Vertex Bi-magic Graphs from Magic and Anti-magic Graphs

A. Amara Jothi*, N.G. David and J. Baskar Babujee

***Abstract*—Let $G(V, E)$ be a graph of order p and size q and let $\lambda : V \cup E \rightarrow \{1, 2, ..., p+q\}$ be a bijective mapping. λ is called a vertex magic total labeling of G if at each vertex x, the vertex weight under this λ, $wt_\lambda(x) = \lambda(x) + \sum_{xy \in E} \lambda(xy) = \alpha$, a constant. λ is called a vertex bi-magic total labeling of G if the vertex weight at each vertex is either α or β, where α, β are two fixed constants. It is called (a, d) vertex anti-magic total labeling of G if the set of vertex weights of all vertices in G is $\{a, a+d, a+2d, ..., a+(p-1)d\}$, where $a, d > 0$ are integers. In this article, we introduce two other variations of bi-magic labeling namely (1, 0) vertex bi-magic and (0, 1) vertex bi-magic and also discuss new techniques of generating (1, 1) vertex bi-magic, (1, 0) vertex bi-magic and (0, 1) vertex bi-magic graphs using some operations on vertex magic and vertex anti-magic graphs.**

***Index Terms*—Vertex magic graph, Vertex bi-magic graph, Vertex anti-magic graph, Bijective function.**

***MSC 2010 Codes* – 05C78**

I. PRELIMINARY

ALL graphs in this article are assumed to be finite, simple and undirected. In [1] MacDougall et al. introduced the notion of vertex magic total labeling (VMTL) and studied the basic properties of this labeling and also given methods of assigning VMTL for several families of graphs, including cycle, paths, complete graphs of odd order and complete bipartite graphs. They also have identified few families of graphs which do not admit VMTL. VMTL has also been extensively studied by many authors [2, 3, 4, 5]. (a, d) vertex anti-magic total labeling (VATL) was introduced by Baca et al. in [6]. In [7] Yegnanarayanan defined several variations of vertex magic labelings (VMLs) and vertex anti-magic labelings (VALs) namely (1, 1) VML, (1, 0) VML, (0, 1) VML, (1, 1)-(a, d) VAL, (1, 0)-(a, d) VAL, (0, 1)-(a, d) VAL also investigated the existence of such labeling on a number of classes of graphs. (1, 1) vertex bi-magic labeling (VBL) was introduced by Baskar Babujee in [8]. For a survey on VML, VAL and VBL of graphs we refer to [9]. In this article, we introduce two other variations of labelings namely (1, 0) VBL and (0, 1) VBL and new variations of vertex bi-magic graphs are discussed using some operations on vertex magic and vertex anti-magic graphs. The following definitions are related to our study.

Definition 1.1 [7, 8] Let $G(p, q)$ be a graph and let $\lambda : V \cup E \rightarrow \{1, 2, ..., p+q\}$ be a bijection. λ is called a (1, 1) VML of G if at each vertex x, the vertex weight under this λ, $wt_\lambda(x) = \lambda(x) + \sum_{xy \in E} \lambda(xy) = \alpha$, a constant. λ is called a (1, 1) VBL of G if the vertex weight at each vertex x is either α or β, where α, β are two fixed constants. It is called (1, 1)-(a, d) VAL of G if the set of vertex weights of all vertices in G is $\{a, a+d, a+2d, ..., a+(p-1)d\}$, where $a, d > 0$ are integers.

Definition 1.2 [7] Let $G(p, q)$ be a graph and let $\lambda : V \rightarrow \{1, 2, ..., p\}$ be a bijection. λ is called a (1, 0) VML of G if at each vertex x, the vertex weight under this λ, $wt_\lambda(x) = \lambda(x) + \sum_{y \in V,\ xy \in E} \lambda(y) = \alpha$, a constant. λ is called a (1, 0) VBL of G if the vertex weight at each vertex is either α or β, where α, β are two fixed constants. It is called (1, 0)-(a, d) VAL of G if the set of vertex weights of all vertices in G is $\{a, a+d, a+2d, ..., a+(p-1)d\}$, where $a, d > 0$ are integers.

Definition 1.3 [7] Let $G(p, q)$ be a graph and let $\lambda : E \rightarrow \{1, 2, ..., q\}$ be a bijection. λ is called a (0, 1) VML of G if at each vertex x, the vertex weight under this λ, $wt_\lambda(x) = \sum_{xy \in E} \lambda(xy) = \alpha$, a constant. λ is called a (0, 1) VBL of G if the vertex weight at each vertex is either α or β, where α, β are two fixed constants. It is called (0, 1)-(a, d) VAL of G if the set of vertex weights of all vertices in G is $\{a, a+d, a+2d, ..., a+(p-1)d\}$, where $a, d > 0$ are integers.

Definition 1.4 [10] Duplication of a vertex $x \in V$ by an edge $x^{'}x^{''}$ in a graph G produces a new graph $G^{'}$ in which the neighborhood of $x^{'}$ and $x^{''}$ are respectively $N(x^{'}) = \{x, x^{''}\}$ and $N(x^{''}) = \{x, x^{'}\}$.

Definition 1.5 Let G_1 and G_2 be any two graphs with order p_1 and p_2, respectively. Then $G_1 \odot G_2$ is obtained by taking G_1, p_1 copies of G_2 and joining i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

Definition 1.6 Let G_1 and G_2 be any two graphs of orders p_1 and p_2, respectively. Then $G_1 + G_2$ is obtained by adding an edge between every vertex of G_1 and G_2.

II. MAIN RESULTS

Theorem 2.1 If G has (1, 1)-(a, 1) VAL, then $G + K_1$ has (1, 1) VBL.

Proof Let G be a (1, 1)-(a, 1) vertex anti-magic graph with p vertices $\{x_i : 1 \leq i \leq p\}$ and q edges. Consider the bijection $\lambda : V \cup E \rightarrow \{1, 2, ..., p+q\}$ on G with the anti-magic property that vertex weight $wt_\lambda(x_i) = a + p - i; 1 \leq i \leq p$ for $a > 0$. Let $G^{\prime}(V^{\prime}, E^{\prime}) = G + K_1$ be the graph with $V^{\prime} = V \cup \{z\}$

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and $E' = E \cup \{zx_i : 1 \leq i \leq p\}$. Consider the bijection $\mu : V' \cup E' \to \{1, 2, ..., p+q, p+q+1, ..., 2p+q+1\}$ on defined as follows:
$\mu(x_i) = \lambda(x_i); 1 \leq i \leq p$, $\mu(e) = \lambda(e)$, for all $e \in E$.
$\mu(z) = 2p+q+1$, and $\mu(zx_i) = p+q+i$, for $1 \leq i \leq p$.
To complete the proof, it is necessary to show the existence of α and β.
For $x_i \in V, 1 \leq i \leq p$ we have
$wt_\mu(x_i) = \mu(zx_i) + wt_\lambda(x_i)$
$= (p+q+i) + (a+p-i) = 2p+q+a = \alpha$.
For $z \in V'$, $wt_\mu(z) = \sum \mu(zx_i) + \mu(z)$
$= \sum_{i=1}^{p}(p+q+i) + (2p+q+1)$
$= (p+q)p + \frac{p(p+1)}{2} + 2p+q+1 = \frac{1}{2}(3p^2+2pq+2q+2)$
$= (p+1)(q+1+\frac{3p}{2}) = \beta$.
Therefore, G' admits (1, 1) VBL for $G+K_1$ with the weights $\alpha = 2p+q+a$ and $\beta = (p+1)(q+1+\frac{3p}{2})$.

Theerem 2.2 Let G be a graph of odd order. If G has (1, 1) VML, then $G+2K_1$ admits (1, 1) VBL.

Proof Let $G(p,q)$ with odd p be a graph that admits (1, 1) VML. Then there exists a mapping $\lambda : V \cup E \to \{1, 2, ..., p+q\}$ such that the weight at each vertex $wt_\lambda(x_i) = a$ is a constant. Consider the new graph $G'(V', E') = G+2K_1$ with $V' = V \cup \{y, z\}$ and $E' = E \cup \{yx_i, zx_i : 1 \leq i \leq p\}$. Define a bijection $\mu : V' \cup E' \to \{1, 2, ..., 3p+q+2\}$ as given below:
Case(i) : When $n = 4k+1, k \in N$
For $i = 2, 4, 6, ..., \frac{p-1}{2}$; $\mu(yx_{2i}) = p+q+2i$, $\mu(yx_{2i+1}) = p+q+2i+1$, $\mu(zx_{2i}) = 3p+q+1-2i$, $\mu(zx_{2i+1}) = 3p+q-2i$.
For $i = 1, 3, 5..., \frac{p-3}{2}$; $\mu(yx_{2i}) = 3p+q+1-2i$, $\mu(yx_{2i+1}) = 3p+q-2i$, $\mu(zx_{2i}) = p+q+2i$, $\mu(zx_{2i+1}) = p+q+1+2i$.
Case(ii) : When $n = 4k-1, k \in N$
For $i = 2, 4, 6, ..., \frac{p-3}{2}$; $\mu(yx_{2i}) = p+q+2i$, $\mu(yx_{2i+1}) = p+q+2i+1$, $\mu(zx_{2i}) = 3p+q+1-2i$, $\mu(zx_{2i+1}) = 3p+q-2i$.
For $i = 1, 3, 5..., \frac{p-1}{2}$; $\mu(yx_{2i}) = 3p+q+1-2i$, $\mu(yx_{2i+1}) = 3p+q-2i$, $\mu(zx_{2i}) = p+q+2i$, $\mu(zx_{2i+1}) = p+q+1+2i$.
$\mu(y) = 3p+q+2$, $\mu(yx_1) = p+q+1$, $\mu(z) = 3p+q+1$ and $\mu(zx_1) = 3p+q$.
In the following, we compute the weights of vertices in G'
$wt_\mu(x_1) = wt_\lambda(x_1) + \mu(yx_1) + \mu(zx_1) = a + (p+q+1) + (3p+q) = 4p+2q+a+1 = \alpha$.
$wt_\mu(x_{i+1}) = wt_\lambda(x_{i+1}) + \mu(zx_{i+1}) + \mu(yx_{2p-i}) = a + (p+q+i+1) + (p+q+2p-i) = 4p+2q+a+1 = \alpha$.
When $n = 4k-1, k \in N$ we have
$wt_\mu(y) = \mu(y) + \mu(yx_1) + \sum_{i\ odd} \mu(yx_{2i}) + \sum_{i\ odd} \mu(yx_{2i+1}) + \sum_{i\ even} \mu(yx_{2i}) + \sum_{i\ even} \mu(yx_{2i+1})$
$= (3p+q+2)+(p+q+1)+\sum_{i\ odd}(3p+q+1-2i)+\sum_{i\ odd}(3p+q-2i) + \sum_{i\ even}(p+q+2i) + \sum_{i\ even}(p+q+1+2i)$
$= 4p+2q+3+(3p+q+1)(\frac{p+1}{4})-2(1+3+...+\frac{p-1}{2})+(3p+q)(\frac{p+1}{4})-2(1+3+...+\frac{p-1}{2})+(p+q)(\frac{p-3}{4})+2(2+4+...+\frac{p-3}{2})+(p+q+1)(\frac{p-3}{4})+2(2+4+...+\frac{p-3}{2})$
$= 4p+2q+3+(6p+2q+1)(\frac{p+1}{4})-4(1+3+...+\frac{p-1}{2})+(2p+2q+1)(\frac{p-3}{4})+4(2+4+...+\frac{p-3}{2})$
$= 4p+2q+3+(6p+2q+1)(\frac{p+1}{4})-(\frac{p+1}{2})^2+(2p+2q+1)(\frac{p-3}{4})+(p-3)(\frac{p+1}{4})$
$= 2p^2+pq+q+\frac{1}{2}(7p+3) = \beta$ and
$wt_\mu(z) = \mu(z) + \mu(zx_1) + \sum_{i\ odd} \mu(zx_{2i}) + \sum_{i\ odd} \mu(zx_{2i+1}) + \sum_{i\ even} \mu(zx_{2i}) + \sum_{i\ even} \mu(zx_{2i+1})$
$= (3p+q+2)+(p+q+1)+\sum_{i\ odd}(3p+q+1-2i)+\sum_{i\ odd}(3p+q-2i) + \sum_{i\ even}(p+q+2i) + \sum_{i\ even}(p+q+1+2i)$
$= 4p+2q+3+(3p+q+1)(\frac{p-1}{4})-2(1+3+...+\frac{p-3}{2})+(3p+q)(\frac{p-1}{4})-2(1+3+...+\frac{p-3}{2})+(p+q)(\frac{p-1}{4})+2(2+4+...+\frac{p-1}{2})+(p+q+1)(\frac{p-1}{4})+2(2+4+...+\frac{p-1}{2})$
$= 4p+2q+3+(8p+4q+2)(\frac{p-1}{4})-4(1+3+...+\frac{p-3}{2})+(2p+2q+1)(\frac{p-1}{4})+4(2+4+...+\frac{p-1}{2})$
$= 4p+2q+3+(8p+4q+2)(\frac{p-1}{4})-(\frac{p-1}{2})^2+(p-1)(\frac{p+3}{4})$
$= 2p^2+pq+q+\frac{1}{2}(7p+3) = \beta$.
When $n = 4k+1, k \in N$ we have
$wt_\mu(y) = \mu(y) + \mu(yx_1) + \sum_{i\ odd} \mu(yx_{2i}) + \sum_{i\ odd} \mu(yx_{2i+1}) + \sum_{i\ even} \mu(yx_{2i}) + \sum_{i\ even} \mu(yx_{2i+1})$
$= (3p+q+2)+(p+q+1)+\sum_{i\ odd}(3p+q+1-2i)+\sum_{i\ odd}(3p+q-2i)$
$+ \sum_{i\ even}(p+q+2i) + \sum_{i\ even}(p+q+1+2i)$
$= 4p+2q+3+(3p+q+1)(\frac{p-1}{4})-2(1+3+...+\frac{p-3}{2})+(3p+q)(\frac{p-1}{4})-2(1+3+...+\frac{p-3}{2})+(p+q)(\frac{p-1}{4})+2(2+4+...+\frac{p-1}{2})+(p+q+1)(\frac{p-1}{4})+2(2+4+...+\frac{p-1}{2})$
$= 4p+2q+3+(8p+4q+2)(\frac{p-1}{4})-(\frac{p-1}{2})^2+(p-1)(\frac{p+3}{4})$
$= 2p^2+pq+p+\frac{1}{2}(7p+3) = \beta$ and
$wt_\mu(z) = \mu(z) + \mu(zx_1) + \sum_{i\ odd} \mu(zx_{2i}) + \sum_{i\ odd} \mu(zx_{2i+1}) + \sum_{i\ even} \mu(zx_{2i}) + \sum_{i\ even} \mu(zx_{2i+1})$
$= (3p+q+1)+(3p+q)+\sum_{i\ odd}(p+q+2i)+\sum_{i\ odd}(p+q+1+2i) + \sum_{i\ even}(3p+q+1-2i) + \sum_{i\ even}(3p+q-2i)$
$= 6p+2q+1+(p+q)(\frac{p-1}{4})-2(1+3+...+\frac{p-3}{2})+(p+q+1)(\frac{p-1}{4})+2(1+3+...+\frac{p-3}{2})+(3p+q+1)(\frac{p-1}{4})-$

$2(2+4+...+\frac{p-1}{2})+(3p+q)(\frac{p-1}{4})-2(2+4+...+\frac{p-1}{2})$
$= 6p+2q+1+(8p+4q+2)(\frac{p-1}{4})+(\frac{p-1}{2})^2-(p-1)(\frac{p+3}{4})$
$= 2p^2+pq+q+\frac{1}{2}(7p+3) = \beta.$
Hence, the resultant graph G' has (1, 1) VBL.

Theorem 2.3 If G has (1, 1)-(a, d) VAL, then $G \odot dK_1$ admits (1, 1) VBL.

Proof Let $G(p, q)$ be a (1, 1)-(a, d) vertex anti-magic graph with $V(G) = \{x_1, x_1, ..., x_n\}$. Let λ be the VAL of G such that $\lambda(x_i) = i$, for $1 \leq i \leq p$ and let the vertex weight be $wt_\lambda(x_i) = a+(i-1)d$, for $1 \leq i \leq p$. Consider the new graph $G'(V', E') = G \odot mK_1$ with $V' = V \cup \{x_i^j : 1 \leq i \leq p, 1 \leq j \leq d\}$ and $E' = E \cup \{e_i^j : 1 \leq i \leq p, 1 \leq j \leq d\}$. Define a vertex bi-magic labeling $\lambda' : V' \cup E' \to \{1, 2, ..., p+q+2dp\}$ on G' in the following way:
For $1 \leq i \leq p, 1 \leq j \leq d$;
$\lambda'(x_i^j) = p+q+2dp-jp+i$, $\lambda'(e_i^j) = p+q+j-(i-1)$.
The vertex weight in G' under λ' is calculated in the following:
For each i, $1 \leq i \leq p$

$$wt_{\lambda'}(x_i) = wt_\lambda(x_i) + \sum_{j=1}^{d} \lambda'(e_i^j)$$
$$= (a+id-d) + \sum_{j=1}^{d}(p+q+jp-i+1)$$
$$= (a+id-d) + (p+q-i+1)d + \frac{pd(d+1)}{2}$$
$= \frac{1}{2}(2a+pd^2+3pd+2qd) = \alpha.$
For x_i^j, $i = 1, 2, ..., p, j = 1, 2, ..., d$, we obtain
$wt_{\lambda'}(x_i^j) = \lambda'(x_i^j) + \lambda'(e_i^j)$
$= (p+q+2dp-jp+i) + p+q+jp-(i-1)$
$= 2p+2q+2dp+1 = \beta.$
Thus, the graph $G \odot mK_1$ admits (1, 1) VBL .

Theorem 2.4 If an odd order graph G has (1, 1)-(a, 2) VAL, then "vertex by edge duplication at all vertices on G" admits (1, 1) VBL.

proof Let G be a (1, 1)-(a, 2) vertex anti-magic graph with $\lambda : V \cup E \to \{1, 2, ..., p+q\}$ such that the vertex weight $wt_\lambda(x_i) = a+2p-2i$, for $1 \leq i \leq p$. Consider the graph obtained by "vertex by edge duplication at all vertices on G" with $V' = V \cup \{x_i', x_i'' : 1 \leq i \leq p\}$ and $E' = E \cup \{e_i', e_i'', e_i''' : 1 \leq i \leq p\}$, where e_i'', e_i''' are adjacent to x_i'. Similarly, e_i', e_i''' are adjacent to x_i''. A bijective mapping $\lambda' : V' \cup E' \to \{1, 2, ..., 6p+q\}$ is given below.
$\lambda'(e_i') = p+q+i$, for $1 \leq i \leq p$; $\lambda'(e_i'') = 2p+q+i$, for $1 \leq i \leq p$;

$$\lambda'(e_i''') = \begin{cases} \{3p+q+\frac{p+3}{2}-\frac{i+1}{2}; \text{if } i \equiv 1 \ (mod\ 2), \\ \qquad 1 \leq i \leq p\} \\ \{4p+q+1-\frac{i}{2}; \text{if } i \equiv 0 \ (mod\ 2), \\ \qquad 2 \leq i \leq p-1.\} \end{cases}$$

$$\lambda'(x_i') = \begin{cases} \{5p+q+1-\frac{i+1}{2}; \text{if } i \equiv 1 \ (mod\ 2), \\ \qquad 1 \leq i \leq p, \} \\ \{4p+q+\frac{p+1}{2}-\frac{i}{2}; \text{if } i \equiv 0 \ (mod\ 2), \\ \qquad 2 \leq i \leq p-1.\} \end{cases}$$

$$\lambda'(x_i'') = \begin{cases} \{6p+q+1-\frac{i+1}{2}; \text{if } i \equiv 1 \ (mod\ 2), \\ \qquad 1 \leq i \leq p, \} \\ \{5p+q+\frac{p+1}{2}-\frac{i}{2}; \text{if } i \equiv 0 \ (mod\ 2), \\ \qquad 2 \leq i \leq p-1.\} \end{cases}$$

The vertex weights under the labeling λ' are
$wt_{\lambda'}(x_i) = wt_\lambda(x_i) + \lambda'(e_i') + \lambda'(e_i'')$
$= (a+2p-2i) + (p+q+i) + (2p+q+i) = a+5p+2q = \alpha, 1 \leq i \leq p.$
When $i \equiv 1 \ (mod\ 2)$, $1 \leq i \leq p$, we obtain
$wt_{\lambda'}(x_i') = \lambda'(x_i') + \lambda'(e_i'') + \lambda'(e_i''')$
$= (5p+q+1-\frac{i+1}{2}) + (2p+q+i) + (3p+q+\frac{p+3}{2}-\frac{i+1}{2})$
$= 10p+3q+\frac{p+3}{2} = \beta.$
$wt_{\lambda'}(x_i'') = \lambda'(x_i'') + \lambda'(e_i') + \lambda'(e_i''')$
$= (6p+q+1-\frac{i+1}{2}) + (p+q+i) + (3p+q+\frac{p+3}{2}-\frac{i+1}{2})$
$= 10p+3q+\frac{p+3}{2} = \beta.$
When $i \equiv 0 \ (mod\ 2)$, $1 \leq i \leq p$, we obtain
$wt_{\lambda'}(x_i') = \lambda'(x_i') + \lambda'(e_i'') + \lambda'(e_i''')$
$= (4p+q+\frac{p+1}{2}-\frac{i}{2}) + (2p+q+i) + (4p+q+1)$
$= 10p+3q+\frac{p+3}{2} = \beta.$
$wt_{\lambda'}(x_i'') = \lambda'(x_i'') + \lambda'(e_i') + \lambda'(e_i''')$
$= (5p+q+\frac{p+1}{2}-\frac{i}{2}) + (p+q+i) + (4p+q+1-\frac{i}{2})$
$= 10p+3q+\frac{p+3}{2} = \beta.$
Hence the resultant graph admits (1, 1) VBL.

Theorem 2.5 If G has (1, 1) VML, then for even $m \geq 2$, $G \odot mK_1$ admits (1, 1) VBL.

proof Let $G(p, q)$ be a graph with (1, 1) VML with $V = \{x_1, x_2, ..., x_p\}$, $E = \{e_1, e_2, ..., e_q\}$. Then there exists a mapping $\mu : V \cup E \to \{1, 2, ..., p+q\}$ such that the vertices of G receives the weight $wt_\mu(x_i) = a$. Consider the graph $G'(V', E') = G \odot mK_1$ with vertex set $V' = V \cup \{x_i^j : 1 \leq i \leq p, 1 \leq j \leq m\}$ and edge set $E' = E \cup \{e_i^j : 1 \leq i \leq p, 1 \leq j \leq m\}$. Define a mapping $\mu' : V' \cup E' \to \{1, 2, ..., p+q+2mp\}$ is as follows:
$\mu'(x_i) = \mu(x_i), 1 \leq i \leq p$ and $\mu'(e) = \mu(e)$, for all $e \in E$.

$$\mu'(x_i^j) = \begin{cases} \{p+q+2mp-jp+i; \text{if } j \equiv 0 \ (mod\ 2), \\ \qquad 1 \leq i \leq p, 2 \leq j \leq m\} \\ \{p+q+2mp-(j-1)p+1-i; \text{if } j \equiv 1 \ (mod\ 2), \\ \qquad 1 \leq i \leq p, 1 \leq j \leq m-1.\} \end{cases}$$

$$\mu'(e_i^j) = \begin{cases} \{jp+q+(p+1)-i; \text{if } j \equiv 0 \ (mod\ 2), \\ \qquad 1 \leq i \leq p, 2 \leq j \leq m\} \\ \{jp+q+i, \text{if } j \equiv 1 \ (mod\ 2), \\ \qquad 1 \leq i \leq p, 1 \leq j \leq m-1.\} \end{cases}$$

The vertex weight under the labeling μ', are obtained in the following:
when $j \equiv 1 \ (mod\ 2), 1 \leq i \leq p, 1 \leq j \leq m$
$wt_{\mu'}(x_i^j) = \mu'(e_i^j) + \mu'(x_i^j)$
$= (jp+q+i) + (p+q+2mp-jp+p+1-i) = 2p+2q+2mp+1 = \alpha.$
when $j \equiv 0 \ (mod\ 2), 1 \leq i \leq p, 1 \leq j \leq m$
$wt_{\mu'}(x_i^j) = \mu'(e_i^j) + f\mu'(x_i^j)$
$= (jp+q+p+1-i) + (p+q+2mp-jp+i) = 2p+2q+2mp+1 = \alpha.$
For a fixed i, $1 \leq i \leq p$;

$wt_{\mu'}(x_i) = \sum_{j\ odd} \mu'(e_i^j) + \sum_{j\ even} \mu'(e_i^j)$
$= \sum_{j\ odd} (jp+q+i) + \sum_{j\ even} (jp+q+p+1-i)$
$= (i+q)(\frac{m}{2}) + p\sum_{j\ odd} j + (q+p+1-i)(\frac{m}{2}) + p\sum_{j\ even} j$
$= \frac{im}{2} + \frac{qm}{2} + p(1+3+...+m-1) + \frac{qm}{2} + \frac{pm}{2} + \frac{m}{2} - \frac{im}{2} + p(2+4+...+m) + a$
$= qm + pm + \frac{m}{2} + \frac{pm^2}{2} + a = \beta.$
Hence the proof.

Theorem 2.6 If an odd order graph G has (1, 1)-(a, 1) VAL, then the graph obtained by "vertex by edge duplication at all vertices on G" admits (1, 1) VBL.

proof Let G be a (1, 1)-(a, 1) vertex anti-magic graph with $\lambda : V \cup E \to \{1, 2, ..., p+q\}$ such that the vertex weights $wt_\lambda(x_i) = a - 1 + i$, for $1 \le i \le p$. Consider the graph G' obtained by "vertex by edge duplication at all vertices on G" with $V' = V \cup \{x_i', x_i'' : 1 \le i \le p\}$ and $E' = E \cup \{e_i', e_i'', e_i''' : 1 \le i \le p\}$, where $e_i' = (x_i, x_i')$, $e_i'' = (x_i, x_i'')$, and $e_i''' = (x_i', x_i'')$. The required mapping $\lambda' : V' \cup E' \to \{1, 2, ..., 6p+q\}$ is given below.
$\lambda'(x_i) = \lambda(x_i), 1 \le i \le p$ and $\lambda'(e) = \lambda(e)$, for all $e \in E$.

$$\lambda'(x_i') = \begin{cases} 5p+q+\frac{p+1}{2}+i; & \text{if } 1 \le i \le \frac{p-1}{2} \\ 5p+q+i-\frac{p-1}{2}; & \text{if } \frac{p+1}{2} \le i \le p. \end{cases}$$

$$\lambda'(x_ix_i') = \begin{cases} p+q+p+1-2i; & \text{if } 1 \le i \le \frac{p-1}{2} \\ p+q+2p+1-2i; & \text{if } \frac{p+1}{2} \le i \le p. \end{cases}$$

$$\lambda'(x_i'') = \begin{cases} 4p+q+p+1-2i; & \text{if } 1 \le i \le \frac{p-1}{2} \\ 4p+q+2p+1-2i; & \text{if } \frac{p+1}{2} \le i \le p. \end{cases}$$

$\lambda'(x_i'x_i'') = 3p+q+i;\ 1 \le i \le p$

$$\lambda'(x_ix_i'') = \begin{cases} 2p+q+\frac{p+1}{2}+i; & \text{if } 1 \le i \le \frac{p-1}{2} \\ 2p+q+i-\frac{p-1}{2}, & \text{if } \frac{p+1}{2} \le i \le p. \end{cases}$$

The bi-magic property of the above assignment labeling is verified below.
For $1 \le i \le \frac{p-1}{2}$;
$wt_{\lambda'}(x_i) = wt_\lambda(x_i) + \lambda'(x_ix_i') + \lambda'(x_ix_i'')$
$= (a-1+i) + (p+q+p+1-2i) + (2p+q+\frac{p+1}{2}+i)$
$= a + 3p + 2q + \frac{3p+1}{2} = \alpha.$
$wt_{\lambda'}(x_i') = \lambda'(x_i') + \lambda'(x_ix_i') + \lambda'(x_i'x_i'')$
$= (5p+q+\frac{p+1}{2}+i) + (p+q+p+1-2i) + (3p+q+i)$
$= 9p + 3q + \frac{3(p+1)}{2} = \beta.$
$wt_{\lambda'}(x_i'') = \lambda'(x_i'') + \lambda'(x_ix_i'') + \lambda'(x_i'x_i'')$
$= (4p+q+p+1-2i) + (2p+q+\frac{p+1}{2}+i) + (3p+q+i)$
$= 9p + 3q + \frac{3(p+1)}{2} = \beta.$
For $\frac{p+1}{2} \le i \le p$;
$wt_{\lambda'}(x_i) = wt_\lambda(x_i) + \lambda'(x_ix_i') + \lambda'(x_ix_i'')$
$= (a-1+i) + (p+q+2p+1-2i) + (2p+q-\frac{p+1}{2}+i)$
$= a + 3p + 2q + \frac{3p+1}{2} = \alpha.$
$wt_{\lambda'}(x_i') = \lambda'(x_i') + \lambda'(x_ix_i') + \lambda'(x_i'x_i'')$
$= (5p+q-\frac{p-1}{2}+i) + (p+q+2p+1-2i) + (3p+q+i)$
$= 9p + 3q + \frac{3(p+1)}{2} = \beta.$
$wt_{\lambda'}(x_i'') = \lambda'(x_i'') + \lambda'(x_ix_i'') + \lambda'(x_i'x_i'')$
$= (4p+q+2p+1-2i) + (2p+q-\frac{p-1}{2}+i) + (3p+q+i)$
$= 9p + 3q + \frac{3(p+1)}{2} = \beta.$
Hence the proof.

Theorem 2.7 The graph $K_p + mK_l,\ p, l \ge 2$ and even m, admits (1, 0) VBL.

proof Let $G(p, q) = K_p + mK_l$, with $V = V_1 \cup V_2$ where $V_1 = \{x_i : 1 \le i \le p\}$ is the vertex set of K_p and $V_2 = \{y_k^j : 1 \le j \le l, 1 \le k \le m\}$ is the vertex set of m disjoint union of K_l. Define a bijection $\mu : V \to \{1, 2, ..., p+ml\}$ as follows:
$\mu(x_i) = i,\ 1 \le i \le p$

$$\mu(y_k^j) = \begin{cases} \{p+(j-1)m+k; \text{if } j \equiv 1\ (mod\ 2), \\ \qquad 1 \le j \le l-1, 1 \le k \le m\} \\ \{p+jm+1-k; \text{if } j \equiv 0\ (mod\ 2), \\ \qquad 2 \le j \le l, 1 \le k \le m.\} \end{cases}$$

Verification:
$wt_\mu(x_i) = \sum_{i=1}^{p} \mu(x_i) + \sum_{k=1}^{m}\sum_{j\ odd} \mu(y_k^j) + \sum_{k=1}^{m}\sum_{j\ even} \mu(y_k^j)$
$= \sum_{i=1}^{p} i + \sum_{k=1}^{m}\sum_{j\ odd} (p+(j-1)m+k) + \sum_{k=1}^{m}\sum_{j\ even} (p+jm+1-k)$
$= \frac{p(p+1)}{2} + \sum_{k=1}^{m}\sum_{j=1}^{l} p + \sum_{k=1}^{m}\sum_{j\ even} 1 + \sum_{k=1}^{m}\sum_{j\ odd} (j-1)m + \sum_{k=1}^{m}\sum_{j\ even} jm + \sum_{k=1}^{m}\sum_{j\ odd} k + \sum_{k=1}^{m}\sum_{j\ even} (-k)$
$= \frac{p(p+1)}{2} + pml + \frac{ml}{2} + m^2(0+2+4+...+(l-2)) + m^2(2+4+...+l) + \frac{ml}{2} - \frac{ml}{2}$
$= \frac{p(p+1)}{2} + pml + \frac{ml}{2} + \frac{m^2l^2}{2} = \alpha$, for each $i,\ 1 \le i \le p$.
For a fixed $k,\ 1 \le k \le m$ and any $j, 1 \le j \le l$.
$wt_\mu(y_k^j) = \sum_{i=1}^{p} \mu(x_i) + \sum_{j\ odd} \mu(y_k^j) + \sum_{j\ even} \mu(y_k^j)$
$= \sum_{i=1}^{p}(i) + \sum_{j\ odd} (p+(j-1)m+k) + \sum_{j\ even} (p+jm+1-k)$
$= \frac{p(p+1)}{2} + (p+k)\frac{l}{2} + m\sum_{j\ odd} (j-1) + (p+1-k)\frac{l}{2} + m\sum_{j\ even} j = \frac{p(p+1)}{2} + \frac{pl}{2} + \frac{kl}{2} + m(0+2+4+...+(l-2)) + \frac{pl}{2} + \frac{l}{2} - \frac{kl}{2} + m(2+4+...+l)$
$= \frac{p(p+1)}{2} + pl + \frac{l}{2} + m[(\frac{l-2}{2})(\frac{l-2}{2}+1) + \frac{l}{2}(\frac{l}{2}+1)]$
$= \frac{p(p+1)}{2} + pl + \frac{l}{2} + \frac{ml^2}{2} = \beta.$
Hence the proof.

Theorem 2.8 Let G be (0, 1)-(a, 1) vertex anti-magic graph. Then $G + K_1$ admits (0, 1) VBL.

proof Let $G(p, q)$ be a (0, 1)-(a, 1) vertex anti-magic graph with $\lambda : E \to \{1, 2, ..., q\}$ such that vertex weight $wt_\lambda(x_i) = a + i - 1$, for $1 \le i \le p,\ a > 0$. Consider the graph $G + K_1$ with vertex set $V' = V \cup \{z\}$ and $E' = E \cup \{zx_i : 1 \le i \le p\}$. Define a mapping

$\mu : E' \to \{1, 2, ..., q, q+1, ..., p+q\}$ as follows:
$\mu(zx_i) = p + q + 1 - i$, for $1 \leq i \leq p$.
The vertex weight under the labeling μ is
$wt_\mu(x_i) = \mu(zx_i) + wt_\lambda(x_i) = (p+q+1-i) + (a+i-1)$
$= a + p + q = \alpha, 1 \leq i \leq p$ and

$$wt_\mu(z) = \sum_{i=1}^{p} wt_\lambda(zx_i)$$
$$= \sum_{i=1}^{p} (p+q+1-i)$$
$$= (p+q+1)p - \frac{p(p+1)}{2} = \frac{1}{2}(p^2 + p + 2pq) = \beta.$$

Hence the proof.

Theorem 2.9 If an even order graph G, has (0, 1)-(a, 2) VAL, then $G' = G + 2K_1$ admits (0, 1) VBL.

proof Let $G(p,q)$ be a (0, 1)-(a, 2) vertex anti-magic graph with $\mu : E \to \{1, 2, ..., q\}$ such that the vertex weight $wt_\mu(x_i) = a - 2 + 2i$, for $1 \leq i \leq p$. Consider the graph $G' = G + 2K_1$ with $V' = V \cup \{y, z\}$ and $E' = E \cup \{yx_i, zx_i : 1 \leq i \leq p\}$. Define a mapping $\mu' : E' \to \{1, 2, ..., 2p+q\}$ as given below:
$\mu'(x_i) = \mu(x_i), 1 \leq i \leq p$ and $\mu'(e) = \mu(e)$, for all $e \in E$.

$$\mu'(yx_i) = \begin{cases} 3p+q+1-i; & \text{if } i \equiv 1 \ (mod\ 2), 1 \leq i \leq p-1 \\ 2p+q+1-i, & \text{if } i \equiv 0 \ (mod\ 2), 2 \leq i \leq p. \end{cases}$$

$$\mu'(zx_i) = \begin{cases} 2p+q+1-i; & \text{if } i \equiv 1 \ (mod\ 2), 1 \leq i \leq p-1 \\ 3p+q+1-i, & \text{if } i \equiv 0 \ (mod\ 2), 2 \leq i \leq p. \end{cases}$$

Verification of the bimagic property for the above labeling.
When i is odd, $wt_{\mu'}(x_i) = wt_\mu(x_i) + \mu'(yx_i) + \mu'(zx_i)$
$= (a-2+2i) + (3p+q+1-i) + (2p+q+1-i)$
$= a + 5p + 2q = \alpha$.
When i is even, $wt_{\mu'}(x_i) = wt_\mu(x_i) + \mu'(yx_i) + \mu'(zx_i)$
$= (a-2+2i) + (2p+q+1-i) + (3p+q+1-i)$
$= a + 5p + 2q = \alpha$.

$$wt_{\mu'}(y) = \sum_{i\ odd} \mu'(yx_i) + \sum_{i\ even} \mu'(yx_i)$$
$$= \sum_{i\ odd} (3p+q+1-i) + \sum_{i\ even} (2p+q+1-i)$$
$$= (3p+q+1)\frac{p}{2} - \sum_{i\ odd} i + (2p+q+1)\frac{p}{2} - \sum_{i\ even} i$$
$$= (3p+q+1)\frac{p}{2} - (\frac{p}{2})^2 + (2p+q+1)\frac{p}{2} - (\frac{p}{2})(\frac{p}{2}+1)$$
$$= 2p^2 + pq + \frac{p}{2} = \beta.$$
$$wt_{\mu'}(z) = \sum_{i\ odd} \mu'(zx_i) + \sum_{i\ even} \mu'(zx_i)$$
$$= \sum_{i\ odd} (p+q+p+1-i) + \sum_{i\ even} (p+q+2p+1-i)$$
$$= (p+q+p+1)\frac{p}{2} - (\frac{p}{2})^2 + (p+q+2p+1)\frac{p}{2} - (\frac{p}{2})(\frac{p}{2}+1)$$
$$= 2p^2 + pq + \frac{p}{2} = \beta.$$

Hence, the graph G' has (0, 1) VBL.

III. Conclusion

In this paper, we demonstrated the methods of obtaining new families of (1, 1) vertex bi-magic, (1, 0) vertex bi-magic and (0, 1) vertex bi-magic graphs from vertex magic and vertex anti-magic graphs by performing graph operations. In future, we would like to extent these ideas to general graphs to obtain new families of graphs which do not admit bimagic labeling.

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