Edge Removal and Edge Addition in k-dependent k-domination of a Graph

D. K. Thakkar and D. D. Pandya

***Abstract*—In this paper, we consider a variant of domination namely k-dependent k-domination. We have derived characterizations of edges which are responsible to increase or decrease k-dependent, k-domination number of a graph. We have also given examples to illustrate the conditions given in theorems.**

***Index Terms*—k-dependent k-dominating Set , minimal k-dependent k-dominating Set , minimum k-dependent k-dominating Set , k-dependent k-domination Number.**

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I. Introduction

SUPPOSE G is a graph. S is a Minimum Dominating set. If we remove an edge from graph then domination number increases, decreases or remains unchanged. Similarly if we add an edge between two non-adjacent vertices then domination number decreases or remains unchanged. This topic has been studied by several authors. Reader may refer to references [1], [2], [3], [4]. In [5], we can see effect of edge removal and vertex removal in domination. [6] contains results related to vertex removal for some variants of domination. We shall prove results regarding edge addition and edge removal for k-dependent k-domination.

II. Some Preliminary Results

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 2.1 Let G be a graph and K be an integer, $K \geq 1$ A subset S of V(G) is said to be a K-dependent K-dominating set if

1. For every vertex v in S, v is adjacent to at most K - 1 vertices of S.
2. For every vertex u outside of S, u is adjacent to at least K vertices of S.

□

Definition 2.2 A K-dependent K-dominating Set S is said to be a minimal K-dependent K-dominating if no proper subset $S' \subseteq S$ is a K-dependent K-dominating Set. □

Definition 2.3 A minimal K-dependent K-dominating with minimum cardinality is called minimum K-dependent K-dominating Set. □

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Definition 2.4 The K-dependent K-domination number $\gamma_{KK}(G)$ of a graph G equals the minimum cardinality of a K-dependent K-dominating Set in G. □

Note. It is useful to partition the Edges of G into three sets according to how their removal affects K-dependent K-domination number.

Let $E = E_{KK}^{+} \bigcup E_{KK}^{-} \bigcup E_{KK}^{0}$ for
$E_{KK}^{+} = e \in E(G)/\gamma_{KK}(G-e) > \gamma_{KK}(G)$
$E_{KK}^{0} = e \in E(G)/\gamma_{KK}(G-e) = \gamma_{KK}(G)$
$E_{KK}^{-} = e \in E(G)/\gamma_{KK}(G-e) < \gamma_{KK}(G)$

III. Main Results

First we consider the case of removing an edge from given graph..

Theorem 3.1 An edge $e = uv \in E_{KK}^{+} \Leftrightarrow$ (for every γ_{KK} set S of graph G) u belongs to S, v does not belong to S and

1. v is adjacent to exactly K vertices of S including u.
2. For every vertex $w \neq u$,w does not belong to S, $w \in N(v)$ then $|N(w) \bigcap S| \geq k$.

Proof:

[Sufficiency]

Let S be a γ_{KK} set of the graph G and condition satisfies. Consider graph $G-e$, where $e = uv$. Now S is not a K-dependent K-dominating set in graph $G-e$. Let T be a γ_{KK} set in the graph $G-e$ with $|T| \leq |S|$.

(a) Let u and v do not belong to T. Now in graph G, T is a K-dependent K-dominating set and u,v do not belong to T. Which contradicts our assumption.

(b) Let $u, v \in T$. If $|N(u) \bigcap T < K-1|$ and $|N(v) \bigcap T < K-1|$ then T is a γ_{KK} set in graph G and u,v belong to T which contradicts our assumption. let $|N(u) \bigcap T| = K-1$ then $|T| - v$ will be K-dependent K-dominating set in the graph G and $|T-v| < |S|$. Which is a contradiction.

(c) Let u belongs to T, v does not belong to T. So $|N(v) \bigcap T| \geq K$ in graph $G-e$. Now in graph G, T is a γ_{KK} set in graph G and $|N(v) \bigcap T| \geq K+1$ in graph G which contradicts our assumption.

So an edge e = uv belongs to E_{KK}^{+}.

[Necessity]

Let $e = uv \in E_{KK}^{+}$ means removal of edge e increase K-dependent K-domination number. So, $\gamma_{KK}(G-e) > \gamma_{KK}(G)$. Let S be any γ_{KK} set of graph G. Then S can not be K-dependent K-dominating set in the graph $G-e$. We have following possibilities.

(1) Let u does not belong to S, v does not belong to S. Then S is a K-dependent K-dominating set in graph $G-e$ also. Hence $\gamma_{KK}(G-[e]) \leqslant \gamma_{KK}(G)$. Which is a contradiction.
(2) Let $u \in S, v \in S$. Then S is a K-dependent K-dominating set in graph $G-e$ also. hence $\gamma_{KK}(G-[e]) \leqslant \gamma_{KK}(G)$. Which is a contradiction.
(3) Let u belongs to S, v does not belong to S. (a) Suppose v is adjacent to at least $K+1$ vertices of S including u in graph G. Then removing $e = uv$ still v will be adjacent to at least K vertices of S and $e = uv$ does not belong to E^{+}_{KK}. Which is a contradiction. So v is adjacent to exactly K vertices of S including u in graph G. (b) Obviously for every vertex $w \neq u$,w does not belong to S, $w \in N(v)$ then $|N(w) \bigcap S| \geq K$

□

Theorem 3.2. An edge $e = uv \in E^{-}_{KK} \Leftrightarrow$ (for any K-dominating set T of graph G with $|T| < \gamma_{KK}(G)$) $u \in T, v \in T$ and $|N(u) \bigcap T| = K-1$ or $|N(v) \bigcap T| = K-1$.

Proof:

[Sufficiency]

Let T be a set which satisfies given condition in theorem. Then T is a K-dependent K-dominating set in graph $G-uv$. Therefore $\gamma_{KK}(G-e) \leq |T| \leq \gamma_{KK}(G)$. Therefore $e = uv \in E^{-}_{KK}$.

[Necessity]

Suppose $e = uv \in E^{-}_{KK}$ means $\gamma_{KK}(G-e) < \gamma_{KK}(G)$. Let T be a minimum K-dependent K-dominating set in graph $G-uv$. Then T can not be a K-dependent K-dominating set in graph G. Since T must be a K-dominating set in graph G, it can not be a K-dependent set in graph G. Therefore $u, v \in T$ and obviously $|N(u) \bigcap T| = K-1$ or $|N(v) \bigcap T| = K-1$ in graph $G-e$.

Illustration 3.3 Here we give an example of a graph which represents K-dependent K-dominating set (k=2) and effect of edge removal.
E^{+}_{22} = [ab, bc, cg, gf, fe, ea]

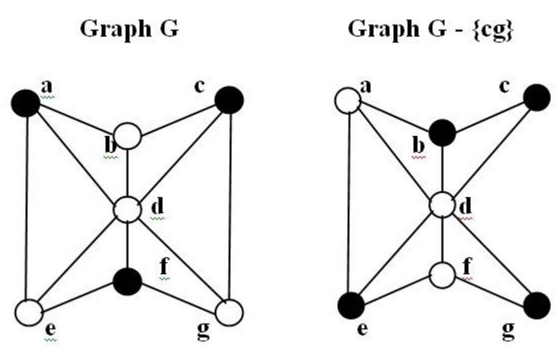

FIGURE 1

Illustration 3.4 Here we give an example of a graph which represents K-dependent K-dominating set (K=2) and effect of edge removal.
E^{0}_{22} = [dc, db, da, de, df, dg]

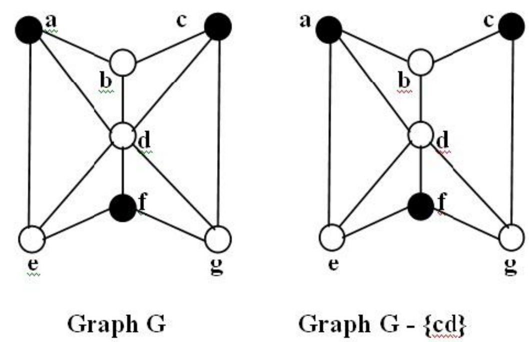

FIGURE 2

Illustration 3.5 Here we give an example of a graph which represents K-dependent K-dominating set (K=2) and effect of edge removal.
E^{-}_{22} = [ad, di]

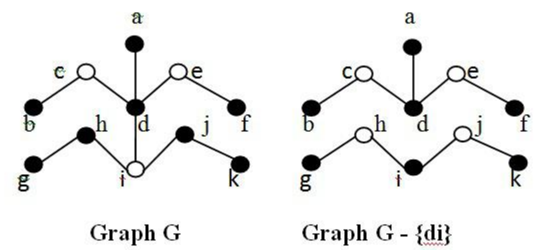

FIGURE 3

Theorem 3.6 Let u and v be two non - adjacent vertices of graph G and $e = uv$ then $\gamma_{KK}(G+e) < \gamma_{KK}(G) \Leftrightarrow$ There is a subset S of V(G) such that $|S| < \gamma_{KK}(G)$ and following conditions hold.

(a) u belongs to S, v does not belong to S and v is adjacent to exactly $K-1$ vertices of S in graph G.
(b) S is a K-dependent K-dominating set in the sub graph $G-v$

Proof:

[Sufficiency]

Let S be subset of V(G) such that $|S| < \gamma_{KK}(G)$ and given conditions hold. Then S is a K-dependent K-dominating set in graph G also with $|S| < \gamma_{KK}(G)$
So, $\gamma_{KK}(G+e) \leq |S| < \gamma_{KK}(G)$.
So, $\gamma_{KK}(G+e) < \gamma_{KK}(G)$.

[Necessity]

Let $\gamma_{KK}(G+e) < \gamma_{KK}(G)$. Suppose S be a γ_{KK} set in graph $G+e$ then $|S| < \gamma_{KK}(G)$. Let u and v be two non - adjacent vertices and $\gamma_{KK}(G+uv) < \gamma_{KK}(G)$.

(1) Suppose u does not belong to S and v does not belong to S. Here S is a K-dependent K-dominating set in graph G also which is not possible as $|S| < \gamma_{KK}(G)$
(2) Suppose u belongs to S and v belongs to S. Here S is a K-dependent K-dominating set in graph G also which is not possible as $|S| < \gamma_{KK}(G)$
(3) Let u belongs to S, v does not belong to S. Now by theorem 3.1 , v is adjacent to exactly K vertices of S including u in graph $G+[e]$. v is adjacent to exactly

$K-1$ vertices of S in graph G.

□

Illustration 3.7 Here we give an example of a graph which represents K-dependent K-dominating set (K=2) and effect of edge addition between two non - adjacent vertices of graph. Here $\gamma_{22}(G+[uv]) < \gamma_{22}(G)$

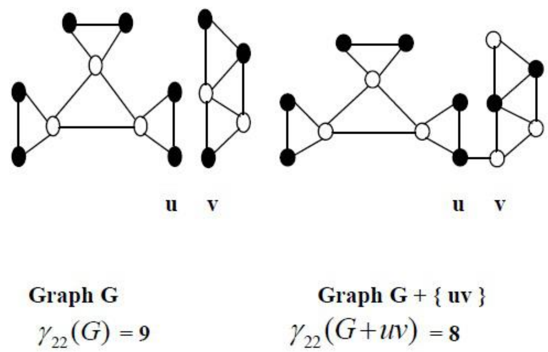

FIGURE 4

Definition 3.8 Let G be a graph and $e = uv$ be an edge of graph G then graph G is said to be γ^{+}_{KK} critical graph if $\gamma_{KK}(G-[uv]) > \gamma_{KK}(G)$ for all edges of graph G. □

Definition 3.9 Let G be a graph and $e = uv$ be an edge of graph G then graph G is said to be γ_{KK} insensitive graph if $\gamma_{KK}(G-[uv]) = \gamma_{KK}(G)$ for all edges of graph G. □

Illustration 3.10 Here we give an example of a graph which is γ_{KK} insensitive graph (K=2) . Here $\gamma_{KK}(G-[uv]) = \gamma_{KK}(G)$ for all edges of graph G.

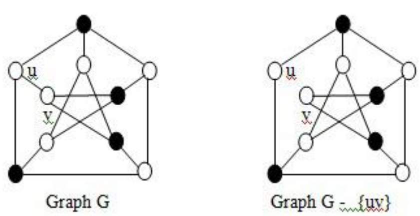

FIGURE 5

Lemma 3.11 Let G be a graph and $deg(v) \geq K+1$ for all v, $K \geq 1$ then there is an edge $e = uv$ such that $\gamma_{KK}(G-[e]) = \gamma_{KK}(G)$

Proof: Let G be a graph and $deg(v) \geq K+1$ for all v, $K \geq 1$ and S is a minimum K-dependent K-dominating set of graph G. Since minimum degree of graph G is K+1 , minimum K-dependent K-dominating set is a proper subset of vertex set of graph G (because there exist a proper K-dependent K-dominating set of graph G for example $V(G) - v$ for any vertex v).

So at least one vertex v does not belong to S. Let $deg(v) \geq K+1$ and vertices $u_1, u_2, u_3, , , , u_K, u_{K+1}$ are adjacent to vertex v .If all vertices $u_1, u_2, u_3, , , , u_K, u_{K+1}$ are in S then $\gamma_{KK}(G-[e]) = \gamma_{KK}(G)$ for edge $e = vu_p$, $p = 1, 2, 3, , , K+1$

If at least one vertex is outside S among all vertices adjacent with v say r then $\gamma_{KK}(G-[e]) = \gamma_{KK}(G)$ for edge $e = [vr]$.

So in all cases, there exist at least one edge $e = uv$ such that $\gamma_{KK}(G-[e]) = \gamma_{KK}(G)$ □

Note: Let G = (V,E) be a graph with $deg(v) \geq 1$ for all $v \in V$, $K \geq 1$ and S be a minimum K-dependent K-dominating set for graph G. If graph G has x vertices outside from S then $|E^{0}_{KK}| \geq x$.

We now consider following two classes of graphs called CER and UER

CER = Change Edge Removal and UER = Unchange Edge Removal

The graphs in CER were called critical graphs and independently characterized by Bauer et al. [1] and Walikar and Acharya [5] for domination in graphs.

We introduce following two classes for K-dependent K-domination.

CER (KK) = Change Edge Removal in K-dependent K-domination and UER (KK) = Unchange Edge Removal in K-dependent K-domination

Definition 3.12

CER(KK) = [graph G where $\gamma_{KK}(G-[e]) > \gamma_{KK}(G)$, for all edge $e = uv$]

UER(KK) = [graph G where $\gamma_{KK}(G-[e]) = \gamma_{KK}(G)$, for all edge $e = uv$]

Illustration 3.13 Let $G = (V, E)$ be a graph with $deg(v) \geq K+1$ for all $v \in V$, $K \geq 1$ then G does not belong to CER (KK) .

soln Let $G = (V, E)$ be a graph with $deg(v) \geq K+1$ for all $v \in V$, $K \geq 1$

Now by lemma 3.11, there is an edge $e = uv$ such that $\gamma_{KK}(G-[e]) = \gamma_{KK}(G)$. So all edges can not be in E^{+}_{KK}. So G does not belong to CER (KK) .

Illustration 3.14 Here we give an example of a graph which is γ^{+}_{KK} critical graph (K=2) . Here $\gamma_{KK}(G-[uv]) > \gamma_{KK}(G)$ for all edges of graph G.

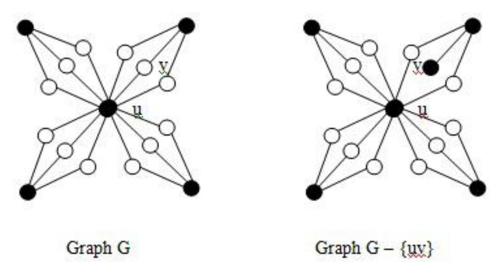

FIGURE 6

We now consider following two classes of graphs called CEA and UEA

CEA = Change Edge Addition and UEA = Unchange Edge Addition

The difficult problem of characterizing graphs in CEA was investigated by Summer and Blitch who called them ”edge domination critical graphs”. They were able to characterize these graphs only in the special cases for which $\gamma(G) = 1$ or 2 for domination.

The graphs in UEA were characterized in terms of their vertex sets by Carrington. We introduce following two classes for K-dependent K-domination.
CEA (KK) = Change Edge Addition in K-dependent K-domination and UEA (KK) = Unchange Edge Addition in K-dependent K-domination

Definition 3.15

CEA(KK) = [graph G where $\gamma_{KK}(G + [e]) < \gamma_{KK}(G)$, for all edge $e = uv$ between two non - adjacent vertices u and v]

UEA(KK) = [graph G where $\gamma_{KK}(G + [e]) = \gamma_{KK}(G)$, for all edge $e = uv$ between two non - adjacent vertices u and v]

Illustration 3.16 Here we give an example of graph which is in UEA(KK) means $\gamma_{KK}(G + [e]) = \gamma_{KK}(G)$ for all edges $e = uv$ (K = 2)

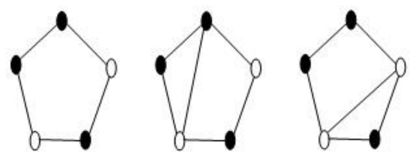

FIGURE 7

Illustration 3.17 Here we give an example of graph which is in CEA(KK) means $\gamma_{KK}(G + [e]) < \gamma_{KK}(G)$ for all edges $e = uv$ (K = 2)

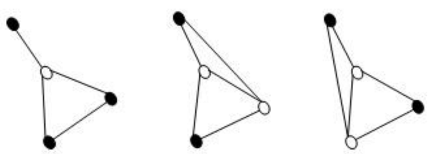

FIGURE 8

IV. Conclusions and Future Scope of Research

Here, we have characterized edges which are responsible to increase or decrease K-dependent K-domination number of graph and considered graphs in UEA (KK), CEA (KK), UER (KK), CER (KK). Similarly there are some vertices which are responsible to increase or decrease K-dependent K-domination number of graph and classes UVR (KK), UVA (KK), CVA (KK), CVR (KK). It will be interesting if one can find relation between these classes.

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