Dominator Coloring of Some Wheel Related Graphs

Samir K. Vaidya and Minal S. Shukla

***Abstract*—A dominator coloring is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G. We investigate dominator chromatic number of the degree splitting graph of wheel W_n and helm H_n.**

***Index Terms*—coloring, domination number, dominator coloring, degree splitting graph.**

***MSC 2010 Codes* – 05C15, 05C38, 05C76**

I. Introduction

THROUGHOUT this work the graph G we mean a simple, finite, connected and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A proper k-coloring of a graph G is a function $f : V(G) \to \{1,\ 2,\ \dots\ k\}$ such that $f(u) \neq f(v)$ for all $uv \in E(G)$. The chromatic number $\chi(G)$ of G is the smallest integer k such that G admits a proper coloring using k colors. The set of vertices with same color is called the color class.

The open neighbourhood of $v \in V(G)$ is the set $N(v) = \{u \in V(G) :\ uv \in E(G)\}$ while the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$.
The set $S \subseteq V(G)$ of vertices in a graph G is called a dominating set if every vertex $v \in V(G)$ is either an element of S or is a neighbour of at least one element of S. The dominating set with minimal cardinality is called the γ- set and its cardinality is the domination number $\gamma(G)$.

A proper coloring is called dominator coloring if each vertex of the graph dominates every vertex of some color class. The dominator chromatic number $\chi_d(G)$ is the smallest integer k such that G admits a dominator coloring with k-colors. It is obvious that every vertex dominates its own color class. The concept of dominator coloring was introduced by Gera *et al.*[5]. The dominator chromatic number for some graph families is obtained by Kavitha and David [7–9] and it has been extensively studied by Arumugam *et al.*[1] while dominator coloring of bipartite graphs is discussed by Gera [4]. The dominator chromatic number of various graphs are investigated by Merouane *et al.* [10]. The dominator coloring for degree splitting graph of various graph families have been investigated by Vaidya and Shukla[11].

Dr. Samir K. Vaidya is a Professor at the Department of Mathematics, Saurashtra University, Rajkot, Gujarat-360005, India. (E-mail: samirkvaidya@yahoo.co.in)

Minal S. Shukla is with Department of Mathematics, Atmiya Institute of Technology and Science, Rajkot, Gujarat-360005, India. (E-mail:shuklaminal19@gmail.com).

II. Some Preliminary Results

We will give brief summary of some definitions and existing results required for the present work.

Definition 2.1. Let G be a graph with $V(G) = S_1 \cup S_2 \cup \ldots \cup S_t \cup T$ where each S_i is a set of all the vertices of the same degree with at least two elements $T = V(G) \setminus \bigcup_{i=1}^{t} S_i$. The *degree splitting* graph of G, denoted by $DS(G)$, is obtained from G by adding vertices $w_1,\ w_2, \dots, w_t$ and joining w_i to each vertex of S_i for $1 \leqslant i \leqslant t$. Note that if $V(G) = \bigcup_{i=1}^{n} S_i$ then $T = \emptyset$.

Definition 2.2. The wheel W_n is defined to be the join of $K_1 + C_n$. The vertex corresponding to K_1 is known as apex and the vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges.

Definition 2.3. The helm H_n is the graph obtained from wheel W_n by attaching a pendant edge to each of its rim vertices. This graph is having three types of vertices: an apex of degree n, n vertices of degree 4 and n pendant vertices.

Proposition 2.4.[6]

$$\chi[W_n] = \begin{cases} 3; & n\ is\ even \\ 4; & n\ is\ odd \end{cases}$$

Proposition 2.5.[6]

Let H be a subgraph of graph G. Then $\chi(G) \geqslant \chi(H)$.

Proposition 2.6. [6]
For any graph G, $\chi(G) \geq 3$ if and only if G has an odd cycle.

Proposition 2.7.[2]

$$\gamma[DS(W_n)] = 2,\ n \geqslant 4$$

III. Main Results

Lemma 3.1.

$$\chi[DS(W_n)] = \begin{cases} 4; & n\ is\ odd \\ 3; & n\ is\ even \end{cases}$$

Proof: Let $v_1,\ v_2, ..., v_n$ be the rim vertices and v be the apex of W_n. Thus $V(W_n) = \{v, v_i :\ 1 \leqslant i \leqslant n\} = S_1 \cup S_2$ where $S_1 = \{v_i; 1 \leqslant i \leqslant n\}$ and $S_2 = \{v\}$. For obtaining $DS(W_n)$ from W_n add a vertex w_1 corresponding to S_1. Thus $V[DS(W_n)] = V(W_n) \cup \{w_1\}$ and $E[DS(W_n)] = E(W_n) \cup \{w_1 v_i / v_i \in S_1 :\ 1 \leqslant i \leqslant n\}$. $|V(DS(W_n)| = n + 2$ and $|E(DS(W_n)| = 3n$.

Case-1: n is odd

According to Proposition-2.5, we claim $\chi[DS(W_n)] \geqslant 4$ as W_n is a subgraph of $DS(W_n)$. If we assign coloring as $f(v) = 1 = f(w_1),\ \ f(v_i) = 2,\ (i = 1,\ 3,\ 5, \dots,\ n - 2),\ f(v_i) = 3,\ (i = 2,\ 4, \dots,\ n - 1),\ f(v_n) = 4$. Thus for proper coloring minimum four colors are essential. Hence $\chi[DS(W_n)] = 4$.

Case-2: n is even

According to Proposition-2.5, we claim $\chi\,[DS(W_n)] \geqslant 3$ as W_n is a subgraph of $DS(W_n)$. If we assign coloring as $f(v) = f(w_1) = 1,\ f(v_i) = 2,\ (i = 1,\ 3, \ldots, n-1),\ f(v_i) = 3,\ (i = 2,\ 4, \ldots, n)$. Thus for proper coloring minimum three colors are essential. Hence $\chi[DS(W_n)] = 3$.

Theorem 3.2. For $n \geqslant 4$

$$\chi_{_d}\,[DS(W_n)] = \chi\,[DS(W_n)] + \gamma\,[DS(W_n)] - 2$$

Proof: We continue with the terminology and notations used in Lemma-3.1 The set $\{v,\ w_1\}$ is the only γ - set of graph $DS(W_n)$. By Proposition-2.7, $\gamma[DS(W_n)] = 2$ and according to Lemma-3.1, $\chi[DS(W_n)] = 4$ when n is odd and $\chi[DS(W_n)] = 3$ when n is even. We initiate the coloring by assigning number of colors equal to $|\gamma(DS(W_n))|$ to the vertices of γ - set. Next we assign the colors to the remaining n vertices using $\chi\,[DS(W_n)] - 2$ number of colors.

As each vertex dominates all the vertices of at least one color class. Therefore this proper coloring pattern give rise to a dominator coloring for the respective graphs. Hence $\chi_{_d}\,[DS(W_n)] = \chi\,[DS(W_n)] + \gamma\,[DS(W_n)] - 2$.

Remark 3.3. We avoid the case when $n = 3$ as $DS(W_3) = K_5$ and $\chi_{_d}[K_n] = n$ is proved by Gera [3]

Lemma 3.4.

$$\gamma\,[DS(H_n)] = \begin{cases} 2; & n = 3 \\ 2; & n = 4 \\ 3; & n \geqslant 5 \end{cases}$$

Proof: Consider H_n with $V(H_n) = \{v,\ v_i,\ u_i:\ 1 \leqslant i \leqslant n\}$ where v is the apex, v_i's are rim vertices and u_i's are pendant vertices. There are three types of vertices in H_n.

(i) vertices of degree 4 namely $v_1,\ v_2,\ ...,\ v_n$

(ii) vertices of degree 1 namely $u_1,\ u_2,\ ...,\ u_n$

(iii) a vertex of degree n namely v (the apex)

Thus $V(H_n) = S_1 \cup S_2 \cup T$, where $S_1 = \{v_1,\ v_2,\ ...,\ v_n\}$, $S_2 = \{u_1,\ u_2,\ ...,\ u_n\}$ and $T = \{v\}$. Now in order to obtain $DS(H_n)$ from H_n, we add $w_1,\ w_2$ corresponding to S_1 and S_2 respectively. Then $V[DS(H_n)] = V(H_n) \cup \{w_1,\ w_2\}$ so $V\,|DS(H_n)| = 2n + 3$ and $E[DS(H_n)] = E(H_n) \cup \{v_i w_1 : 1 \leqslant i \leqslant n\} \cup \{u_i w_2 : 1 \leqslant i \leqslant n\}$ so $E\,|DS(H_n)| = 5n$.

Case-1: For $n = 3$

There are three types of vertices in the graph $DS(H_3)$- three vertices of degree two, three vertices of degree five and two vertices of degree three. As there are three vertices of maximum degree five we have to take at least one vertex with degree five in dominating set. Then there are three sets which can be minimal dominating set. They are $S_1 = \{v_1,\ w_1\}$, $S_2 = \{v_2,\ w_1\}$ and $S_3 = \{v_3,\ w_1\}$ as $N[S_1] = V[DS(H_3)]$, $N[S_2] = V[DS(H_3)]$ and $N[S_3] = V[DS(H_3)]$. Therefore S_1, S_2 and S_3 are minimal dominating sets with $|S_1| = |S_2| = |S_3| = 2$. Hence $\gamma[DS(H_3)] = 2$.

Case-2: For $n = 4$

The set $S = \{w_1,\ w_2\}$ is the only dominating set of $DS(H_4)$ as $N\,[S] = V\,[DS(H_4)]$. Hence $\gamma[DS(H_4)] = 2$.

Case-3: For $n \geqslant 5$

The graph $DS(H_n)$ has none of the vertex having degree $n+1$. This implies that $\gamma[DS(H_n)] \geqslant 2$. As $N[w_1] = \{w_1,\ v_1,\ v_2, \ldots, v_n\}$, $N[w_2] = \{w_2,\ u_1,\ u_2, ..., u_n\}$ and $N[v] = \{v,\ v_1,\ v_2, ..., v_n\}$. The set $S = \{w_1,\ w_2,\ v\}$ is the set with minimum cardinality for which $N[S] = V[DS(H_n)]$ with $|S| = 3$. Hence $\gamma\,[DS(H_n)] = 3$.

Lemma 3.5.

$$\chi[DS(H_n)] = \begin{cases} 4; & odd\ n \geqslant 3 \\ 4; & n = 4 \\ 3; & even\ n \geqslant 6 \end{cases}$$

Proof: We continue with the terminology and notations used in Lemma-3.4.

Case-1: For odd $n \geqslant 3$

By Proposition-2.6, $\chi[DS(H_n)] \geqslant 3$ as $DS(H_n)$ contains an odd cycle. By assigning proper coloring to the vertices as $f(v) = f(w_1) = f(w_2) = 1,\ f(v_i) = 2,\ (i = 1,\ 3, \ldots,\ n-2),\ f(v_i) = 3,\ (i = 2,\ 4, \ldots,\ n-1),\ f(v_i) = 4, (i = n)\ f(u_i) = 3,\ (i = 1,\ 3, \ldots, n),\ f(u_i) = 2,\ (i = 2,\ 4, \ldots,\ n-1)$. Thus for proper coloring minimum four colors are essential. Hence $\chi[DS(H_n)] = 4$.

Case-2: For $n = 4$

By Proposition-2.6, $\chi[DS(H_4)] \geqslant 3$ as $DS(H_4)$ contains an odd cycle. By assigning proper coloring to the vertices as $f(v) = f(w_1) = 1,\ f(w_2) = 2,\ f(v_1) = f(v_3) = 3,\ f(v_2) = f(v_4) = 4,\ f(u_1) = f(u_3) = 4,\ f(u_2) = f(u_4) = 3$.Thus for proper coloring minimum four colors are essential. Hence $\chi[DS(H_4)] = 4$.

Case-3: For even $n \geqslant 6$

In this case $\chi[DS(H_n)] \geqslant 2$ as $DS(H_n)$ contains an even cycle. By assigning proper coloring to the vertices as $f(v) = f(w_1) = f(w_2) = 1,\ f(v_i) = 2,\ (i = 1,\ 3, \ldots,\ n-1),\ f(v_i) = 3,\ (i = 2,\ 4, \ldots,\ n)\ f(u_i) = 3,\ (i = 1,\ 3, \ldots, n-1),\ f(u_i) = 2,\ (i = 2,\ 4, \ldots,\ n)$. Thus for proper coloring minimum three colors are essential. Hence $\chi[DS(H_n)] = 3$.

Theorem 3.6. For $n \geqslant 3$

$$\chi_{_d}\,[DS(H_n)] = \chi\,[DS(H_n)] + \gamma[DS(H_n)] - 1$$

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Proof: We continue with the terminology and notations used in Lemma-3.4.

Case-1: When $n = 3$:

In this case the set $\{v_1,\ w_1\}$ is one of the γ- sets of graph $DS(H_3)$. According to Lemma-3.4, $\gamma[DS(H_3)] = 2$ and by Lemma-3.5, $\chi[DS(H_3)] = 4$. Therefore we prescribe the optimal coloring for $DS(H_3)$ by assigning number of colors equal to $|\gamma(DS(H_3))|$ to the vertices of γ- set. Next we assign the colors to the remaining vertices using $\chi[DS(H_3)] - 1$ number of colors. We summarize the coloring pattern as $f(v_1) = 1,\ f(w_1) = 2,\ f(v_2) = 3,\ f(v_3) = 4,\ f(v) = 5 = f(w_2),\ f(u_1) = f(u_2) = f(u_3) = 5$. This coloring satisfies the condition of dominator coloring. Thus $\chi_{_d}[DS(H_3)] = 5$.

Case-2: When $n = 4$:

In this case the set $\{v,\ w_1\}$ is the only γ -set of $DS(H_4)$. By Lemma-3.4, $\gamma[DS(H_4)] = 2$ and according to Lemma-3.5, $\chi[DS(H_4)] = 4$. Therefore we prescribe the optimal coloring for $DS(H_4)$ by assigning number of colors equal to $|\gamma(DS(H_4))|$ to the vertices of γ- set. Next we assign the colors to the remaining vertices using $\chi[DS(H_4)] - 1$ number of colors. We summarize the coloring pattern as $f(v) = 1,\ f(w_1) = 2,\ f(v_1) = 3 = f(v_3),\ f(v_2) = 4 = f(v_4),\ f(w_2) = 5,\ f(u_1) = f(u_3) = 4,\ f(u_2) = f(u_4) = 3$. This coloring satisfies the condition of dominator coloring. Thus $\chi_d[DS(H_4)] = 5$.

Case-3: When $n \geqslant 5$:

In this case the set $\{v,\ w_1,\ w_2\}$ is one of the γ- sets of graph $DS(H_n)$. According to Lemma-3.4, $\gamma[DS(H_n)] = 3$ and by Lemma-3.5, $\chi[DS(H_n)] = 4$ for odd n and $\chi[DS(H_n)] = 3$, for even n. We initiate the coloring by assigning number of colors equal to $|\gamma(DS(H_n))|$ to the vertices of γ -set. Next we assign the colors to the remaining vertices using $\chi[DS(H_n)] - 1$ number of colors.

As each vertex dominates all the vertices of at least one color class. Therefore this proper coloring pattern give rise to a dominator coloring for the respective graphs. Hence $\chi_d[DS(H_n)] = \chi[DS(H_n)] + \gamma[DS(H_n)] - 1$.

IV. Concluding Remarks

The dominator chromatic numbers of W_n and H_n are known while we investigate the same for the degree splitting graphs of wheel W_n and helm H_n.

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