Abstract—The n-self adjoint operator is a generalization of well known self-adjoint operator. We prove that any normal, self-adjoint operator on a Hilbert space $H$ is self-adjoint. We also discuss the properties of $S_n$, the set of all n-self adjoint operators on a Hilbert space $H$, which are similar to self-adjoint operators. We obtained expected results, for some special cases with $\dim H < \infty$.

Index Terms—Matrices, Hilbert space, self adjoint operators and n-self adjoint operators.

MSC 2010 Codes – 47B38, 46C05

I. INTRODUCTION

The development of the theory of n-self adjoint operators in infinite dimensional Hilbert space is motivated by its connections with differential equations, particularly conjugate point theory and disconjugacy, which was first observed by Helton [1]. The 3-self adjoint operators are studied in [1], [2], [3], [4].

Let $H$ be a Hilbert space and $BL(H)$ denote the set of bounded, linear operators on $H$. An operator $A \in BL(H)$ is self adjoint if $A = A^*$. The concept of n-self adjoint operators has been defined in [5] as follows:

Definition 1.1 An operator $A \in BL(H)$ is said to be n-self adjoint, if

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} A^k A^{n-k} = 0$$

where $A^0$ and $A^{*0}$ are identity operators. $\square$

We shall denote the set of all n-self adjoint operators by $S_n$.

It is clear that $S_1$ is the set of self adjoint operators.

In this paper, first we prove that if n-self adjoint operator is normal then it becomes self-adjoint (Theorem 2.4). Then we discuss algebraic structure of $S_n$. It is well known that the set of all self-adjoint operators on $H$ is a real, linear subspace of $BL(H)$ and it is closed under multiplication, with some condition [6]. We study similar results for $S_n$ with $\dim H < \infty$, in section 3.

For the basic concepts and results of matrices and operators, we refer to [6] and [7].

II. INFINITE DIMENSIONAL SPACE

In this section, we consider an infinite dimensional Hilbert space $H$. In the main theorem, we prove that n-self adjoint operators together with normality give self adjointness.

We know that $BL(H)$ is a Banach algebra with the norm defined as

$$\|A\| = \sup \{ \|Ax\| : x \in H, \|x\| = 1 \} \quad A \in BL(H).$$

First we give an example of an operator in $S_3$ which is not self adjoint.

Example 2.1. Define $T$ and $N$ on $l^2$ as follows:

$T(x) = (\alpha x)$ where $\alpha = \{\alpha(i)\}$ with $\alpha(2i) = \alpha(2i-1) = \frac{1}{i}$

and

$N(x) = \{N(x)(i)\}$, where

$$N(x,t) = \begin{cases} \frac{x(i+1)}{\beta(i)} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

with $\beta(i) \neq 0 \forall i, \beta(i) \in \mathbb{C}$ for some $i$ and $\frac{1}{\beta(i)} \to 0$ as $i \to \infty$.

Define $A$ on $l^2$ by $A = T + N$. Then $A \in S_3$, but $A$ is not self adjoint.

It is very well known that the set of self-adjoint operators is closed in $BL(H)$. We proved the similar result for $S_n$.

We fix the notation $a_k = (-1)^k \binom{n}{k}$.

Theorem 2.2. $S_n$ is closed in $BL(H)$.

Proof. Let $\{A_j\}$ be a sequence in $S_n$ such that $A_j \to A$ in $BL(H)$. Then $\|A_j - A\| \to 0$. We shall show that $A \in S_n$.

As $A_j \in S_n$, $\sum_{k=0}^{n} a_k A_j^k A_j^{n-k} = 0$, for each $j$. Further

$$\| \sum_{k=0}^{n} a_k A_j^k A_j^{n-k} - \sum_{k=0}^{n} a_k A^k A^{n-k} \| \leq \sum_{k=0}^{n} a_k \| A_j^k A_j^{n-k} - A^k A^{n-k} \|$$

Since $A_j \to A$ and adjoint operation is continuous, $A_j^* \to A^*$ in $BL(H)$. Also, as multiplication is jointly continuous, $A_j^k A_j^{n-k} \to A^k A^{n-k}$ in $BL(H)$. So, we get

$$\| \sum_{k=0}^{n} a_k A_j^k A_j^{n-k} \| \to 0 \quad \text{and} \quad \sum_{k=0}^{n} a_k A^k A^{n-k} = 0.$$ 

Hence $A \in S_n$. Consequently, $S_n$ is closed in $BL(H)$.

To prove the main result, we shall need the following basic result [6].

Proposition 2.3 Let $A \in BL(H)$ be a normal operator. Then, $\|A^2\| = \|A\|^2$. Consequently, $\|A^{2k}\| = \|A\|^{2k}$ for each $k \in \mathbb{N}$.

Let $N_R$ denote the set of all normal operators on $H$.

Theorem 2.4. $S_n \cap N_R = S_1$. 

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Proof. Let $A \in S_n \cap N_R$. Then $AA^* = A^*A$ and 
\[ \sum_{k=0}^{n} a_k A^{*k} A^{n-k} = 0. \]
Since the operators $A$ and $A^*$ commute, 
\[ \sum_{k=0}^{n} a_k A^{*k} A^{n-k} = (A - A^*)^n. \]
So $(A - A^*)^n = 0$.
Taking $B = A - A^*$, we get $B$ normal and also $B^n = 0$. So $B^{2n} = 0$. Using Proposition 2.3 for $B$, we get $\|B\| = 0$ or $B = 0$, i.e., $A = A^*$. Thus $A \in S_1$. Therefore $S_n \cap N \subset S_1$. It is clear that $S_1 \subset S_n \subset N_R$. Consequently, $S_n \cap N_R = S_1$.

Note that the set $S_n$ is not algebraically closed, even if $\text{dim} H$ is finite, which we shall discuss in Section 3.

The following result can be easily verified from the definition.

**Theorem 2.5.** If $A \in S_n$ and $\alpha \in \mathbb{R}$, then $\alpha A \in S_n$.

With some extra care, we get the following result.

**Theorem 2.6.** Let $A \in S_n$, $B \in S_1$ and $A$ commutes with $B$. Then $AB \in S_n$.

**Proof.** As $A \in S_n$, \[ \sum_{k=0}^{n} a_k A^{*k} A^{n-k} = 0. \]
Also $B = B^*$ and $AB = BA$. So $(AB)^{*k} = A^{*k} B^k$ and $(AB)^{n-k} = A^{*k} B^{n-k} = B^{n-k} A^{*k}$. Thus 
\[ \sum_{k=0}^{n} a_k (AB)^{*k} (AB)^{n-k} = \sum_{k=0}^{n} a_k (A^{*k} B^k) (B^{n-k} A^{*k}) = 0. \]
Hence $AB \in S_n$.

III. **Algebraic Properties**

The theory of $n$-self adjoint operators on an infinite dimensional Hilbert space differs significantly from that of $n$-self adjoint matrices. For example, the main characterization (Theorem 3.1) fails for infinite dimensional space.

Throughout this section, we assume that $H$ is a Hilbert space with $\text{dim} H = p < \infty$. So, an operator $A \in BL(H)$ is considered as a matrix $A = [a_{ij}]$ and $I$ denote the identity operator or matrix. We discussed the algebraic properties of $S_n$.

We start with the following very useful characterization of $S_n$.

**Theorem 3.1.**[5] An operator $A \in BL(H)$, with $\text{dim} H < \infty$, is in $S_n$ if and only if $A = T + N$, where $T$ is self adjoint, $TN = NT$ and $N^{[\frac{1}{3}]} = 0$ ([$r$] denote the integer part of $r$).

**Remarks 3.2.**
1) It is clear that $S_2n = S_{2n-1}$.
2) Let $A \in S_n$ with $A = T + N$. Then, $N^{[\frac{1}{3}]} = 0$ and so, for $m \geq n$, $N^{[\frac{1}{m}]} = 0$. Thus $A \in S_m$, $\forall m \geq n$.
3) $S_1 = S_2 \subset S_3 \subset S_4 \subset \ldots \subset S_{2n-1} = S_{2n}$.

First, we show that certain matrices are always in $S_n$.

**Theorem 3.3.** Let $T = \alpha I$ with $\alpha \in \mathbb{R}$ and $N$ be an upper (lower) triangular matrix. Then $A = T + N$ is in $S_n$, $\forall n \geq 2p - 1$.

**Proof.** Take $A = T + N$. It is clear that $T$ is self adjoint and $TN = (\alpha I) N = NT$. Further, if $n \geq 2p - 1$, then $\frac{1}{n} \geq \left(\frac{1}{2n} + \frac{1}{4n}\right) = \frac{1}{2p}$. Since $N$ is an upper (lower) triangular matrix, we get $N^{[\frac{1}{2p}]} = 0$. Therefore, by Theorem 3.1, $A \in S_n$, if $n \geq 2p - 1$.

Since addition, scalar multiplication and multiplication of upper triangular matrices are upper triangular, we get the following result.

**Theorem 3.4.** Let $A, B \in S_n$ be as in Theorem 3.3. Then $A + B$ and $AB$ are in $S_n$.

Unfortunately, $S_n$ is not closed under addition, even for $n = 3$.

**Example 3.5.** Let $A = \begin{pmatrix} 3 & i \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 0 \\ 2i & 4 \end{pmatrix}$.

Then $A, B \in S_3$ but $C = A + B \notin S_3$, as 
\[ \sum_{k=0}^{3} (-1)^k \begin{pmatrix} 3k+1 \\ k \end{pmatrix} C^{3-k} = \begin{pmatrix} 0 & -24i \\ -24i & 0 \end{pmatrix} \neq 0. \]

The next example shows that even with $B = B^*$, $A + B$ may not be in $S_3$.

**Example 3.6.** Let $A = \begin{pmatrix} 2 & i \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

Then $B = B^*$ and $A \in S_3$. But, $A + B$ is not in $S_3$.

However, for special types of $A$ and $B$ in $S_n$, we prove that $A + B \in S_n$.

**Theorem 3.7.** Let $A \in S_n$ with $A = T + N$, $B = T_0 + \left(N \right)^t$ with $T_0$ self adjoint and $\left(N \right)^t$ the conjugate transpose of $N$. Then, $A + B$ is self adjoint. Consequently, $A + B \in S_n$.

**Proof.** Let $A = T + N$ and $B = T_0 + \left(N \right)^t$ with $T$ and $T_0$ self adjoint, $TN = NT$ and $N^{[\frac{1}{2}]} = 0$. Then, $A + B = (T + T_0) + (N + \left(N \right)^t)$. But, for any matrix $N$, $(N + \left(N \right)^t)$ is self adjoint. So, $A + B$ is self adjoint. Hence $A + B \in S_n$.

**Theorem 3.8.** Let $A \in S_n$. Then $A^k \in S_n$, for each $k \in N$.

**Proof.** Let $A \in S_n$. Then $A = T + N$ with $T = T^*$, $TN = NT$ and $N^{[\frac{1}{2}]} = 0$. Now, $A^2 = T^2 + 2NT + N^2 = T^2 + N(2T + N) = T^2 + NP_2(T, N)$. And, $\left[NP_2(T, N)\right]^{[\frac{1}{2}]} = N^{[\frac{1}{m}]} P_2(T, N)^{[\frac{1}{m}]}$, so $T$ and $N$ commutes. So, $\left[NP_2(T, N)\right]^{[\frac{1}{2}]} = 0$. Also, $T^2 \left[NP_2(T, N)\right] = \left[NP_2(T, N)\right] T^2$. Further, it is clear that $T^2$ is self adjoint and So, by Theorem 3.1, $A^2 \in S_n$. Now $A^3 = (T + N)A^2 = (T + N)(T^2 + NP_2(T, N)) = T^3 + NP_3(T, N)$, where $P_3(T, N)$ is a polynomial in $T$ and $N$. In general, for each $k$, it can be checked that $A^k = T^k + NP_k(T, N)$. So,
as we have seen earlier, $T^k[NP_k(T, N)] = [NP_k(T, N)]T^k$. Also, $T^k$ is self adjoint and $NP_k(T, N)[\frac{1}{k^2}] = 0$. So, by Theorem 3.1, $A^k \in S_n$.

In general, $S_n$ is not closed under multiplication (Example 3.6).

The next result shows that $S_n$ is closed under adjoint.

**Theorem 3.9.** If $A \in S_n$, then $A^* \in S_n$.

**Proof.** Let $A \in S_n$. Then $A = T + N$ with $T = T^*$, $TN = NT$ and $N[\frac{1}{k^2}] = 0$. Now, $A^* = T^* + N^* = T + N^*$. Also, $(N^*)[\frac{1}{k^2}] = 0$ and $TN = NT \Rightarrow N^*T = TN^*$. Thus, by Theorem 3.1, $A^* \in S_n$.

**IV. Conclusion**

We discussed the algebraic properties of the n-self adjoint operators, $S_n$. We obtained two results similar to self adjoint operators (Theorems 2.2 and 2.5). But, the sum or product of two n-self adjoint operators may not be n-self adjoint operator, even if $\dim H < \infty$. However, with special form, we get partial results (Theorems 3.3, 3.5 and 3.7). Also, we proved that with normality, n-self adjoint operators become self adjoint. It will be interesting to find conditions under which $S_n$ is closed under addition and multiplication.

**References**


