Edge Product Cordial Labeling in the Context of Some Graph Operations

S. K. Vaidya and C. M. Barasara

***Abstract*—For a graph $G = (V(G), E(G))$ a function $f : E(G) \rightarrow \{0, 1\}$ is called an edge product cordial labeling of G if the induced vertex labeling function defined by the product of incident edge labels be such that the edges with label 1 & label 0 differ by at most 1 and the vertices with label 1 & label 0 also differ by at most 1. We investigate edge product cordial labeling in the context of some graph operations.**

***Index Terms*—Cordial labeling, Product cordial labeling, Edge product cordial labeling.**

***MSC 2010 Codes* - 05C38, 05C76, 05C78.**

I. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. For all standard terminology and notations we follow Balakrishnan and Ranganathan [1].

Definition 1.1 : A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called *a vertex labeling (or an edge labeling)*.

For an extensive survey on graph labeling and bibliographic references we refer to Gallian [2].

Definition 1.2 : For a graph $G = (V(G), E(G))$, an edge labeling function $f : E(G) \rightarrow \{0, 1\}$ induces a vertex labeling function $f^* : V(G) \rightarrow \{0, 1\}$ defined as $f^*(v) = \prod f(e_i)$ for $\{e_i \in E(G)/e_i$ is incedent to $v\}$.

Now denoting the number of vertices of G having label i under f^* as $v_f(i)$ and the number of edges of G having label i under f as $e_f(i)$.

Then f is called *edge product cordial labeling* of graph G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called *edge product cordial* if it admits edge product cordial labeling.

The concept of edge product cordial labeling was introduced by Vaidya and Barasara [3] in which they have investigated several results on this newly defined concept. The edge product cordial labeling for some snake related graphs are also reported in Vaidya and Barasara [4].

Here we investigate some new results on edge product cordial labeling in the context of some graph operations.

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II. Main Results

Definition 2.1 : For a simple connected graph G the *square of graph* G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Theorem 2.2 : C_n^2 is not an edge product cordial graph except for $n = 3$.

Proof : We will consider following four cases.

Case 1: When $n = 3$.

C_3^2 is same as C_3 and as reported in [3] it is edge product cordial graph.

Case 2: When n is odd and $n > 3$.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least n edges out of $2n$ edges. The edges with label 0 will give rise at least $\left\lceil \frac{n}{2} \right\rceil + 1$ vertices with label 0 and at most $\left\lfloor \frac{n}{2} \right\rfloor - 1$ vertices with label 1 out of total n vertices. Therefore $|v_f(0) - v_f(1)| \geq 3$. Thus the vertex condition for edge product cordial graph is violated.

Case 3: When $n = 4$.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to 3 edges out of 6 edges. The edges with label 0 will give rise at least 3 vertices with label 0 and at most 1 vertices with label 1 out of total 4 vertices. Therefore $|v_f(0) - v_f(1)| \geq 2$. Thus the vertex condition for edge product cordial graph is violated.

Case 4: When n is even and $n > 4$.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least n edges out of $2n$ edges. The edges with label 0 will give rise at least $\frac{n}{2} + 2$ vertices with label 0 and at most $\frac{n}{2} - 2$ vertices with label 1 out of total n vertices. Therefore $|v_f(0) - v_f(1)| \geq 4$. Thus the vertex condition for edge product cordial graph is violated.

Therefore C_n^2 is not an edge product cordial graph except for $n = 3$.

Theorem 2.3 : P_n^2 is not an edge product cordial graph for even n.

Proof : In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $\left\lfloor \frac{2n-3}{2} \right\rfloor$ edges out of $2n - 3$ edges. The edges with label 0 will give rise at least $\frac{n}{2} + 1$ vertices with label 0 and at most $\frac{n}{2} - 1$ vertices with label 1 out of total n vertices. Therefore $|v_f(0) - v_f(1)| \geq 2$. Thus the vertex condition for edge product cordial graph is violated.

Therefore P_n^2 is not an edge product cordial graph for even n.

Theorem 2.4 : P_n^2 is edge product cordial graph for odd n.

Proof : Let P_n be the graph with vertices $v_1, v_2, \ldots, v_n$. We have $|V(P_n^2)| = n$ and $|E(P_n^2)| = 2n - 3$.

We define $f : E(P_n^2) \rightarrow \{0, 1\}$ as follows.

$$f(v_i v_{i+1}) = 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f(v_i v_{i+2}) = 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f(v_i v_j) = 0; \qquad \text{otherwise.}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(1) + 1 = \frac{n+1}{2}$$
$$e_f(0) + 1 = e_f(1) = n - 1$$

Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Therefore P_n^2 is edge product cordial graph for odd n.

Illustration 2.5 : The graph P_7^2 and its edge product cordial labeling is shown in Fig. 1.

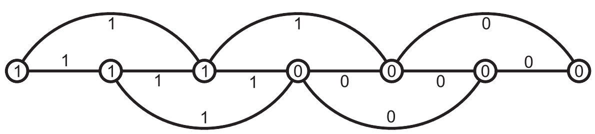

Fig. 1

Definition 2.6 : The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''.

Theorem 2.7 : $D_2(C_n)$ is not an edge product cordial graph.

Proof : We will consider following two cases.

<u>**Case 1:** When n is even.</u>

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $2n$ edges out of $4n$ edges. The edges with label 0 will give rise at least $\frac{n}{2}+n$ vertices with label 0 and at most $\frac{n}{2}$ vertices with label 1 out of total $2n$ vertices. Therefore $|v_f(0)-v_f(1)| \geq n$. Thus the vertex condition for edge product cordial graph is violated.

<u>**Case 2:** When n is odd.</u>

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $2n$ edges out of $4n$ edges. The edges with label 0 will give rise at least $\left\lceil \frac{n}{2} \right\rceil + n$ vertices with label 0 and at most $\left\lfloor \frac{n}{2} \right\rfloor$ vertices with label 1 out of total $2n$ vertices. Therefore $|v_f(0) - v_f(1)| \geq n + 1$. Thus the vertex condition for edge product cordial graph is violated.

Therefore $D_2(C_n)$ is not an edge product cordial graph.

Theorem 2.8 : $D_2(P_n)$ is not an edge product cordial graph.

Proof : In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $2n-2$ edges out of $4n - 4$ edges. The edges with label 0 will give rise at least $n + 1$ vertices with label 0 and at most $n - 1$ vertices with label 1 out of total $2n$ vertices. Therefore $|v_f(0)-v_f(1)| \geq 2$. Thus the vertex condition for edge product cordial graph is violated.

Therefore $D_2(P_n)$ is not an edge product cordial graph.

Definition 2.9 : The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

Theorem 2.10 : $M(C_n)$ is not an edge product cordial graph.

Proof : We will consider following two cases.

<u>**Case 1:** When n is even.</u>

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $\frac{3n}{2}$ edges out of $3n$ edges. The edges with label 0 will give rise at least $n+1$ vertices with label 0 and at most $n-1$ vertices with label 1 out of total $2n$ vertices. Therefore $|v_f(0)-v_f(1)| \geq 2$. Thus the vertex condition for edge product cordial graph is violated.

<u>**Case 2:** When n is odd.</u>

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $\frac{3n-1}{2}$ edges out of $3n$ edges. The edges with label 0 will give rise at least $n+1$ vertices with label 0 and at most $n-1$ vertices with label 1 out of total $2n$ vertices. Therefore $|v_f(0)-v_f(1)| \geq 2$. Thus the vertex condition for edge product cordial graph is violated.

Therefore $M(C_n)$ is not an edge product cordial graph.

Theorem 2.11 : $M(P_n)$ is an edge product cordial graph.

Proof : Let P_n be the graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_{n-1}$. For the graph $M(P_n)$ added vertices corresponding to edges are $e_1, e_2, \ldots, e_{n-1}$. We have $|V(M(P_n))| = 2n - 1$ and $|E(M(P_n))| = 3n - 4$.

To define $f : E(M(P_n)) \rightarrow \{0, 1\}$ we will consider following two cases.

<u>**Case 1:** When n is even.</u>

$$f(e_i e_{i+1}) = 1; \quad 1 \leq i \leq \frac{n}{2} - 2,$$
$$f(e_i e_j) = 0; \quad \text{otherwise},$$
$$f(v_i e_i) = 1; \quad 1 \leq i \leq \frac{n}{2},$$
$$f(v_{i+1} e_i) = 1; \quad 1 \leq i \leq \frac{n}{2} - 1,$$
$$f(v_n e_{n-1}) = 1;$$
$$f(v_i e_j) = 0; \quad \text{otherwise.}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(1) + 1 = n$$
$$e_f(0) = e_f(1) = \frac{3n-4}{2}$$

<u>**Case 2:** When n is odd.</u>

$$f(e_i e_{i+1}) = 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1,$$
$$f(e_i e_j) = 0; \quad \text{otherwise},$$
$$f(v_i e_i) = 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f(v_{i+1} e_i) = 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1,$$
$$f(v_n e_{n-1}) = 1;$$
$$f(v_i e_j) = 0; \quad \text{otherwise.}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(1) + 1 = n$$
$$e_f(0) = e_f(1) + 1 = \frac{3n-3}{2}$$

Thus in each case we have $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$.

Therefore $M(P_n)$ is an edge product cordial graph.

Illustration 2.12 : The graph $M(P_7)$ and its edge product cordial labeling is shown in Fig. 2.

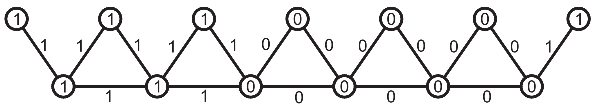

Fig. 2

Definition 2.13 : The *total graph* of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G. The total graph of G is denoted by $T(G)$.

Theorem 2.14 : $T(C_n)$ is not an edge product cordial graph.

Proof : In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $2n$ edges out of $4n$ edges. The edges with label 0 will give rise at least $n+2$ vertices with label 0 and at most $n-2$ vertices with label 1 out of total $2n$ vertices. Therefore $|v_f(0)-v_f(1)| \geq 4$. Thus the vertex condition for edge product cordial graph is violated.

Therefore $T(C_n)$ is not an edge product cordial graph.

Theorem 2.15 : $T(P_n)$ is an edge product cordial graph.

Proof : Let P_n be the graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_{n-1}$. For the graph $T(P_n)$ added vertices corresponding to edges are $e_1, e_2, \ldots, e_{n-1}$. We have $|V(T(P_n))| = 2n-1$ and $|E(T(P_n))| = 4n-5$.

To define $f : E(T(P_n)) \rightarrow \{0,1\}$ we will consider following two cases.

<u>**Case 1:** When n is even.</u>

$$\begin{aligned}
f(v_iv_{i+1}) &= 1; \quad 1 \leq i \leq \frac{n}{2}, \\
f(v_iv_j) &= 0; \quad \text{otherwise}, \\
f(e_ie_{i+1}) &= 1; \quad 1 \leq i \leq \frac{n}{2}-1, \\
f(e_ie_j) &= 0; \quad \text{otherwise}, \\
f(v_ie_i) &= 1; \quad 1 \leq i \leq \frac{n}{2}, \\
f(v_{i+1}e_i) &= 1; \quad 1 \leq i \leq \frac{n}{2}-1, \\
f(v_ie_j) &= 0; \quad \text{otherwise}.
\end{aligned}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(1) + 1 = n$$
$$e_f(0) + 1 = e_f(1) = 2n-2$$

<u>**Case 2:** When n is odd.</u>

$$\begin{aligned}
f(v_iv_{i+1}) &= 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
f(v_iv_j) &= 0; \quad \text{otherwise}, \\
f(e_ie_{i+1}) &= 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
f(e_ie_j) &= 0; \quad \text{otherwise}, \\
f(v_ie_i) &= 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
f(v_{i+1}e_i) &= 1; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
f(v_ie_j) &= 0; \quad \text{otherwise}.
\end{aligned}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(1) + 1 = n$$
$$e_f(0) + 1 = e_f(1) = 2n-2$$

Thus in each case we have $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$.

Therefore $T(P_n)$ is an edge product cordial graph.

Illustration 2.16 : The graph $T(P_7)$ and its edge product cordial labeling is shown in Fig. 3.

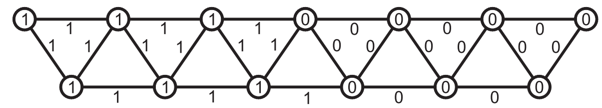

Fig. 3

Definition 2.17 : The *splitting graph* of a graph G is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The resultant graph is denoted by $S'(G)$.

Theorem 2.18 : $S'(C_n)$ is not an edge product cordial graph.

Proof : We will consider following two cases.

<u>**Case 1:** When n is even.</u>

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $\frac{3n}{2}$ edges out of $3n$ edges. The edges with label 0 will give rise at least $n + \left\lceil \frac{n}{4} \right\rceil$ vertices with label 0 and at most $n - \left\lceil \frac{n}{4} \right\rceil$ vertices with label 1 out of total $2n$ vertices. Therefore $|v_f(0)-v_f(1)| \geq 2\left\lceil \frac{n}{4} \right\rceil$. Thus the vertex condition for edge product cordial graph is violated.

<u>**Case 2:** When n is odd.</u>

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $\left\lfloor \frac{3n}{2} \right\rfloor$ edges out of $3n$ edges. The edges with label 0 will give rise at least $2n - \left\lfloor \frac{3n+1}{4} \right\rfloor$ vertices with label 0 and at most $\left\lfloor \frac{3n+1}{4} \right\rfloor$ vertices with label 1 out of total $2n$ vertices. Therefore $|v_f(0)-v_f(1)| \geq 2\left\lceil \frac{n-1}{4} \right\rceil$. Thus the vertex condition for edge product cordial graph is violated.

Therefore $S'(C_n)$ is not an edge product cordial graph.

Theorem 2.19 : $S'(P_n)$ is not an edge product cordial graph for odd n.

Proof : In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $\frac{3n-3}{2}$ edges out of $3n-3$ edges. The edges with label 0 will give rise at least $n+1$ vertices with label 0 and at most $n-1$ vertices with label 1 out of total $2n$ vertices. Therefore $|v_f(0)-v_f(1)| \geq 2$. Thus the vertex condition for edge product cordial graph is violated.

Therefore $S'(P_n)$ is not an edge product cordial graph for odd n.

Theorem 2.20 : $S'(P_n)$ is edge product cordial graph for even n.

Proof : Let P_n be the graph with vertices $v_1, v_2, \ldots, v_n$. For graph $S'(P_n)$ added vertices corresponding to $v_1, v_2, \ldots, v_n$ are $v'_1, v'_2, \ldots, v'_n$. We have $|V(S'(P_n))| = 2n$ and $|E(S'(P_n))| = 3n - 3$.

We define $f : E(S'(P_n)) \rightarrow \{0, 1\}$ follows.

$$
\begin{aligned}
&f(v_i v_{i+1}) = 1; && 1 \leq i \leq \frac{n}{2} - 1,\\
&f(v_i v_j) = 0; && \text{otherwise},\\
&f(v'_i v_{i+1}) = 1; && 1 \leq i \leq \frac{n}{2},\\
&f(v'_{i+1} v_i) = 1; && 1 \leq i \leq \frac{n}{2} - 1,\\
&f(v'_n v_{n-1}) = 1; &&\\
&f(v'_i v_j) = 0; && \text{otherwise}.
\end{aligned}
$$

In view of the above labeling patten we have

$$
\begin{aligned}
v_f(0) = v_f(1) = n\\
e_f(0) = e_f(1) - 1 = \frac{3n}{2} - 2
\end{aligned}
$$

Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Therefore $S'(P_n)$ is edge product cordial graph for even n.

Illustration 2.21 : The graph $S'(P_6)$ and its edge product cordial labeling is shown in Fig. 4.

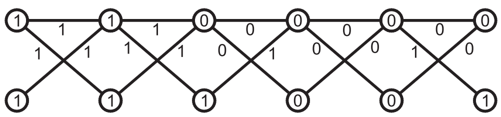

Fig. 4

III. Concluding Remarks

An edge analogue of cordial labeling named as E - cordial labeling is also available in literature. In which $f : E(G) \rightarrow \{0, 1\}$ is called an E - cordial labeling of G if the induced vertex labeling function defined by the sum modulo 2 of incident edge labels be such that the edges with label 1 & label 0 differ by at most 1 and the vertices with label 1 & label 0 also differ by at most 1. We have observed that E - cordial and edge product cordial labelings of a graph are two independent concepts. A graph may possess one or both of these labeling or neither as exhibited below.

1) $T(P_n)$ is E - cordial as well as edge product cordial.
2) Trees with n vertices and $n \equiv 2$ (mod 4) is not E - cordial but it is edge product cordial.
3) The complete bipartite graph $K_{m,n}$ for $m+n \not\equiv 2$ (mod 4) and $m, n \geq 2$ is E - cordial but not edge product cordial.
4) The complete graph K_n for $n \equiv 2$ (mod 4) is neither E - cordial nor edge product cordial.

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