The $b$-Chromatic Number of Helm and Closed Helm

S. K. Vaidya and M. S. Shukla

Abstract—The helm $H_n$ is the graph obtained from wheel $W_n = C_n + K_1$ by attaching a pendant edge to each rim vertex while the closed helm is the graph obtained from helm by joining each pendant vertex to form a cycle. We investigate $b$-chromatic numbers for helm and closed helm.

Index Terms—$b$-Coloring, $b$-Continuity, $b$-Spectrum.

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I. INTRODUCTION

We begin with a simple, finite, connected and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A coloring of the vertices of $G$ is a mapping $f : V(G) \rightarrow \mathbb{N}$. For every vertex $v \in V(G)$, $f(v)$ is called the color of $v$.

If any two adjacent vertices have different colors then $f$ is called proper coloring. The chromatic number $\chi(G)$ is the smallest integer $k$ for which $G$ admits a proper coloring using $k$ colors. The set of vertices with a particular color is called a color class.

A $b$-coloring by $b$ colors is a proper coloring of the vertices of $G$ such that in each color class there exists a vertex which has neighbours in all the other $k-1$ color classes. In other words each color class contains a vertex which has at least one neighbour in all the other color classes. Such vertex is called a color dominating vertex. It is obvious that every coloring of a graph $G$ by $\chi(G)$ colors is a $b$-coloring of $G$.

The $b$-chromatic number $\varphi(G)$ is the largest integer $k$ such that $G$ admits a $b$-coloring with $k$ colors. The concept of $b$-coloring was introduced by Irving and Manlove [1] and showed that the problem of determining $\varphi(G)$ is NP-hard for general graphs but it is polynomial-time solvable for trees. In the same paper they have introduced the concepts of $b$-continuity and $b$-spectrum.

If the $b$-coloring exists for every integer $k$ satisfying $\chi(G) \leq k \leq \varphi(G)$ then $G$ is called $b$-continuous and the $b$-spectrum $\varphi_{b}(G)$ of a graph $G$ is the set of $k$ integers(colors) for which $G$ has a $b$-coloring. A graph $G$ is tight if it has exactly $m(G)$ vertices of degree $m(G) - 1$. It has been proved by Havet et al. [2] that all tight graph are $b$-colorable. The bounds for the $b$-chromatic number for various graphs are established in the work of Kouider and Maheo [3] while $b$-chromatic number for Peterson graph and power of a cycle is discussed by Chandrakumar and Nicholas [4].

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The $b$-continuity of chordal graphs is discussed by Faik [5]. Also the discussion on $b$-coloring of central graph of some graphs is reported in Thilagavathi et al. [6].

Many results on $b$-coloring and $b$-continuity are reported in Alkhattee [7] and the $b$-chromatic number for cartesian product of some families of graphs is explored by Balakrishnan et al. [8] while Vaidya and Shukla [9,10] have investigated $b$-chromatic number of some cycle and wheel related graphs.

Proposition 1.1 [1]: If graph $G$ admits a $b$-coloring with $m$-colors, then $G$ must have at least $m$ vertices with degree at least $m - 1$ (Since each color class has a $b$-vertex).

Proposition 1.2 [11]: If the graph $G$ contains $K_n$ as a subgraph, $\chi(G) \geq n$.

Definition 1.3 [1]: The $m$-degree of a graph $G$, denoted by $m(G)$, is the largest integer $m$ such that $G$ has $m$ vertices of degree at least $m - 1$.

Proposition 1.4 [11]: For any graph $G$, $\chi(G) \geq 3$ if and only if $G$ has an odd cycle.

Proposition 1.5 [3]: $\chi(G) \leq \varphi(G) \leq m(G)$.

Definition 1.6: A vertex of $G$ with degree at least $m(G) - 1$ is called a dense vertex.

Definition 1.7: Let $G$ be a tight graph. The $b$-closure of $G$, denoted by $G^*$, is the graph with vertex set $V(G^*) = V(G)$ and edge set $E(G^*) = E(G) \cup \{uv ; u$ and $v$ are vertices with a common dense neighbour\}.

Proposition 1.8 [2]: Let $G$ be a tight graph. Then $\varphi(G) = m(G)$ if and only if $\chi(G^*) = m(G)$.

Definition 1.9: The wheel $W_n$ is defined to be the join of $K_1 + C_n$. The vertex corresponding to $K_1$ is known as apex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges.

Definition 1.10: The helm $H_n$ is the graph obtained from wheel $W_n$ by attaching a pendant edge to each rim vertex. It contains three types of vertices: an apex of degree $n$, $n$ vertices of degree $4$ and $n$ pendant vertices.

Definition 1.11: A closed helm $CH_n$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to form a cycle. It contains three types of vertices: an apex of degree $n$, $n$ vertices of degree $4$ and $n$ vertices of degree $3$.

We continue to recognize apex of wheel as the apex of respective graphs corresponding to Definitions 1.10 and 1.11.
II. MAIN RESULTS

Lemma 2.1: For the helm graph $H_n$

$\chi(H_n) = \begin{cases} 
3, & n \text{ is even} \\
4, & n \text{ is odd}.
\end{cases}$

Proof: For $H_n$, let $\{e_1, e_2, \ldots, e_n\}$ be the spoke edges of $H_n$ and $\{e_{n+1}, e_{n+2}, \ldots, e_{3n}\}$ be the rim edges of the cycle in $H_n$ while $\{e_{2n+1}, e_{2n+2}, \ldots, e_{3n}\}$ be the pendant edges of $H_n$.

Moreover $\{v_1, u_2, \ldots, u_n\}$ be the pendant vertices of $H_n$ and $\{v_1, v_2, \ldots, v_n\}$ be the vertices of degree 4. Denote the apex of $H_n$ as $v$. Also $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$.

To prove the result we consider the following cases.

Case 1: $n$ is even.

In this case $H_n$ contains a cycle $C_3$. Then by Proposition 1.4, $\chi(H_n) \geq 3$. If we assign a proper coloring as $f(v) = 3$, $f(v_{2k-1}) = 1$, $f(v_{2k}) = 2$, $f(u_{2k-1}) = 2$, $f(u_{2k}) = 1; k \in N$ then $\chi(H_n) = 3$.

Case 2: $n$ is odd.

In this case $H_n$ contains $K_3$. Then by Proposition 1.2, $\chi(H_n) \geq 3$. As $v$ is one of the vertex of $K_3$ and we have used already 3-colors for proper coloring. So to color $v$ assign new color. Hence $\chi(H_n) = 4$.

Theorem 2.2: For the helm graph $H_n$

$\varphi(H_n) = \begin{cases} 
4, & n = 3 \\
5, & n = 4 \\
5, & n = 5 \\
5, & n = 6 \\
5, & n \geq 7.
\end{cases}$

Proof: We continue with the terminology and notations used in Lemma 2.1 and consider the following cases.

Case 1: For $n = 3$

In this case, $|V(H_3)| = 7$ and $|E(H_3)| = 9$. Also $m(H_3) = 4$. Then by Proposition 1.5, $\varphi(H_3) \leq 4$. Further more the graph contains $K_3$ then according to Proposition 1.2, $\varphi(H_3) \geq 3$.

If possible $H_3$ has $b$-coloring using four colors with $c = \{1, 2, 3, 4\}$. To assign the proper coloring define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v) = 4, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(u_1) = 2, f(u_2) = 1, f(u_3) = 4$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v$. Thus $\varphi(H_3) = 4$.

Case 2: For $n = 4$

In this case, $|V(H_4)| = 9$, $|E(H_4)| = 12$ and $m(H_4) = 5$. As $H_4$ contains exactly $m(H_4)$ vertices of degree $m(H_4) - 1$ then $H_4$ is a tight graph. Also $\chi(H_4) = m(H_4) = 5$. Then by Proposition 1.8, $\varphi(H_4) = 5$.

For $b$-coloring consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v) = 5, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(u_1) = 4, f(u_2) = 3, f(u_3) = 4, f(v_4) = 1, f(u_5) = 2$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4, cdv(5) = v$. Thus $\varphi(H_4) = 5$.

Case 3: For $n = 5$

In this graph $|V(H_5)| = 11$ and $|E(H_5)| = 15$. Also $m(H_5) = 5$. Then by Proposition 1.5, $\varphi(H_5) \leq 5$. Suppose that $H_5$ does have a $b$-chromatic 5-coloring.

Now consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v) = 5, f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(u_1) = 3, f(u_2) = 4, f(u_3) = 4, f(u_4) = 2, f(u_5) = 2$ which in turn forces to assign $f(v_5) = 1$.

This proper coloring gives the color dominating vertices for color classes 1, 2 and 5 but not for 3, 4 which is contradiction to our assumption. Thus $\varphi(H_5) \neq 5$.

Hence we can color the graph by four colors. For $b$-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function as $f : V \to \{1, 2, 3, 4\}$ as $f(v) = 4, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 1, f(v_5) = 2, f(u_1) = 3, f(u_2) = 4, f(u_3) = 1, f(u_4) = 2, f(u_5) = 3$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v$. Thus $\varphi(H_5) = 4$.

Case 4: For $n = 6$

For the graph $H_6$, $|V(H_6)| = 13$ and $|E(H_6)| = 18$. Also $m(H_6) = 5$. Then by Proposition 1.5 $\varphi(H_6) \leq 5$. Suppose that $H_6$ does have a $b$-chromatic 5-coloring.

Now consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v) = 5, f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 1, f(v_6) = 2, f(v_7) = 3, f(u_1) = 4, f(u_2) = 1, f(u_3) = 3, f(u_4) = 1, f(u_5) = 2, f(u_6) = 3$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_5, cdv(5) = v$. Thus $\varphi(H_6) = 5$.

Case 5: For $n \geq 7$

In this case $|V(H_7)| = 15$ and $|E(H_7)| = 21$. Also $m(H_7) = 5$. Then by Proposition 1.5 $\varphi(H_7) \leq 5$. Suppose that $H_7$ does have a $b$-chromatic 5-coloring.

Now consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring we define the color function as $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v) = 5, f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v_7) = 1, f(u_1) = 1, f(u_2) = 3, f(u_3) = 4, f(u_4) = 1, f(u_5) = 5, f(u_6) = 1, f(u_7) = 2$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v$. Thus $\varphi(H_7) = 5$.

For $n \geq 7$:

We repeat the colors as in the graph $H_7$ for the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and for the remaining vertices assign the colors as $f(v) = 5, f(v_{2k+6}) = 2, f(v_{2k+7}) = 1, f(u_{2k+6}) = 1, f(u_{2k+7}) = 2; k \in N$.

Hence $\varphi(H_n) = 5, n \geq 7$.

Theorem 2.3: $H_n$ is 5-colorable.

Proof: To prove this result we continue with the terminology and notations used in Lemma 2.1 and consider the following cases.

Case 1: $n = 3$

In this case, $H_3$ is 5-colorable as $\chi(H_3) = \varphi(H_3) = 4$.
Case 2: $n = 4$

By Lemma 2.1, $\chi(H_4) = 3$ and by Theorem 2.2, $\varphi(H_4) = 5$. It is obvious that $b$-coloring for the graph $H_4$ is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the $b$-coloring of the graph $H_4$ is as follows.

Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function $f : V \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 2$, $f(u_1) = 3$, $f(u_2) = 1$, $f(u_3) = 1$, $f(u_4) = 1$.

This proper coloring gives the color dominating vertices as $\text{cdv}(1) = v_1$, $\text{cdv}(2) = v_2$, $\text{cdv}(3) = v_3$, $\text{cdv}(4) = v$. Thus $H_4$ is four colorable. Hence $b$-coloring exists for every integer $k$ satisfying $\chi(H_4) \leq k \leq \varphi(H_4)$ (Here $k = 3, 4, 5$). Consequently $H_4$ is $b$-continuous.

Case 3: $n = 5$

In this case the graph $H_5$ is $b$-continuous as $\chi(H_5) = \varphi(H_5) = 4$.

Case 4: $n = 6$

In this case by Lemma 2.1, $\chi(H_6) = 3$ and by Theorem 2.2, $\varphi(H_6) = 5$. It is obvious that $b$-coloring for the graph $H_6$ is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the $b$-coloring of the graph $H_6$ is as follows.

Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function $f : V \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 1$, $f(v_5) = 2$, $f(v_6) = 3$, $f(u_1) = 3$, $f(u_2) = 1$, $f(u_3) = 1$, $f(u_4) = 1$, $f(u_5) = 1$, $f(u_6) = 1$.

This proper coloring gives the color dominating vertices as $\text{cdv}(1) = v_1$, $\text{cdv}(2) = v_2$, $\text{cdv}(3) = v_3$, $\text{cdv}(4) = v$. Thus $H_6$ is four colorable. Hence $b$-coloring exists for every integer $k$ satisfying $\chi(H_6) \leq k \leq \varphi(H_6)$ (Here $k = 3, 4, 5$). Hence $H_6$ is $b$-continuous.

Case 5: $n \geq 7$ For $n = 7$, $\chi(H_7) = 4$ by Lemma 2.1 and $\varphi(H_7) = 5$ by Theorem 2.2.

It is obvious that $b$-coloring for the graph $H_7$ is possible using the number of colors $k = 4, 5$. Hence $b$-coloring exists for every integer $k$ satisfying $\chi(H_7) \leq k \leq \varphi(H_7)$ (Here $k = 4, 5$).

For odd $n > 7$

In this case the graph $H_n$ is obviously $b$-continuous from $\chi(H_n) \leq k \leq \varphi(H_n)$ as $\chi(H_n) = 4$ and $\varphi(H_n) = 5$.

For even $n > 7$

In this case we repeat the color assignment as in case $n = 6$ discussed above for the vertices $\{v, v_1, v_2, v_3, v_4, v_5, v_6, u_1, u_2, u_3, u_4, u_5, u_6\}$ and for the remaining vertices give the color as follows:

When $k = 4$

$f(v) = 4$, $f(v_{2k+5}) = 1$, $f(v_{2k+6}) = 2$, $f(u_{2k+4}) = 1$; $k \in N$. Hence $H_n$ is $b$-continuous.

Any coloring with $\chi(G)$ colors is a $b$-coloring, we state the following obvious result.

Corollary 2.4:

$S_b(H_n) = S_b(CH_n) = \begin{cases} \{4\}, & n = 3, 5 \\ \{3, 4, 5\}, & n = 4, 6 \\ \{4, 5\}, & \text{odd } n \geq 7 \\ \{3, 4, 5\}, & \text{even } n > 7. \end{cases}$

Lemma 2.5: For the closed helm graph $CH_n$

$\chi(CH_n) = \begin{cases} 3, & n \text{ even} \\ 4, & n \text{ odd}. \end{cases}$

Proof: For $CH_n$, the edge set of $CH_n$ is defined as $E(CH_n) = E(H_n) \cup \{u_{12}, u_{23}, ..., u_{n-1}\}$. The vertex set of $CH_n$ is $V(CH_n) = V(H_n)$ where $v$ be the apex, $v_1$, $v_2$, ..., $v_n$ be the vertices of degree four and $u_1$, $u_2$, ..., $u_n$ be the vertices of degree three.

Now for every $n \in \mathbb{N}$, $n \geq 3$.

$\chi(CH_n) = 4$.

To prove this result we consider the following cases.

Case 1: $n$ is even.

In this case the $CH_n$ contains all the cycles then by Proposition 1.4, $\chi(CH_n) \geq 3$. Now for proper coloring of $CH_n$, we need to assign color to only $v$, as $CH_n - v$ is 2-colorable. Hence $\chi(CH_n) = 3$.

Case 2: $n$ is odd.

In this case $CH_n$ contains $K_3$. Then by Proposition 1.2, $\chi(CH_n) \geq 3$. As $v$ is one of the vertex of $K_3$ and we have used already 3-colors for proper coloring. So to color $v$ assign new color. Hence $\chi(CH_n) = 4$.

Theorem 2.6: For the closed helm graph $CH_n$

$\varphi(CH_n) = \begin{cases} 4, & n = 3 \\ 5, & n = 4 \\ 4, & n = 5 \\ 5, & n = 6 \\ 5, & n \geq 7. \end{cases}$

Proof: To prove the result we continue with the terminology and notations used in Lemma 2.5 and consider the following cases.

Case 1: $n = 3$

A closed helm $CH_3$ has, $|V(CH_3)| = 7$ and $|E(CH_3)| = 12$. Also $m(CH_3) = 4$. Then by Proposition 1.5, $\varphi(CH_3) \leq 4$. Further more the graph contains $K_3$, $\varphi(CH_3) \geq 3$.

We suppose that $CH_3$ has $b$-coloring using four colors. Now consider the set of colors $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function $f : V \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(u_1) = 2$, $f(u_2) = 1$, $f(u_3) = 4$.

This proper coloring gives the color dominating vertices as $\text{cdv}(1) = v_1$, $\text{cdv}(2) = v_2$, $\text{cdv}(3) = v_3$, $\text{cdv}(4) = v$. Thus $\varphi(CH_3) = 4$.

Case 2: For $n = 4$

In this case, $|V(CH_4)| = 9$, $|E(CH_4)| = 16$ and $m(CH_4) = 5$. As $CH_4$ contains exactly $m(CH_4)$ vertices of degree $m(CH_4) - 1$ then $CH_4$ is a tight graph. Also $\chi(CH_4^{*}) = m(CH_4) = 5$.

Then by Proposition 1.8, $\varphi(CH_4) = 5$.

For $b$-coloring consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring we define the color function as $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v) = 5$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 4$, $f(u_1) = 3$, $f(u_2) = 4$, $f(u_3) = 1$, $f(u_4) = 2$.

This proper coloring gives the color dominating vertices as $\text{cdv}(1) = v_1$, $\text{cdv}(2) = v_2$, $\text{cdv}(3) = v_3$, $\text{cdv}(4) = v_4$, $\text{cdv}(5) = v$. Thus $\varphi(CH_4) = 5$. 
Case 3: For $n = 5$

In this case $|V(CH_5)| = 11$ and $|E(CH_5)| = 20$. Also $m(CH_5) = 5$. Then by Proposition 1.5, $\varphi(CH_5) \leq 5$. Suppose that $CH_5$ does have a $b$-chromatic 5-coloring.

Now consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring we define the color function as $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v) = 5$, $f(u_1) = 4$, $f(u_2) = 1$, $f(v_3) = 2$, $f(v_4) = 3$, $f(u_5) = 4$, $f(u_6) = 2$ which in turn forces to assign $f(v_5) = 1$ or 2.

This proper coloring gives the color dominating vertices for $n = 5$.

Case 4: For $n = 6$

For the graph $CH_6$, $|V(CH_6)| = 13$ and $|E(CH_6)| = 24$. Also $m(CH_6) = 5$. Then by Proposition 1.5, $\varphi(CH_6) \leq 5$.

Suppose that $CH_6$ does have a $b$-chromatic 5-coloring.

Now consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring we define the color function as $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v) = 5$, $f(u_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(u_4) = 1$, $f(v_5) = 3$, $f(u_6) = 1$, $f(u_7) = 2$, $f(u_8) = 3$, $f(u_9) = 3$, $f(u_{10}) = 3$, $f(u_{11}) = 3$, $f(u_{12}) = 2$, $f(u_{13}) = 3$, $f(u_{14}) = 3$, $f(u_{15}) = 3$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1$, $cdv(2) = v_2$, $cdv(3) = v_3$, $cdv(4) = v_4$, $cdv(5) = v_6$. Thus $\varphi(CH_6) = 5$.

Case 5: For $n \geq 7$

In this case $|V(CH_7)| = 15$ and $|E(CH_7)| = 28$. Also $m(CH_7) = 5$. Then by Proposition 1.5, $\varphi(CH_7) \leq 5$.

Suppose that $CH_7$ does have a $b$-chromatic 5-coloring.

Now consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring we define the color function as $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v) = 5$, $f(u_1) = 4$, $f(v_2) = 1$, $f(v_3) = 2$, $f(v_4) = 3$, $f(u_5) = 4$, $f(u_6) = 1$, $f(u_7) = 2$, $f(u_8) = 3$, $f(u_9) = 3$, $f(u_{10}) = 3$, $f(u_{11}) = 3$, $f(u_{12}) = 2$, $f(u_{13}) = 3$, $f(u_{14}) = 3$, $f(u_{15}) = 3$. This proper coloring gives the color dominating vertices as $cdv(1) = v_2$, $cdv(2) = v_3$, $cdv(3) = v_4$, $cdv(4) = v_5$, $cdv(5) = v_6$. Thus $\varphi(CH_7) = 5$.

For $n \geq 7$:

We repeat the colors as in the graph $CH_7$ for the vertices $\{v, v_1, v_2, v_3, v_4, v_5, v_6, v_7, u_1, u_2, u_3, u_4, u_6, u_7\}$ and for the remaining vertices assign the colors as $f(v) = 5$, $f(v_{2k+6}) = 1$, $f(v_{2k+7}) = 2$, $f(u_{2k+6}) = 2$, $f(u_{2k+7}) = 4$; $k \in N$.

Hence $\varphi(CH_n) = 5$, $n \geq 7$

Theorem 2.7: $CH_n$ is $b$-continuous.

Proof: To prove this result we continue with the terminology and notations used in Lemma 2.5 and consider the following cases.

Case 1: $n = 3$

In this case the graph $CH_3$ is $b$-continuous as $\chi(CH_3) = \varphi(CH_3) = 4$.

Case 2: $n = 4$

By Lemma 2.5, $\chi(CH_4) = 3$ and by Theorem 2.6, $\varphi(CH_4) = 4$. It is obvious that $b$-coloring for the graph $CH_4$ is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the $b$-coloring of the graph $CH_4$ is as follows. Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function $f : V \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(u_4) = 2$, $f(u_5) = 3$, $f(u_6) = 4$, $f(u_7) = 4$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1$, $cdv(2) = v_2$, $cdv(3) = v_3$, $cdv(4) = v_4$. Thus $CH_4$ is four colorable. Hence $b$-coloring exists for every integer $k$ satisfying $\chi(CH_4) \leq k \leq \varphi(CH_4)$ (Here $k = 3, 4, 5$). Consequently $CH_4$ is $b$-continuous.

Case 3: $n = 5$

In this case the graph $CH_5$ is $b$-continuous as $\chi(CH_5) = \varphi(CH_5) = 4$.

Case 4: $n = 6$

By Lemma 2.5, $\chi(CH_6) = 3$ and by Theorem 2.6, $\varphi(CH_6) = 5$. It is obvious that $b$-coloring for the graph $CH_6$ is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the $b$-coloring of the graph $CH_6$ is as follows.

Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function $f : V \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 1$, $f(v_5) = 2$, $f(v_6) = 3$, $f(u_7) = 3$, $f(u_8) = 3$, $f(u_9) = 3$, $f(u_{10}) = 3$, $f(u_{11}) = 3$, $f(u_{12}) = 2$, $f(u_{13}) = 3$, $f(u_{14}) = 3$, $f(u_{15}) = 1$, $f(u_{16}) = 2$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1$, $cdv(2) = v_2$, $cdv(3) = v_3$, $cdv(4) = v_4$. Thus $CH_6$ is four colorable. Hence $b$-coloring exists for every integer $k$ satisfying $\chi(CH_6) \leq k \leq \varphi(CH_6)$. Hence $CH_6$ is $b$-continuous.

Case 5: $n \geq 7$

For $n = 7$, from Lemma 2.5, we have $\chi(CH_7) = 4$ and by Theorem 2.6, $\varphi(CH_7) = 5$. It is obvious that $b$-coloring for the graph $CH_7$ is possible using the number of colors $k = 4, 5$. Hence $b$-coloring exists for every integer $k$ satisfying $\chi(CH_7) \leq k \leq \varphi(CH_7)$ (Here $k = 4, 5$).

For odd $n \geq 7$

In this case the graph $CH_n$ is obviously $b$-continuous from $\chi(CH_n) \leq k \leq \varphi(CH_n)$ as $\chi(CH_n) = 4$ and $\varphi(CH_n) = 5$.

For even $n > 7$

In this case we repeat the color assignment as in case $n = 6$ discussed above for the vertices $\{v, v_1, v_2, v_3, v_4, v_5, v_6, u_1, u_2, u_3, u_4, u_5, u_6\}$ and for the remaining vertices give the color as follows.

When $k = 4$

$f(v) = 4$, $f(v_{2k+5}) = 2$, $f(v_{2k+6}) = 3$, $f(u_{2k+5}) = 1$, $f(u_{2k+6}) = 2$ where $k \in N$. Hence $CH_n$ is $b$-continuous.
III. CONCLUSIONS

We have obtained \( b \)-chromatic number for the larger graphs obtained from the standard graphs. We have determined the \( b \)-chromatic number of helm and closed helm which are obtained by adding edges in wheel.

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