Solving the k^{th} Term of Natividad's Fibonacci-like Sequence

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***Abstract*—In this paper, the formula for the k^{th} term of Natividad's Fibonacci-like sequence was obtained.**

***Index Terms*—Fibonacci sequence, Fibonacci-like sequence, k^{th} term of a sequence.**

***MSC 2010 Codes* – 11B39, 11B50**

I. Introduction

IN [1], Natividad introduced a Fibonacci-like sequence by inserting terms between to numbers a and b where $a < b$. That is given two numbers a and b the terms $x_1,\ x_2,\ \ldots, x_{n-1}$ will be inserted so that the sequence $a,\ x_1,\ x_2,\ \ldots, x_{n-1},\ b$ will satisfy the Fibonacci sequence's recurrence relation

$$x_k = x_{k-1} + x_{k-2},\ n \geq 2$$

with initial values $x_0 = a$ and $x_1 = \frac{b-aF_n}{F_{n+1}}$ where F_k is the kth Fibonacci number.

The term x_1 was derived in [1]. Now we will be interested in the general formula of the terms of this sequence.

II. Main Result

Theorem 2.1: The k^{th} term of Natividad'd Fibonacci-like sequence is

$$x_k = \frac{bF_k - a(F_nF_k - F_{n+1}F_{k-1})}{F_{n+1}}$$

where F_j is the j^{th} Fibonacci number.

Proof:

Since the sequence is Fibonacci-like then its k^{th} is of the form

$$x_k = c_1\phi^k + c_2(1-\phi)^k$$

where $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio.

Using the initial values, we will come up with the system

$$\begin{cases} c_1 + c_2 = a \\ c_1\phi + c_2(1-\phi) = x_1 \end{cases}$$

Solving this yields

$$c_1 = \frac{x_1 - a(1-\phi)}{\sqrt{5}}$$

$$c_2 = \frac{a\phi - x_1}{\sqrt{5}}$$

Hence

$$\begin{aligned} x_k &= \left[\frac{x_1 - a(1-\phi)}{\sqrt{5}}\right]\phi^k + \left[\frac{a\phi - x_1}{\sqrt{5}}\right](1-\phi)^k \\ &= \left[\frac{x_1\phi^k + a\phi^{k-1}}{\sqrt{5}}\right] + \left[\frac{-a(1-\phi)^{k-1} - x_1(1-\phi)^k}{\sqrt{5}}\right] \\ &= x_1\left[\frac{\phi^k - (1-\phi)^k}{\sqrt{5}}\right] + a\left[\frac{\phi^{k-1} - (1-\phi)^{k-1}}{\sqrt{5}}\right] \end{aligned}$$

Note that the m^{th} Fibonacci number is given by

$$F_m = \frac{\phi^m - (1-\phi)^m}{\sqrt{5}}$$

From this, the result will be

$$\begin{aligned} x_k &= x_1F_k + aF_{k-1} \\ &= \left[\frac{b - F_na}{F_{n+1}}\right]F_k + aF_{k-1} \\ &= \frac{bF_k - aF_nF_k + aF_{n+1}F_{k-1}}{F_{n+1}} \\ &= \frac{bF_k - a(F_nF_k - F_{n+1}F_{k-1})}{F_{n+1}} \end{aligned}$$

This completes the proof. □

III. Conclusion

The formula for the k^{th} term of Natividad's Fibonacci-like sequence was obtained. Through the relationship of Fibonacci-like sequences to the Fibonacci numbers and systems of linear equations, the formula was obtained.

References

[1] L. Natividad, "Deriving a formula in solving Fibonacci-like sequence", *International Journal of Mathematics and Scientfic Computing*, vol.1, no.1, pp. 19-21, 2011.

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