The b-Chromatic Number of Helm and Closed Helm

S. K. Vaidya and M. S. Shukla

***Abstract*—The helm H_n is the graph obtained from wheel $W_n = C_n + K_1$ by attaching a pendant edge to each rim vertex while the closed helm is the graph obtained from helm by joining each pendant vertex to form a cycle. We investigate b-chromatic numbers for helm and closed helm.**

***Index Terms*—b-Coloring, b-Continuity, b-Spectrum.**
***MSC 2010 Codes* - 05C15.**

I. Introduction

We begin with a simple, finite, connected and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A coloring of the vertices of G is a mapping $f : V(G) \rightarrow \mathbb{N}$. For every vertex $v \in V(G)$, $f(v)$ is called the color of v.

If any two adjacent vertices have different colors then f is called proper coloring. The chromatic number $\chi(G)$ is the smallest integer k for which G admits a proper coloring using k colors. The set of vertices with a particular color is called a color class.

A *b-coloring* by k colors is a proper coloring of the vertices of G such that in each color class there exists a vertex which has neighbours in all the other *k-1* color classes. In other words each color class contains a vertex which has at least one neighbour in all the other color classes. Such vertex is called a color dominating vertex. It is obvious that every coloring of a graph G by $\chi(G)$ colors is a b-coloring of G.

The b-chromatic number $\varphi(G)$ is the largest integer k such that G admits a b-coloring with k colors. The concept of b-coloring was introduced by Irving and Manlove [1] and showed that the problem of determining $\varphi(G)$ is NP-hard for general graphs but it is polynomial-time solvable for trees. In the same paper they have introduced the concepts of b-continuity and b-spectrum.

If the b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ then G is called b-continuous and the b-spectrum $S_b(G)$ of a graph G is the set of k integers(colors) for which G has a b-coloring. A graph G is tight if it has exactly $m(G)$ vertices of degree $m(G)-1$. It has been proved by Havet *et al.* [2] that all tight graph are b-colorable. The bounds for the b-chromatic number for various graphs are established in the work of Kouider and Maheo [3] while b-chromatic number for Peterson graph and power of a cycle is discussed by Chandrakumar and Nicholas [4].

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The b-continuity of chordal graphs is discussed by Faik [5]. Also the discussion on b-coloring of central graph of some graphs is reported in Thilagavathi *et al.* [6].

Many results on b-coloring and b-continuity are reported in Alkhateeb [7] and the b-chromatic number for cartesian product of some families of graphs is explored by Balakrishnan *et al.* [8] while Vaidya and Shukla [9, 10] have investigated b-chromatic number of some cycle and wheel related graphs.

Proposition 1.1 [1]: If graph G admits a b-coloring with m-colors, then G must have at least m vertices with degree at least $m-1$ (Since each color class has a b-vertex).

Proposition 1.2 [11]: If the graph G contains K_n as a subgraph, $\chi(G) \geq n$.

Definition 1.3 [1]: The m-degree of a graph G, denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m-1$.

Proposition 1.4 [11]: For any graph G, $\chi(G) \geq 3$ if and only if G has an odd cycle.

Proposition 1.5 [3]: $\chi(G) \leq \varphi(G) \leq m(G)$.

Definition 1.6: A vertex of G with degree at least $m(G)-1$ is called a dense vertex.

Definition 1.7: Let G be a tight graph. The b-closure of G, denoted by G^*, is the graph with vertex set $V(G^*) = V(G)$ and edge set $E(G^*) = E(G) \cup \{uv$; u and v are vertices with a common dense neighbour$\}$.

Proposition 1.8 [2]: Let G be a tight graph. Then $\varphi(G) = m(G)$ if and only if $\chi(G^*) = m(G)$.

Definition 1.9: The wheel W_n is defined to be the join of $K_1 + C_n$. The vertex corresponding to K_1 is known as apex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges.

Definition 1.10: The helm H_n is the graph obtained from wheel W_n by attaching a pendant edge to each rim vertex. It contains three types of vertices: an apex of degree n, n vertices of degree 4 and n pendant vertices.

Definition 1.11: A closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle. It contains three types of vertices: an apex of degree n, n vertices of degree 4 and n vertices of degree 3.

We continue to recognize apex of wheel as the apex of respective graphs corresponding to Definitions 1.10 and 1.11.

II. MAIN RESULTS

Lemma 2.1 : For the helm graph H_n

$$\chi(H_n) = \begin{cases} 3, & n \text{ is even} \\ 4, & n \text{ is odd.} \end{cases}$$

Proof : For H_n, let $\{e_1, e_2, ..., e_n\}$ be the spoke edges of H_n and $\{e_{n+1}, e_{n+2}, ..., e_{2n}\}$ be the rim edges of the cycle in H_n while $\{e_{2n+1}, e_{2n+2}, ..., e_{3n}\}$ be the pendant edges of H_n.

Moreover $\{u_1,\ u_2, ...,\ u_n\}$ be the pendant vertices of H_n and $\{v_1,\ v_2, ...,\ v_n\}$ be the vertices of degree 4. Denote the apex of H_n as v. Also $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. To prove the result we consider the following cases.

Case 1: n is even.

In this case H_n contains a cycle C_3. Then by Proposition 1.4, $\chi(H_n) \geq 3$. If we assign a proper coloring as $f(v) = 3,\ f(v_{2k-1}) = 1,\ f(v_{2k}) = 2,\ f(u_{2k-1}) = 2,\ f(u_{2k}) = 1; k \in N$ then $\chi(H_n) = 3$.

Case 2: n is odd.

In this case H_n contains K_3. Then by Proposition 1.2, $\chi(H_n) \geq 3$. As v is one of the vertex of K_3 and we have used already 3-colors for proper coloring. So to color v assign new color. Hence $\chi(H_n) = 4$.

Theorem 2.2 : For the helm graph H_n

$$\varphi(H_n) = \begin{cases} 4, & n = 3 \\ 5, & n = 4 \\ 4, & n = 5 \\ 5, & n = 6 \\ 5, & n \geq 7. \end{cases}$$

Proof : We continue with the terminology and notations used in Lemma 2.1 and consider the following cases.

Case 1: For $n = 3$

In this case, $|V(H_3)| = 7$ and $|E(H_3)| = 9$. Also $m(H_3) = 4$. Then by Proposition 1.5, $\varphi(H_3) \leq 4$. Further more the graph contains K_3, then according to Proposition 1.2, $\varphi(H_3) \geq 3$.

If possible H_3 has b-coloring using four colors with $c = \{1,\ 2,\ 3,\ 4\}$. To assign the proper coloring define the color function $f : V \to \{1,\ 2,\ 3,\ 4\}$ as $f(v) = 4,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(u_1) = 2,\ f(u_2) = 1,\ f(u_3) = 2$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v$. Thus $\varphi(H_3) = 4$.

Case 2: For $n = 4$

In this case, $|V(H_4)| = 9$, $|E(H_4)| = 12$ and $m(H_4) = 5$. As H_4 contains exactly $m(H_4)$ vertices of degree $m(H_4) - 1$ then H_4 is a tight graph. Also $\chi(H^*{}_4) = m(H_4) = 5$. Then by Proposition 1.8, $\varphi(H_4) = 5$.

For b-coloring consider the color class $c = \{1,\ 2,\ 3,\ 4,\ 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1,\ 2,\ 3,\ 4,\ 5\}$ as $f(v) = 5,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 4,\ f(u_1) = 3,\ f(u_2) = 4,\ f(u_3) = 1,\ f(u_4) = 2$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v_4,\ cdv(5) = v$. Thus $\varphi(H_4) = 5$.

Case 3: For $n = 5$

In this graph $|V(H_5)| = 11$ and $|E(H_5)| = 15$. Also $m(H_5) = 5$. Then by Proposition 1.5, $\varphi(H_5) \leq 5$. Suppose that H_5 does have a b-chromatic 5-coloring.

Now consider the color class $c = \{1,\ 2,\ 3,\ 4,\ 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1,\ 2,\ 3,\ 4,\ 5\}$ as $f(v) = 5,\ f(v_1) = 4,\ f(v_2) = 1,\ f(v_3) = 2,\ f(v_4) = 3,\ f(u_1) = 3,\ f(u_2) = 4,\ f(u_3) = 4,\ f(u_4) = 2,\ f(u_5) = 2$ which in turn forces to assign $f(v_5) = 1$.

This proper coloring gives the color dominating vertices for color classes $1, 2$ and 5 but not for $3, 4$ which is contradiction to our assumption. Thus $\varphi(H_5) \neq 5$.

Hence we can color the graph by four colors. For b-coloring consider the color class $c = \{1,\ 2,\ 3,\ 4\}$ and to assign the proper coloring we define the color function as $f : V \to \{1,\ 2,\ 3,\ 4\}$ as $f(v) = 4,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 1,\ f(v_5) = 2,\ f(u_1) = 3,\ f(u_2) = 1,\ f(u_3) = 2,\ f(u_4) = 2,\ f(u_5) = 3$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v$. Thus $\varphi(H_5) = 4$.

Case 4: For $n = 6$

For the graph H_6, $|V(H_6)| = 13$ and $|E(H_6)| = 18$. Also $m(H_6) = 5$. Then by Proposition 1.5 $\varphi(H_6) \leq 5$. Suppose that H_6 does have a b-chromatic 5-coloring.

Now consider the color class $c = \{1,\ 2,\ 3,\ 4,\ 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1,\ 2,\ 3,\ 4,\ 5\}$ as $f(v) = 5,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 4,\ f(v_5) = 2,\ f(v_6) = 4,\ f(u_1) = 3,\ f(u_2) = 4,\ f(u_3) = 1,\ f(u_4) = 1,\ f(u_5) = 1,\ f(u_6) = 1$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v_4,\ cdv(5) = v$. Thus $\varphi(H_6) = 5$.

Case 5: For $n \geq 7$

In this case $|V(H_7)| = 15$ and $|E(H_7)| = 21$. Also $m(H_7) = 5$. Then by Proposition 1.5 $\varphi(H_7) \leq 5$. Suppose that H_7 does have a b-chromatic 5-coloring.

Now consider the color class $c = \{1,\ 2,\ 3,\ 4,\ 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1,\ 2,\ 3,\ 4,\ 5\}$ as $f(v) = 5,\ f(v_1) = 4,\ f(v_2) = 1,\ f(v_3) = 2,\ f(v_4) = 3,\ f(v_5) = 4,\ f(v_6) = 2,\ f(v_7) = 1,\ f(u_1) = 1,\ f(u_2) = 3,\ f(u_3) = 4,\ f(u_4) = 1,\ f(u_5) = 1,\ f(u_6) = 1,\ f(u_7) = 2$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_2,\ cdv(2) = v_3,\ cdv(3) = v_4,\ cdv(4) = v_5,\ cdv(5) = v$. Thus $\varphi(H_7) = 5$.

For $n > 7$:

We repeat the colors as in the graph H_7 for the vertices $\{v, v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6,\ v_7,\ u_1,\ u_2,\ u_3,\ u_4,\ u_6,\ u_7,\}$ and for the remaining vertices assign the colors as $f(v) = 5,\ f(v_{2k+6}) = 2,\ f(v_{2k+7}) = 1,\ f(u_{2k+6}) = 1,\ f(u_{2k+7}) = 2; k \in N$.

Hence $\varphi(H_n) = 5,\ n \geq 7$.

Theorem 2.3 : H_n is b-continuous.

Proof : To prove this result we continue with the terminology and notations used in Lemma 2.1 and consider the following cases.

Case 1: $n = 3$

In this case, H_3 is b-continuous as $\chi(H_3) = \varphi(H_3) = 4$.

Case 2: $n = 4$

By Lemma 2.1, $\chi(H_4) = 3$ and by Theorem 2.2, $\varphi(H_4) = 5$. It is obvious that b-coloring for the graph H_4 is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the b-coloring of the graph H_4 is as follows.

Consider the color class $c = \{1,\ 2,\ 3,\ 4\}$ and to assign the proper coloring we define the color function $f : V \rightarrow \{1,\ 2,\ 3,\ 4\}$ as $f(v) = 4,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 2,\ f(u_1) = 3,\ f(u_2) = 1,\ f(u_3) = 1,\ f(u_4) = 1$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v$. Thus H_4 is four colorable. Hence b-coloring exists for every integer k satisfying $\chi(H_4) \leq k \leq \varphi(H_4)$ (Here $k = 3, 4, 5$). Consequently H_4 is b-continuous.

Case 3: $n = 5$

In this case the graph H_5 is b-continuous as $\chi(H_5) = \varphi(H_5) = 4$.

Case 4: $n = 6$

In this case by Lemma 2.1, $\chi(H_6) = 3$ and by Theorem 2.2, $\varphi(H_6) = 5$. It is obvious that b-coloring for the graph H_6 is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the b-coloring of the graph H_6 is as follows.

Consider the color class $c = \{1,\ 2,\ 3,\ 4\}$ and to assign the proper coloring we define the color function $f : V \rightarrow \{1,\ 2,\ 3,\ 4\}$ as $f(v) = 4,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 1,\ f(v_5) = 2,\ f(v_6) = 3,\ f(u_1) = 3,\ f(u_2) = 1,\ f(u_3) = 1,\ f(u_4) = 2,\ f(u_5) = 1,\ f(u_6) = 1$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v$. Thus H_6 is four colorable. Hence b-coloring exists for every integer k satisfying $\chi(H_6) \leq k \leq \varphi(H_6)$ (Here $k = 3, 4, 5$). Hence H_6 is b-continuous.

Case 5: $n \geq 7$ For $n = 7$, $\chi(H_7) = 4$ by Lemma 2.1 and $\varphi(H_7) = 5$ by Theorem 2.2.

It is obvious that b-coloring for the graph H_7 is possible using the number of colors $k = 4, 5$. Hence b-coloring exists for every integer k satisfying $\chi(H_7) \leq k \leq \varphi(H_7)$ (Here $k = 4, 5$).

For odd $n > 7$

In this case the graph H_n is obviously b-continuous from $\chi(H_n) \leq k \leq \varphi(H_n)$ as $\chi(H_n) = 4$ and $\varphi(H_n) = 5$.

For even $n > 7$ In this case we repeat the color assignment as in case $n = 6$ discussed above for the vertices $\{v,\ v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6\ ,\ u_1, u_2,\ u_3,\ u_4,\ u_5,\ u_6\}$ and for the remaining vertices give the color as follows:

When $k = 4$

$f(v) = 4,\ f(v_{2k+5}) = 1,\ f(v_{2k+6}) = 2,\ f(u_{2k+5}) = 2,\ f(u_{2k+6}) = 1; k \in N$. Hence H_n is b-continuous.

Any coloring with $\chi(G)$ colors is a b-coloring, we state the following obvious result.

Corollary 2.4 :

$$S_b(H_n) = S_b(CH_n) = \begin{cases} \{4\}, & n = 3, 5 \\ \{3,\ 4,\ 5\}, & n = 4, 6 \\ \{4,\ 5\}, & odd\ n \geq 7 \\ \{3,\ 4,\ 5\}, & even\ n > 7. \end{cases}$$

Lemma 2.5 : For the closed helm graph

$$\chi(CH_n) = \begin{cases} 3, & n\ even \\ 4, & n\ odd. \end{cases}$$

Proof : For CH_n, the edge set of CH_n is defined as $E(CH_n) = E(H_n) \cup \{u_1u_2,\ \ u_2u_3, ...,\ \ u_{n-1}u_n\}$ and vertex set of CH_n is $V(CH_n) = V(H_n)$ where v be the apex, $v_1,\ v_2,\ ....\ , v_n$ be the vertices of degree four and $u_1,\ u_2,\ ....\ , u_n$ be the vertices of degree three.

Also $|V(CH_n)| = 2n + 1$ and $|E(CH_n)| = 4n$.

To prove this result we consider the following cases.

Case 1: n is even.

In this case the CH_n contains an odd cycles then by Proposition 1.4, $\chi(CH_n) \geq 3$. Now for proper coloring of CH_n, we need to assign color to only v, as $CH_n - v$ is 2-colorable. Hence $\chi(CH_n) = 3$.

Case 2: n is odd.

In this case CH_n contains K_3. Then by Proposition 1.2, $\chi(CH_n) \geq 3$. As v is one of the vertex of K_3 and we have used already 3-colors for proper coloring. So to color v assign new color. Hence $\chi(CH_n) = 4$.

Theorem 2.6 : For the closed helm graph CH_n

$$\varphi(CH_n) = \begin{cases} 4, & n = 3 \\ 5, & n = 4 \\ 4, & n = 5 \\ 5, & n = 6 \\ 5, & n \geq 7. \end{cases}$$

Proof : To prove the result we continue with the terminology and notations used in Lemma 2.5 and consider the following cases.

Case 1: $n = 3$

A closed helm CH_3 has, $|V(CH_3)| = 7$ and $|E(CH_3)| = 12$. Also $m(CH_3) = 4$. Then by Proposition 1.5, $\varphi(CH_3) \leq 4$. Further more the graph contains K_3, $\varphi(CH_3) \geq 3$.

We suppose that CH_3 has b-coloring using four colors. Now consider the set of colors $c = \{1,\ 2,\ 3,\ 4\ \}$ and to assign the proper coloring we define the color function $f : V \rightarrow \{1,\ 2,\ 3,\ 4\}$ as $f(v) = 4,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(u_1) = 2,\ f(u_2) = 1,\ f(u_3) = 4$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v$. Thus $\varphi(CH_3) = 4$.

Case 2: For $n = 4$

In this case, $|V(CH_4)| = 9$, $|E(CH_4)| = 16$ and $m(CH_4) = 5$. As CH_4 contains exactly $m(CH_4)$ vertices of degree $m(CH_4) - 1$ then CH_4 is a tight graph. Also $\chi(CH^*{}_4) = m(CH_4) = 5$.

Then by Proposition 1.8, $\varphi(CH_4) = 5$.

For b-coloring consider the color class $c = \{1,\ 2,\ 3,\ 4,\ 5\}$ and to assign the proper coloring we define the color function as $f : V \rightarrow \{1,\ 2,\ 3,\ 4,\ 5\}$as $f(v) = 5,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 4,\ f(u_1) = 3,\ f(u_2) = 4,\ f(u_3) = 1,\ f(u_4) = 2$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v_4,\ cdv(5) = v$. Thus $\varphi(CH_4) = 5$.

Case 3: For $n = 5$

In this case $|V(CH_5)| = 11$ and $|E(CH_5)| = 20$. Also $m(CH_5) = 5$. Then by Proposition 1.5, $\varphi(CH_5) \leq 5$. Suppose that CH_5 does have a b-chromatic 5-coloring.

Now consider the color class $c = \{1,\ 2,\ 3,\ 4,\ 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1,\ 2,\ 3,\ 4,\ 5\}$ as $f(v) = 5,\ f(v_1) = 4,\ f(v_2) = 1,\ f(v_3) = 2,\ f(v_4) = 3,\ f(u_1) = 3,\ f(u_2) = 4,\ f(u_3) = 1,\ f(u_4) = 4,\ f(u_5) = 2$ which in turn forces to assign $f(v_5) = 1$ or 2.

This proper coloring gives the color dominating vertices for color classes $1, 2$ and 5 but not for $3, 4$ which is contradiction to our assumption. Thus $\varphi(CH_5) \neq 5$.

Hence we can color the graph by four colors. For b-coloring consider the color class $c = \{1,\ 2,\ 3,\ 4\}$ and to assign the proper coloring we define the color function as $f : V \to \{1,\ 2,\ 3,\ 4\}$ as $f(v) = 4,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 1,\ f(v_5) = 2,\ f(u_1) = 3,\ f(u_2) = 1,\ f(u_3) = 2,\ f(u_4) = 3,\ f(u_5) = 1$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v$. Thus $\varphi(CH_5) = 4$.

Case 4: For $n = 6$

For the graph CH_6, $|V(CH_6)| = 13$ and $|E(CH_6)| = 24$. Also $m(CH_6) = 5$. Then by Proposition 1.5, $\varphi(CH_6) \leq 5$.

Suppose that CH_6 does have a b-chromatic 5-coloring. Now consider the color class $c = \{1,\ 2,\ 3,\ 4,\ 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1,\ 2,\ 3,\ 4,\ 5\}$ as $f(v) = 5,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 4,\ f(v_5) = 1,\ f(v_6) = 4,\ f(u_1) = 3,\ f(u_2) = 4,\ f(u_3) = 1,\ f(u_4) = 2,\ f(u_5) = 3,\ f(u_6) = 2$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v_4,\ cdv(5) = v$. Thus $\varphi(CH_6) = 5$.

Case 5: For $n \geq 7$

In this case $|V(CH_7)| = 15$ and $|E(CH_7)| = 28$. Also $m(CH_7) = 5$. Then by Proposition 1.5, $\varphi(CH_7) \leq 5$. Suppose that CH_7 does have a b-chromatic 5-coloring.

Now consider the color class $c = \{1,\ 2,\ 3,\ 4,\ 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1,\ 2,\ 3,\ 4,\ 5\}$ as $f(v) = 5,\ f(v_1) = 4,\ f(v_2) = 1,\ f(v_3) = 2,\ f(v_4) = 3,\ f(v_5) = 4,\ f(v_6) = 1,\ f(v_7) = 3,\ f(u_1) = 1,\ f(u_2) = 3,\ f(u_3) = 4,\ f(u_4) = 1,\ f(u_5) = 2,\ f(u_6) = 3,\ f(u_7) = 4$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_2,\ cdv(2) = v_3,\ cdv(3) = v_4,\ cdv(4) = v_5,\ cdv(5) = v$. Thus $\varphi(CH_7) = 5$.

For $n > 7$:

We repeat the colors as in the graph CH_7 for the vertices $\{v, v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6,\ v_7,\ u_1,\ u_2,\ u_3,\ u_4,\ u_6,\ u_7,\}$ and for the remaining vertices assign the colors as $f(v) = 5,\ f(v_{2k+6}) = 1,\ f(v_{2k+7}) = 2,\ f(u_{2k+6}) = 2,\ f(u_{2k+7}) = 4; k \in N$.

Hence $\varphi(CH_n) = 5,\ n \geq 7$

Theorem 2.7 : CH_n is b-continuous.

Proof :To prove this result we continue with the terminology and notations used in Lemma 2.5 and consider the following cases.

Case 1: $n = 3$

In this case the graph CH_3 is b-continuous as $\chi(CH_3) = \varphi(CH_3) = 4$.

Case 2: $n = 4$

By Lemma 2.5, $\chi(CH_4) = 3$ and by Theorem 2.6, $\varphi(CH_4) = 5$. It is obvious that b-coloring for the graph CH_4 is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the b-coloring of the graph CH_4 is as follows. Consider the color class $c = \{1,\ 2,\ 3,\ 4\}$ and to assign the proper coloring we define the color function $f : V \to \{1,\ 2,\ 3,\ 4\}$ as $f(v) = 4,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 2,\ f(u_1) = 3,\ f(u_2) = 4,\ f(u_3) = 1,\ f(u_4) = 4$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v$. Thus CH_4 is four colorable. Hence b-coloring exists for every integer k satisfying $\chi(CH_4) \leq k \leq \varphi(CH_4)$ (Here $k = 3, 4, 5$). Consequently CH_4 is b-continuous.

Case 3: $n = 5$

In this case the graph CH_5 is b-continuous as $\chi(CH_5) = \varphi(CH_5) = 4$.

Case 4: $n = 6$

By Lemma 2.5, $\chi(CH_6) = 3$ and by Theorem 2.6, $\varphi(CH_6) = 5$. It is obvious that b-coloring for the graph CH_6 is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the b-coloring of the graph CH_6 is as follows.

Consider the color class $c = \{1,\ 2,\ 3,\ 4\}$ and to assign the proper coloring we define the color function $f : V \to \{1,\ 2,\ 3,\ 4\}$ as $f(v) = 4,\ f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 1,\ f(v_5) = 2,\ f(v_6) = 3,\ f(u_1) = 3,\ f(u_2) = 1,\ f(u_3) = 2,\ f(u_4) = 3,\ f(u_5) = 1,\ f(u_6) = 2$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1,\ cdv(2) = v_2,\ cdv(3) = v_3,\ cdv(4) = v$. Thus CH_6 is four colorable. Hence b-coloring exists for every integer k satisfying $\chi(CH_6) \leq k \leq \varphi(CH_6)$. Hence CH_6 is b-continuous.

Case 5: $n \geq 7$

For $n = 7$, from Lemma 2.5, we have $\chi(CH_7) = 4$ and by Theorem 2.6, $\varphi(CH_7) = 5$. It is obvious that b-coloring for the graph CH_7 is possible using the number of colors $k = 4, 5$. Hence b-coloring exists for every integer k satisfying $\chi(CH_7) \leq k \leq \varphi(CH_7)$ (Here $k = 4, 5$).

For odd $n > 7$

In this case the graph CH_n is obviously b-continuous from $\chi(CH_n) \leq k \leq \varphi(CH_n)$ as $\chi(CH_n) = 4$ and $\varphi(CH_n) = 5$.

For even $n > 7$

In this case we repeat the color assignment as in case $n = 6$ discussed above for the vertices $\{v,\ v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6\ ,\ u_1, u_2,\ u_3,\ u_4,\ u_5,\ u_6\}$ and for the remaining vertices give the color as follows.

When $k = 4$

$f(v) = 4,\ f(v_{2k+5}) = 2,\ f(v_{2k+6}) = 3,\ f(u_{2k+5}) = 1,\ f(u_{2k+6}) = 2$ where $k \in N$. Hence CH_n is b-continuous.

III. Conclusions

We have obtained b-chromatic number for the larger graphs obtained from the standard graphs. We have determined the b-chromatic number of helm and closed helm which are obtained by adding edges in wheel.

IV. Acknowledgement

Our thanks are due to the anonymous referee for careful reading of the paper.

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