_2)$ exist then one of n_1 or n_2 must be an abundant number.

IV. PROBLEMS

We begin this section by an example.

Example 4.1 It is easy to verify that 6 and 28 being perfect numbers and 12 and 5 being $4 - abundant$ and $4 - deficient$ numbers respectively are solutions of $\sigma(n_1) + \sigma(n_2) = 2(n_1 + n_2)$ which agrees to theorem 3.1. □

Problem 4.1 From theorem 3.1, we know that if $n_1 \neq 1$ is a $k - deficient$ number solution to the sum of divisor equation (1), then n_2 as a solution must be $k - abundant$ number. However, given a $k - deficient$ number is there an easy way or are there any methods available in finding a $k - abundant$ number? Similarly, given a $k - abundant$ number is there an easy way or are there any methods available in finding a $k - deficient$ number?

Definition 4.1 A positive integer n is called quasiperfect if it satisfies $\sigma(n) = 2n + 1$. □

Problems 2 and 3 below are related.

Problem 4.2 Are there any solutions (n_1, n_2) to equation (1), such that n_1 is an almost perfect number and n_2 is a quasiperfect number or the other way around?

Problem 4.3 Is there a quasiperfect number?[4]

Problem 4.4 Consider the sum of divisor equation $\sigma(n_1) + \sigma(n_2) = 2(n_1n_2)$. From proposition 3.2, we see that if other solutions to this equation exist then one of n_1 or n_2 must be an abundant number. Is there really a solution to this equation other than the one we find in theorem 3.2?

Definition 4.2 A number theoretic function (function whose domain is the set of positive integer) f is said to be additive if $f(n_1 + n_2) = f(n_1) + f(n_2)$ for positive integers n_1 and n_2. □

The last problem concerns about perfect numbers.

Problem 4.5 The sum of divisor σ is not additive. Is it possible that there are perfect numbers n_1 and n_2 for which σ is additive so that $\sigma(n_1 + n_2) = \sigma(n_1) + \sigma(n_2) = 2(n_1 + n_2)$? In other words, is there a perfect number that is a sum of two other perfect numbers?

V. CONCLUSION

By considering the set of of abundant, deficient and perfect numbers; the problem in finding (theoretically) the solutions of the two sum of divisor equation $\sigma(n_1) + \sigma(n_2) = 2(n_1 + n_2)$ and $\sigma(n_1) + \sigma(n_2) = 2n_1n_2$ is not that difficult. The solutions of the sum of divisor equation $\sigma(n_1) + \sigma(n_2) = 2(n_1 + n_2)$ are given by the following sets:

1) $S_1 = \{(n_1, n_2)|n_1, n_2 \in P\}$ where P is the set of perfect numbers.
2) $S_2 = \{(n_1, n_2)|n_1 \in D, n_2 \in A\}$ where D and A are respectively the set of all deficient and abundant numbers.
3) $S_3 = \{(n_1, n_2)|n_1 \in A, n_2 \in D\}$.

While a solution of the sum of divisor equation $\sigma(n_1) + \sigma(n_2) = 2n_1n_2$ is given by $P_1 = \{(n_1, n_2)|n_1 = 1, n_2 \in APN\}$ where APN is the set of almost perfect number and the set $P_2 = \{(n_1, n_2)|n_1 \in APN, n_2 = 1\}$. And if there are others it must be that abundant numbers are involve.

ACKNOWLEDGMENT

The author would like to thank everyone who helped him in the creation and publication of this manuscript. The reviewer for helpful comments and suggestions for the improvization of the manuscript. His colleagues at the Department of Mathematics and Physics headed by Dr. Amelia Z. Daquioag, the dean of the College of Arts and Sciences Dr. Renato G. Reyes and the Central Luzon State University headed by Dr. Tereso Abella and Dr. Ruben Sevilleja for the support. His faculty friends Julius Fergy T. Rabago, Aldous Caesar F. Bueno, Eduard C. Taganap and Sir Melchor Cupatan, his former Prof. at the University of the Philippines Baguio Sir Dexter Jane L. Indong for being his motivation and inspiration. Lastly the author is indebted to his family and to his love Josephine Joy V. Tolentino for the unconditional support and for cheering him up whenever he was down.

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