Summation Identities for k-Fibonacci and k-Lucas Numbers using Matrix Methods

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***Abstract*—In this paper we defined general matrices $M_k(n, m)$, $T_{k,n}$ and $S_k(n, m)$ for k-Fibonacci number. Using these matrices we find some new summation properties for k-Fibonacci and k-Lucas numbers.**

***Index Terms*—k-Fibonacci number, k-Lucas number, Recurrence relation, Matrix algebra.**

***MSC 2010 Codes* – 11B39, 11B37,11B99**

I. Introduction

MANY authors defined generalized Fibonacci numbers by varying initial conditions and recurrence relation [1-7]. This paper represents an interesting investigation about some special relations between matrices, $k-$ Fibonacci numbers and $k-$ Lucas numbers.This investigation is valuable to obtain new $k-$ Fibonacci ,$k-$ Lucas identities by different methods.This paper contributes to $k-$ Fibonacci,$k-$ Lucas numbers literature,and encourage many researchers to investigate the properties of such number numbers.

This paper is organized as follows. Section II contains some preliminary results. In Section III, the matrix $M_k(n, m)$ is defined and using it some properties of $k-$ Fibonacci and $k-$ Lucas numbers are derived. In Section IV, the Matrix $T_{k,n}$ is defined and using it some properties of $k-$ Fibonacci and $k-$ Lucas numbers are derived. In Section V, the Matrix $S_k(n, m)$ is defined and using it some properties of $k-$ Fibonacci and $k-$ Lucas numbers are derived.

II. Some Preliminary Results

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 1. *The $k-$ Fibonacci number $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined as, $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$, with $F_{k,0} = 0, F_{k,1} = 1$,for $n \geq 1$*

Definition 2. *The $k-$ Lucas number $\{L_{k,n}\}_{n\in\mathbb{N}}$ is defined as, $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$, with $L_{k,0} = 2, L_{k,1} = k$,for $n \geq 1$*

Characteristic equation of the initial recurrence relation is,

$$r^2 - kr - 1 = 0, \tag{1}$$

and characteristic roots are

$$r_1 = \frac{k + \sqrt{k^2 + 4}}{2}$$

and

$$r_2 = \frac{k - \sqrt{k^2 + 4}}{2},$$

Characteristic roots verify the properties,

$$r_1 - r_2 = \sqrt{k^2 + 4} = \sqrt{\Delta}, \tag{2}$$

$$r_1 + r_2 = k, \tag{3}$$

$$r_1.r_2 = -1. \tag{4}$$

Binet forms for $F_{k,n}$ and $L_{k,n}$ are

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \tag{5}$$

and

$$L_{k,n} = r_1^n + r_2^n \tag{6}$$

From the definition of the $k-$Fibonacci numbers,one may deduce the value of any k-Fibonacci number by simple substitution on the corresponding $F_{k,n}$. For example,the seventh element of the $4-$Fibonacci number, $F_{4,7}$ is 5473. By doing $k = 1; 2; 3; \cdots$ the respective k-Fibonacci numbers are obtained. Sequence $\{F_{1,n}\}$ is the classical Fibonacci number and $\{F_{2,n}\}$ is the Pell number. It is worthy to be noted that only the first 10 $k-$Fibonacci numbers are referenced in The On-Line Encyclopedia of Integer Sequences [10] with the numbers given in Table 1. For k even with $12 \leq k \geq 62$ sequences $\{F_{k,n}\}$ are referenced without the first term $F_{k,0} = 0$ in [10].

The first $11k$-Fibonacci sequences as numbered in The On-Line Encyclopedia of Integer Sequences [10]:

$F_{1,n}$	$A000045$
$F_{2,n}$	$A000129$
$F_{3,n}$	$A006190$
$F_{4,n}$	$A001076$
$F_{5,n}$	$A052918$
$F_{6,n}$	$A005668$
$F_{7,n}$	$A054413$
$F_{8,n}$	$A041025$
$F_{9,n}$	$A099371$
$F_{10,n}$	$A041041$
$F_{11,n}$	$A049666$

The most commonly used matrix in relation to the recurrence relation (1) is

$$M = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \tag{7}$$

which for $k = 1$ reduces to the ordinary Q- matrix studied in [8, 14]. In this paper, we define more general matrices $M_k(n, m), T_k(n), S_k(n, m)$for Q-matrix . We use these matrices

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to develop various summation identities involving terms from the numbers $F_{k,n}$ and $L_{k,n}$.

Several identities for $F_{k,n}$ and $L_{k,n}$ are proved using Binet forms in [11-14]. Some of these are listed below.

$$F_{k,n+1} + F_{k,n-1} = L_{k,n} \tag{8}$$

$$F_{k,n+1} + L_{k,n-1} = \Delta F_{k,n} \tag{9}$$

$$F_{k,2n} - 2(-1)^n = \Delta F_{k,n}^2 \tag{10}$$

$$F_{k,m+n} - (-1)^m L_{k,n-m} = F_{k,m} L_{k,n} \tag{11}$$

$$L_{k,m+n} - (-1)^m L_{k,n-m} = \Delta F_{k,m} F_{k,n} \tag{12}$$

$$F_{k,m+n} F_{k,n-m} - F_{k,n}^2 = (-1)^{n-m+1} F_{k,m}^2 \tag{13}$$

$$L_{k,m+n} L_{k,n-m} - L_{k,n}^2 = (-1)^{n-m} F_{k,m}^2 \tag{14}$$

$$F_{k,m+n} F_{k,r+m} - (-1)^m F_{k,n} F_{k,r} = F_{k,m} F_{k,n+r+m}$$

$$L_{k,mn} L_{k,n} + \Delta F_{k,mn} F_{k,n} = 2L_{k,(m+1)n} \tag{15}$$

$$F_{k,mn} L_{k,n} + L_{k,mn} F_{k,n} = 2F_{k,(m+1)n} \tag{16}$$

$$L_{k,m+n}^2 + (-1)^{m-1} L_{k,n}^2 = \Delta F_{k,2n+m} F_{k,m} \tag{17}$$

$$L_{k,m+n} L_{k,n} + (-1)^{m+1} L_{k,n-m} L_{k,n} = \Delta F_{k,2n} F_{k,m}$$

$$F_{k,m+2rn} F_{k,2n+m} + (-1)^{m+1} F_{k,2rn} F_{k,2n} = F_{k,2(r+1)n+m} F_{k,m}$$

In [14], matrix M is generalized using PMI

$$M^n = \begin{bmatrix} F_{k,n+1} & F_{k,n} \\ F_{k,n} & F_{k,n-1} \end{bmatrix}$$

where n is an integer.

III. The Matrix $M_k(n, m)$

We now give a generalization of the matrix M and use it to produce summation identities involving terms from the sequences $F_{k,n}$ and $L_{k,n}$.

Definition 3.

$$M_k(n,m) = \begin{bmatrix} F_{k,n+m} & (-1)^{m+1} F_{k,n} \\ F_{k,n} & (-1)^{m+1} F_{k,n-m} \end{bmatrix} \tag{18}$$

where m and n are integers.

Theorem 4. *Let $M_k(n,m)$ be a matrix as in (18). Then*

$$M_k(n,m)^r = F_{k,m}^r \begin{bmatrix} F_{k,rn+m} & (-1)^{m+1} F_{k,rn} \\ F_{k,rn} & (-1)^{m+1} F_{k,rn-m} \end{bmatrix}$$

Proof: This result can be easily established using the Principle of Mathematical Induction. ■

We find that characteristic equation of $M_k(n,m)$ is

$$\lambda^2 - F_{k,m} F_{k,n} \lambda + (-1)^n F_{k,m}^2 = 0 \tag{19}$$

and by Cauchy-Hamilton theorem

$$M_k(n,m)^2 - F_{k,m} F_{k,n} M_k(n,m) + (-1)^n F_{k,m}^2 I = 0$$

Multiplying both sides of equation (25) by $M_k(n,m)^t$ gives

$$(F_{k,m} F_{k,n} M_k(n,m) - (-1)^n F_{k,m}^2 I)^r M_k(n,m)^t$$

$$= M_k(n,m)^{2r+t}$$

and expanding gives

$$\sum_{i=0}^{i=r} \binom{r}{i} (-1)^{(r-1)(n+1)} F_{k,m}^{2r-1} F_{k,n}^i M_k(n,m)^{i+t}$$

$$= M_k(n,m)^{2r+t}$$

Using (18) to equate upper left entries gives

$$\sum_{i=0}^{i=r} \binom{r}{i} (-1)^{(r-1)(n+1)} L_{k,n}^i F_{k,(i+t)n+m}$$

$$= F_{k,(2r+t)n+m}$$

In similar way we can obtain

$$\sum_{i=0}^{i=r} \binom{r}{i} (-1)^{n(r-i)} F_{k,2in+m}$$

$$= L_{k,n}^r F_{k,rn+m}$$

$$\sum_{i=0}^{i=2r} \binom{2r}{i} (-1)^{2nr-i(n-1)} F_{k,2in+m}$$

$$= \Delta^r F_{k,n}^{2r} F_{k,2rn+m}$$

$$\sum_{i=0}^{i=2r+1} \binom{2r+1}{i} (-1)^{n(2r-i+1)+i+1} F_{k,2in+m}$$

$$= \Delta^r F_{k,n}^{2r+1} L_{k,(2r+1)n+m}$$

$$\sum_{i=0}^{i=2r} \binom{2r}{i} (-1)^i 2^i L_{k,n}^{2r-i} F_{k,in+m}$$

$$= \Delta^r F_{k,n}^{2r} F_{k,m}$$

IV. The Matrix $T_{k,n}$

We now give another generalization of the matrix M and use it to produce summation identities involving terms from the sequences $F_{k,n}$ and $L_{k,n}$.

Definition 5.

$$T_{k,n} = \begin{bmatrix} L_{k,n} & F_{k,n} \\ \Delta F_{k,n} & L_{k,n} \end{bmatrix} \tag{20}$$

where n is an integer.

Theorem 6. *Let $T_{k,n}$ be a matrix as in (20) then*

$$T_{k,n}^m = 2^{m-1} \begin{bmatrix} L_{k,nm} & F_{k,nm} \\ \Delta F_{k,nm} & L_{k,nm} \end{bmatrix} \tag{21}$$

Proof: : Using Principle of Mathematical induction (PMI),from (20) it is clear that the result is true for $m = 1$. Assume that the result is true for m (IH)

$$T_{k,n}^m = 2^{m-1} \begin{bmatrix} L_{k,nm} & F_{k,nm} \\ \Delta F_{k,nm} & L_{k,nm} \end{bmatrix}$$

Now consider

$$T_{k,n}^{m+1} = T_{k,n}^{m} T_{k,n}$$

$$= 2^{m-1} \begin{bmatrix} L_{k,nm} & F_{k,nm} \\ \Delta F_{k,nm} & L_{k,nm} \end{bmatrix} \cdot \begin{bmatrix} L_{k,n} & F_{k,n} \\ \Delta F_{k,n} & L_{k,n} \end{bmatrix}$$

$$= 2^{m-1} \begin{bmatrix} L_{k,mn}L_{k,n} + \Delta F_{k,nm}F_{k,n} & L_{k,mn}F_{k,n} + F_{k,mn}L_{k,n} \\ \Delta(L_{k,mn}F_{k,n} + F_{k,mn}L_{k,n}) & L_{k,mn}L_{k,n} + \Delta F_{k,nm}F_{k,n} \end{bmatrix}$$

Using (14) and (15) gives

$$= 2^{m} \begin{bmatrix} L_{k,n(m+1)} & F_{k,n(m+1)} \\ \Delta F_{k,n(m+1)} & L_{k,n(m+1)} \end{bmatrix}$$

Hence proof. ∎

The characteristic equation of $T_{k,n}$ is

$$\lambda^2 - 2L_{k,n}\lambda + 4(-1)^n = 0 \tag{22}$$

and by Cauchy-Hamilton theorem

$$T_{k,n}^2 - 2L_{k,n}T_{k,n} + 4(-1)^n I = 0 \tag{23}$$

Using (16) and (17) gives,we see that

$$T_{k,n}^{m} T_{k,t} = 2^{m} \begin{bmatrix} L_{k,nm+t} & F_{k,nm+t} \\ \Delta F_{k,nm+t} & L_{k,nm+t} \end{bmatrix} \tag{24}$$

Consider the case $n = 1$,we can show by PMI

$$T_{k,1}^{m} = 2^{m-1}(F_{k,m}T_{k,1} + 2F_{k,m-1}I \tag{25}$$

where $m \geq 2$

Equation (25) produces

$$\sum_{i=0}^{i=r} \binom{r}{i} F_{k,n-1}^{r-i} F_{k,n}^{i} F_{k,i+s+t} = F_{k,nr+s+t} \tag{26}$$

The methods applied to $M_k(n,m)$ in previous section when applied to $T_{k,n}$ produce most of the summation identities that we have obtained so far.

V. The Matrix $S_k(n,m)$

We now give one more generalization of the matrix M and use it to produce summation identities involving terms from the sequences $F_{k,n}$ and $L_{k,n}$.

Definition 7.

$$S_k(n,m) = \begin{bmatrix} L_{k,n+m} & (-1)^{m+1}L_{k,n} \\ L_{k,n} & (-1)^{m+1}L_{k,n-m} \end{bmatrix} \tag{27}$$

where n, m are integers.

Theorem 8. *Let $S_k(n,m)$ be a matrix as in (27) then for all integer r*

$$S_k(n,m)^{2r}$$

$$= F_{k,m}^{2r-1}\Delta^r \begin{bmatrix} F_{k,2rn+m} & (-1)^{m+1}F_{k,2rn} \\ F_{k,2rn} & (-1)^{m+1}F_{k,2rn-m} \end{bmatrix}$$

$$S_k(n,m)^{2r-1}$$

$$= F_{k,m}^{2r-2}\Delta^{r-1} \begin{bmatrix} L_{k,2(r-1)n+m} & (-1)^{m+1}L_{k,2(r-1)n} \\ L_{k,2(r-1)n} & (-1)^{m+1}L_{k,2(r-1)n-m} \end{bmatrix}$$

Proof: This theorem can be established using the Principle of Mathematical Induction. ∎

The characteristic equation of $S_k(n,m)$ is

$$\lambda^2 - \Delta F_{k,n}F_{k,m}\lambda - \Delta(-1)^n F_{k,n}^2 = 0$$

and by Cauchy-Hamilton theorem

$$S_k(n,m)^2 - \Delta F_{k,n}F_{k,m}S_k(n,m) - \Delta(-1)^n F_{k,n}^2 I = 0$$

Manipulating above equation gives

$$\Delta F_{k,m}(F_{k,n}S_k(n,m) + (-1)^n F_{k,m}I) = S_k(n,m)^2$$

and

$$(2S_k(n,m) - \Delta F_{k,n}F_{k,m}I) = \Delta F_{k,m}^2 L_{k,n}^2$$

Hence

$$\Delta^r F_{k,m}^r (F_{k,n}S_k(n,m) + (-1)^n F_{k,m}I)^r = S_k(n,m)^{2r}$$

$$(2S_k(n,m) - \Delta F_{k,n}F_{k,m}I)^{2r} = \Delta^r F_{k,m}^{2r} L_{k,n}^{2r} I$$

$$(2S_k(n,m) - \Delta F_{k,n}F_{k,m}I)^{2r+1}$$

$$= \Delta^r F_{k,m}^{2r} L_{k,n}^{2r} (2S_k(n,m) - \Delta F_{k,n}F_{k,m}I)$$

Now expanding previous four equations and equating upper left entries of the relevant matrices gives respectively to

$$\sum_{i=0,i-even}^{i=r} \binom{r}{i} (-1)^{n(r-i)+1} \Delta^{\frac{i-1}{2}} F_{k,n}^{i} L_{k,in+m}$$

$$+ \sum_{i=0,i-odd}^{i=r} \binom{r}{i} (-1)^{n(r-i)} \Delta^{\frac{i}{2}} F_{k,n}^{i} F_{k,in+m}$$

$$= F_{k,2nr+m}$$

$$\sum_{i=0,i-even}^{i=2r} \binom{2r}{i} 2^i \Delta^{\frac{2r-i}{2}} F_{k,n}^{2r-i} F_{k,in+m}$$

$$- \sum_{i=0,i-odd}^{i=2r-1} \binom{2r}{i} 2^i \Delta^{\frac{2r-1-i}{2}} F_{k,n}^{2r-i} L_{k,in+m}$$

$$= L_{k,n}^{2r} F_{k,m}$$

$$\sum_{i=0,i-odd}^{i=2r+1} \binom{2r+1}{i} 2^i \Delta^{\frac{2r+3-i}{2}} F_{k,n}^{2r+1-i} F_{k,in+m}$$

$$- \sum_{i=0,i-even}^{i=2r-1} \binom{2r+1}{i} 2^i \Delta^{\frac{2r+2-i}{2}} F_{k,n}^{2r+1-i} L_{k,in+m}$$

$$= \Delta L_{k,n}^{2r+1} F_{k,m}$$

VI. Conclusion

The matrices $M_k(n,m)$,$T_{k,n}$ and $S_k(n,m)$ are defined for k Fibonacci and k Lucas numbers and some new summation properties are established. Same approach can be applied for generalized Fibonacci numbers.

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