Generalized Cartesian Product of Graphs

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***Abstract*—In graph theory, different types of products of two graphs had been studied, e.g., Cartesian product, Tensor product, Strong product, etc. In this paper we generalize the concept of Cartesian product of graphs.We define 2 - Cartesian product and more generally r - Cartesian product of two graphs. We study these product mainly for path graphs. We also discuss connectedness of the new graph.**

***Index Terms*—Cartesian product of graphs, r-Cartesian product of graphs, Grid graph.**

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I. Preliminaries

THE Cartesian product of two graphs ia well-known and studied in detail ([1], [2], [3], [4], [5]). In this paper, we generalize this product, using the concept of distance.

Let $G = (V(G), E(G))$ be a finite, simple and connected graph. For a connected graph G, $d_G(u, u')$ is the length of the shortest path between u and u' in G. A maximal connected subgraph of G is known as component of G.

If the graph G is a disjoint union of r similar components H, then we shall denote it by,

$$G = \left[\bigcup_{\substack{\circ \\ i=1}}^{r} H^{(i)} \right]$$

Definition 1.1 The grid graph $G = G_{(m),(n)}$ is defined as the graph with vertex set V is $\{(u_i, v_j); i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\}$ and the edge set E is as follows.

$$E(G) = \left[\bigcup_{i=1}^{m} \{(u_i, v_j) \leftrightarrow (u_i, v_{j+1}) : 1 \leq j \leq n-1\} \right]$$

$$\bigcup \left[\bigcup_{j=1}^{n} \{(u_i, v_j) \leftrightarrow (u_{i+1}, v_j) : 1 \leq i \leq m-1\} \right]$$

We recall the usual Cartesian product of graphs.

Definition 1.2 [1] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple connected graphs. The Cartesian product $G_1 \times G_2$ of G_1 and G_2, is a graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') in V are adjacent in $G_1 \times G_2$ if one of the following conditions are satisfied:

(i) $uu' \in E_1$ and $v = v'$ in G_2, i.e., $d_{G_1}(u, u') = 1$ and $d_{G_2}(v, v') = 0$,

(ii) $u = u'$ in G_1 and $vv' \in E_2$, i.e., $d_{G_1}(u, u') = 0$ and $d_{G_2}(v, v') = 1$.

We generalize the above conditions (i) and (ii), by first considering $d_{G_1}(u, u') = 2$ ($d_{G_2}(v, v') = 2$) and then extend it for $d_{G_1}(u, u') = r$ ($d_{G_2}(v, v') = r$).

Throughout this paper, we fix these notations. For the basic terminology, concept and results of graph theory, we refer to ([6], [7], [1]).

II. 2 - Cartesian product of graphs

In this section we define 2 - Cartesian product of graphs and we discuss it for path graph P_m and then for general bipartite graph.

Definition 2.1 The $2-$ Cartesian product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times_2 G_2 = (V, E)$ with the vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \times_2 G_2$ if one of the following conditions are satisfied:

(i) $d_{G_1}(u, u') = 2$ and $d_{G_2}(v, v') = 0$,

(ii) $d_{G_1}(u, u') = 0$ and $d_{G_2}(v, v') = 2$.

It is clear that if we replace distance 2 by 1, then we get usual Cartesian product of graphs.

Let P_m and P_n be two path graphs with,
$V(P_m) = \{u_1, u_2, \ldots, u_m\}$ and
$E(P_m) = \{(u_1u_2), (u_2u_3), \ldots, (u_{m-1}u_m)\}$ and
$V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and
$E(P_n) = \{(v_1v_2), (v_2v_3), \ldots, (v_{n-1}v_n)\}$. Then,

$$V(P_m \times_2 P_n) = \{(u_i, v_j) : u_i \in V(P_m) \text{ and } v_j \in V(P_n)\}.$$

Proposition 2.2 For $m, n \geq 3$,

(a) $P_m \times_2 P_n = \left[\bigcup_{\substack{\circ \\ i=1}}^{4} \left(G_{(\frac{m}{2}),(\frac{n}{2})} \right)^{(i)} \right]$; if m and n are even integers.

(b) If m is even and n is odd, then

$$P_m \times_2 P_n = \left[\bigcup_{\substack{\circ \\ i=1}}^{2} \left(G_{(\frac{m}{2}),(\frac{n+1}{2})} \right)^{(i)} \right] \bigcup_{\circ} \left[\bigcup_{\substack{\circ \\ j=1}}^{2} \left(G_{(\frac{m}{2}),(\frac{n-1}{2})} \right)^{(j)} \right]$$

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(c) $P_m \times_2 P_n = \Big[G_{(\frac{m+1}{2}),(\frac{n+1}{2})} \Big] \bigcup_{\circ} \Big[G_{(\frac{m+1}{2}),(\frac{n-1}{2})} \Big] \bigcup_{\circ} \Big[G_{(\frac{m-1}{2}),(\frac{n+1}{2})} \Big] \bigcup_{\circ} \Big[G_{(\frac{m-1}{2}),(\frac{n-1}{2})} \Big];$

if m and n both are odd.

Proof:

(a) Let $G_1 = P_m = P_{2l}$ and $G_2 = P_n = P_{2t}$; $l, t \in \mathbb{N}$. Then, $d_{G_1}(u_i, u_{i+2}) = 2$; and $d_{G_2}(v_j, v_j) = 0$; for $i = 1, 2, \ldots (2l-2)$ and each $j = 1, 2, \ldots 2t$.
So, we have following connected components for each j.

Thus we get total $4t = 2n$ path graphs $P_{\frac{m}{2}}$.
Next, we fixed first coordinate and change the second coordinate, then (u_i, v_j) is adjacent to (u_i, v_{j+2}) for $j = 1, 2, \ldots 2t-2$, and each $i = 1, 2, \ldots 2l$. These give grid graphs $G_{(\frac{m}{2}),(\frac{n}{2})}$ for $i = 1, 2$ and $j = 1, 2$. since m and n both are even, we get four grid graphs of same size and steps, i.e., $G_{(\frac{m}{2}),(\frac{n}{2})}$.

(b) Let $G_2 = P_n = P_{2t+1}$; $t \in \mathbb{N}$. Then we have one more edge v_{2t} to v_{2t+1} in G_2. So, we have edge (u_i, v_{2t-1}) to (u_i, v_{2t+1}) in the resultant graph for $i = 1, 2, \ldots 2l$. Therefor only two components will be extended by these edges vertically. Thus two grid graphs corresponding to $j = 1$ with $i = 1, 2$ will change and we get $G_{(\frac{m}{2}),(\frac{n+1}{2})}$. Similarly for $j = 2$, we get $G_{(\frac{m}{2}),(\frac{n-1}{2})}$.

(c) In this case m is also odd. So, as in case of (b) we get four different size grid graphs.

Remark 2.3 It is clear that if $m = 1$, then $P_1 \times_2 P_n$ has two path components and for $m = 2$, $P_2 \times_2 P_n$ has four path components.

Note that, if G and H are two connected graphs, then $G \times H$ is always connected. But, as we have seen this result does not remain true for 2 - Cartesian product (Proposition 2.2). Next, we prove that the number of components in $G \times_2 H$ is related with bipartiteness of G and H.

Proposition 2.4 Let G and H be two connected graphs. Then,

(a) $G \times_2 H$ has four components if G and H both are bipartite graph.
(b) $G \times_2 H$ has two components if G is a nonbipartite graph and H is a bipartite graph.
(c) $G \times_2 H$ is connected if G and H both are nonbipartite graphs.

Proof:

(a) Let $G = (U_1 \cup U_2, E)$ and $H = (V_1 \cup V_2, F)$ be two connected and bipartite graphs with partite sets U_1 , U_2 and V_1 , V_2 respectively. Then,
$V(G \times_2 H) = W_{11} \cup W_{12} \cup W_{21} \cup W_{22}$ where,
$W_{ij} = U_i \times V_j$; for i= 1,2 with j= 1,2.
We know that if two vertices lie in the same partite sets of $G(H)$, then distance between them is even, otherwise odd. So, we get at least four disconnected subgraphs corresponding to W_{ij} $i, j = 1, 2$.
Let (u, v) and (u', v') be in W_{11}. Since G and H are connected bipartite graphs, there is a path P_1 between u to u' in G and path P_2 between v to v' in H and length of both paths are even. Suppose,
The path $P_1 : u = u_0 \to u_1 \to u_2 \to \ldots \to u_m = u'$
The path $P_2 : v = v_0 \to v_1 \to v_2 \to \ldots \to v_n = v'$.
Using path P_1 and P_2 we get a path from (u, v) to (u', v') in $G \times_2 H$ as follows: Assume that $m \le n$,
$(u_0, v_0) \to (u_2, v_0) \to \ldots \to (u_m, v_0) \to (u_m, v_2) \to$
$\to (u_m, v_4) \to \ldots \to (u_m, v_n)$.
This shows that W_{11} gives connected component. Similarly each W_{ij} will give connected subgraph. Therefore we have exactly four connected components in $G \times_2 H$.

(b) Let $G = (U, E_1)$ and $H = (V_1 \cup V_2, E_2)$ be two graphs. Then the vertex set $V(G \times_2 H) = [U \times V_1] \cup [U \times V_2]$. It is clear that $U \times V_1$ and $U \times V_2$ give two disconnected subgraphs in $G \times_2 H$.
Fixed $U \times V_1$. Let (u, v) and (u', v') be two vertices in $U \times V_1$.
As G and H both are connected graphs, there is at least one path P_1 between u and u' in G and path P_2 between v and v' in H as in case(a).
If m is an even integer, then there is a path P between (u, v) and (u', v') in $U \times V_1$ as in case (a).
Let m be an odd integer. As G is a nonbipartite graph, there is a walk W between u and u' of even length in G. So, using walk W and path P_2, there is a walk between (u, v) and (u', v') in $U \times V_1$. So, there is a path between (u, v) and (u', v') in $U \times V_1$. Therefore $U \times V_1$ gives a connected subgraph in the graph $G \times_2 H$.
Similarly $U \times V_2$ gives a connected component and hence $G \times_2 H$ has 2 components.

(c) As in case of (b), using a walk of even length between v and v' in H, there is a path between (u, v) and (u', v') in $G \times_2 H$, for any (u, v) and (u', v') in $G \times_2 H$. Hence $G \times_2 H$ is connected.

Next, we discuss $G \times_2 H$ for complete bipartite graphs G and H and show that $G \times_2 H$ can be obtained in terms of complete graphs.

Proposition 2.5 Let $K_{m,n}$ and $K_{s,t}$ be two complete bipartite graphs,with partite sets U_1 , U_2 and V_1 , V_2 respectively. Also $|U_1| = m$, $|U_2| = n$, $|V_1| = s$ and $|V_2| = t$, with $m, n, s, t \ge 2$. Then $K_{m,n} \times_2 K_{s,t}$ has four components as follows:

$$\left\{ \left[\bigcup_{u \in U_1} \{u\} \times K_s \right] \bigcup \left[\bigcup_{v \in V_1} K_m \times \{v\} \right] \right\}$$

$$\bigcup_{\circ} \left\{ \left[\bigcup_{u \in U_1} \{u\} \times K_t \right] \bigcup \left[\bigcup_{v \in V_2} K_m \times \{v\} \right] \right\}$$

$$\bigcup_{\circ}\left\{\left[\bigcup_{u\in U_2}\{u\}\times K_s\right]\bigcup\left[\bigcup_{v\in V_1}K_n\times\{v\}\right]\right\}$$

$$\bigcup_{\circ}\left\{\left[\bigcup_{u\in U_2}\{u\}\times K_t\right]\bigcup\left[\bigcup_{v\in V_2}K_n\times\{v\}\right]\right\}$$

Proof. Let $G = K_{m,n} = (U_1 \cup U_2, E)$ and $H = K_{s,t} = (V_1 \cup V_2, F)$ be two complete bipartite graphs. Then $V(G \times_2 H) = W_{11} \cup W_{12} \cup W_{21} \cup W_{22}$, where $W_{ij} = U_i \times V_j$; for $i = 1, 2$ with $j = 1, 2$.

It is clear that $d_G(u, u') = 2$ if and only if u and u' are in same U_i $(i = 1, 2)$. Thus (u, v) and (u', v) can not be adjacent in $G \times_2 H$ if u and u' are in different U_i. So, there are at least four disconnected subgraphs in $G \times_2 H$, with vertex sets $W_{ij}(i, j = 1, 2)$.

Fix $W_{11} = U_1 \times V_1$. For each $v \in V_1$, (u, v) and (u', v) are adjacent and so it gives $K_m \times \{v\}$. Similarly, for $u \in U_1$ we get the graph $\{u\} \times K_s$.

Thus we get a component in $G \times_2 H$ with vertex set W_{11} consisting of edge disjoint m copies of K_s with s copies of K_m as

$$\left[\bigcup_{u\in U_1}\{u\}\times K_s\right]\bigcup\left[\bigcup_{v\in V_1}K_m\times\{v\}\right]$$

By similar arguments we get other three components corresponding to vertex set W_{12} ,W_{21} and W_{22}.

III. r - CARTESIAN PRODUCT OF GRAPHS

In this section, we extend the distance up to r and define r - Cartesian product of two graphs. Mainly, we discuss it for path graphs. We also show that the number of components and bipartiteness are not related, if $r > 2$.

Definition 3.1 The r - Cartesian product or generalized Cartesian product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times_r G_2 = (V, E)$ with the vertex set $V = V_1 \times V_2$ and the edge set E defined as follows:
Two vertices (u, v) and (u', v') are adjacent in $G_1 \times_r G_2$ if one of the following conditions are satisfied:

(i) $d_{G_1}(u, u') = r$ and $d_{G_2}(v, v') = 0$,
(ii) $d_{G_1}(u, u') = 0$ and $d_{G_2}(v, v') = r$.

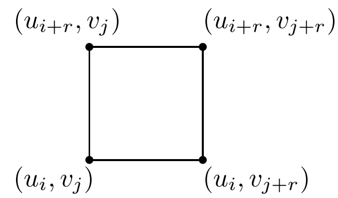
Now onwards we use r - Cartesian product for generalized Cartesian product. It is clear that for $r = 1$, we get usual Cartesian product of graphs.

Proposition 3.2 For $r \in \mathbb{N}$, $1 \le k_1, k_2 \le r$,
$P_{r+k_1} \times_r P_{r+k_2} =$

$$\left[\bigcup_{\substack{\circ\\ i=1}}^{k_1k_2}\left(G_{(2),(2)}\right)^{(i)}\right]\bigcup_{\circ}\left[\bigcup_{\substack{\circ\\ i=1}}^{(r-k_1)k_2}\left(P_2\right)^{(i)}\right]$$

$$\bigcup_{\circ}\left[\bigcup_{\substack{\circ\\ i=1}}^{k_1(r-k_2)}\left(P_2\right)^{(i)}\right]\bigcup_{\circ}\left[\bigcup_{\substack{\circ\\ i=1}}^{(r-k_1)(r-k_2)}\left(P_1\right)^{(i)}\right]$$

Proof : Let $G_1 = P_{r+k_1}$ and $G_2 = P_{r+k_2}$ with $1 \le k_1, k_2 \le r$.
Here $d_{G_1}(u_i, u_{i+r}) = r$; for $i = 1, 2, \ldots, k_1$. So, for each v in $V(G_2)$, we get k_1 edges (u_i, v) to (u_{i+r}, v) in $G_1 \times_r G_2$. Similarly for each u in $V(G_1)$, we get the k_2 edges (u, v_j) to (u, v_{j+r}); for $j = 1, 2, \ldots, k_2$ in the resultant graph. These k_1 and k_2 edges together give k_1k_2 copies of grid graph $G_{(2),(2)}$ as follows:



In G_1 graph the remaining $(r - k_1)$ vertices give P_2 graph with edges $v_j \to v_{j+r}$; for $j = 1, 2, \ldots, k_2$. Thus, we get $k_2(r - k_1)$ copies of P_2 in $G_1 \times_r G_2$. Similarly the $(r - k_2)$ vertices of G_2 with $u_i \to u_{i+r}$; for $i = 1, 2, \ldots, k_1$ gives $k_1(r - k_2)$ copies of P_2 in the resultant graph.
Finally, taking these $(r - k_1)$ vertices of G_1 with $(r - k_2)$ vertices of G_2 gives isolated vertices in $G_1 \times_r G_2$. Thus the number of components in $G_1 \times_r G_2$ is $k_1k_2 + k_2(r - k_1) + k_1(r - k_2) + (r - k_1)(r - k_2) = r^2$.

Next we discuss general case.

Proposition 3.3 Let $r, m, n \in \mathbb{N}$,with $m, n > 1$ and $1 \le k_1, k_2 \le r$.

$$P_{rm+k_1} \times_r P_{rn+k_2} =$$

$$\left[\bigcup_{\substack{\circ\\ i=1}}^{k_1k_2}\left(G_{(m+1),(n+1)}\right)^{(i)}\right]$$

$$\bigcup_{\circ}\left[\bigcup_{\substack{\circ\\ i=1}}^{(r-k_1)k_2}\left(G_{(m),(n+1)}\right)^{(i)}\right]$$

$$\bigcup_{\circ}\left[\bigcup_{\substack{\circ\\ i=1}}^{k_1(r-k_2)}\left(G_{(m+1),(n)}\right)^{(i)}\right]$$

$$\bigcup_{\circ}\left[\bigcup_{\substack{\circ\\ i=1}}^{(r-k_1)(r-k_2)}\left(G_{(m),(n)}\right)^{(i)}\right].$$

Proof: Let $G_1 = P_{rm+k_1}$ and $G_2 = P_{rn+k_2}$.
Here, $d_{G_1}(u_i, u_{i+r}) = r$;
for $i = t, (t+r), (t+2r) \ldots, t+(m-1)r$ with $t = 1, 2, \ldots, k_1$.
These gives $P_{(m+1)}$ for $t = 1, 2, \ldots, k_1$ as follows:

$$u_t \rightarrow u_{t+r} \rightarrow u_{t+2r} \rightarrow \dots \rightarrow u_{t+(m-1)r} \rightarrow u_{t+mr}.$$

Also for $t = k_1 + 1, k_1 + 2, \dots, k_1 + (r - k_1) = r$ we have $P_{(m+1)}$ as follows:

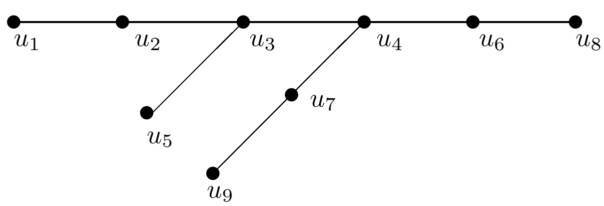
$$u_{(k_1+1)} \rightarrow u_{(k_1+1)+r} \rightarrow u_{(k_1+1)+2r} \rightarrow \dots \rightarrow$$
$$\rightarrow \dots \rightarrow u_{(k_1+1)+(m-2)r} \rightarrow u_{(k_1+1)+(m-1)r}.$$

So, for each v in $V(G_2)$ we get the k_1 copies of P_{m+1} and $(r - k_1)$ copies of P_m in $G_1 \times_r G_2$.
Similarly for each u in $V(G_1)$ we get the k_2 copies of P_{n+1} and $(r - k_2)$ of P_n in the resultant graph. So, as we have seen in Proposition 3.2, we get $k_1 k_2$ copies of $G_{(m+1),(n+1)}$.
Similarly, k_1 copies of P_{m+1} together with $(r - k_2)$ copies of P_n give $k_1(r - k_2)$ copies of $G_{(m+1),(n)}$. Also k_2 copies of P_{n+1} together with $(r - k_1)$ copies of P_m give $k_2(r - k_1)$ copies of $G_{(m),(n+1)}$.
Finally $(r - k_1)$ copies of P_m together with $(r - k_2)$ copies of P_n give $(r - k_1)(r - k_2)$ copies of $G_{(m),(n)}$. So, the numbers of components in the resultant graph is r^2.

The result similar to the Proposition 2.5 is not true in $G \times_r H$ if $r > 2$.

Examples 3.4

(1) If $G = H = C_8$, then the graph $C_8 \times_3 C_8$ is a connected graph.
(2) Let $G = P_5$ and $H = C_6$. Then $P_5 \times_3 C_6$ has nine components but the graph $P_5 \times_3 C_8$ has three components. Note that here one graph is a tree.
(3) Let $G = H = T$ be as follows.



Then the graph $T \times_3 T$ is a connected graph.

This shows that for bipartite graphs G and H, the number of components are not fix in $G \times_3 H$, even if G and H both are trees.

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