n-Self Adjoint Operators

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***Abstract*—The n-self adjoint operator is a generalization of well known self-adjoint operator. We prove that any normal, self-adjoint operator on a Hilbert space H is self-adjoint. We also discuss the properties of $\mathbb{S}_n$ -the set of all n-self adjoint operators on a Hilbert space H, which are similar to self-adjoint operators. We obtained expected results, for some special cases with $dimH < \infty$.**

***Index Terms*—Matrices, Hilbert space, self adjoint operators and n-self adjoint operators.**

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I. Introduction

The development of the theory of n-self adjoint operators in infinite dimensional Hilbert space is motivated by its connections with differential equations, particularly conjugate point theory and disconjugacy, which was first observed by Helton [1]. The 3-self adjoint operators are studied in ([1], [2], [3], [4]).

Let H be a Hilbert space and $BL(H)$ denote the set of bounded, linear operators on H. An operator $A \in BL(H)$ is self adjoint if $A = A^*$.

The concept of n-self adjoint operators has been defined in [5] as follows:

Definition 1.1 An operator $A \in BL(H)$ is said to be n-self adjoint, if

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} A^{*k} A^{n-k} = 0$$

where A^0 and A^{*0} are identity operators. □

We shall denote the set of all n-self adjoint operators by $\mathbb{S}_n$.

It is clear that $\mathbb{S}_1$ is the set of self adjoint operators.

In this paper, first we prove that if n-self adjoint operator is normal then it becomes self-adjoint (Theorem 2.4). Then we discuss algebraic structure of $\mathbb{S}_n$. It is well known that the set of all self-adjoint operators on H is a real, linear subspace of $BL(H)$ and it is closed under multiplication, with some condition [6]. We study similar results for $\mathbb{S}_n$ with $dimH < \infty$, in section 3.

For the basic concepts and results of matrices and operators, we refer to [6] and [7].

II. Infinite Dimensional Space

In this section, we consider an infinite dimensional Hilbert space H. In the main theorem, we prove that n-self adjoint operators together with normality give self adjointness.

We know that $BL(H)$ is a Banach algebra with the norm defined as

$$\| A \| = sup\{\| Ax \| : x \in H, \| x \| = 1\}, \quad A \in BL(H).$$

First we give an example of an operator in $\mathbb{S}_3$ which is not self adjoint.

Example 2.1. Define T and N on l^2 as follows:
$T(x) = (\alpha x)$ where $\alpha = \{\alpha(i)\}$ with $\alpha(2i) = \alpha(2i-1) = \frac{1}{i}$ and
$N(x) = \{N(x)(i)\}$, where

$$N(x,t) = \begin{cases} \frac{x(i+1)}{\beta(i)} & \text{if } \; i \text{ is odd} \\ 0 & \text{if } \; i \text{ is even} \end{cases}$$

with $\beta(i) \neq 0 \; \forall \; i, \; \beta(i) \in \mathbb{C}$ for some i and $\frac{1}{\beta(i)} \to 0$ as $i \to \infty$.
Define A on l^2 by $A = T + N$. Then $A \in \mathbb{S}_3$, but A is not self adjoint.

It is very well known that the set of self-adjoint operators is closed in $BL(H)$. We proved the similar result for $\mathbb{S}_n$.

We fix the notation $a_k = (-1)^k \binom{n}{k}$.

Theorem 2.2. $\mathbb{S}_n$ is closed in $BL(H)$.

Proof. Let $\{A_j\}$ be a sequence in $\mathbb{S}_n$ such that $A_j \to A$ in $BL(H)$. Then $\| A_j - A \| \to 0$. We shall show that $A \in \mathbb{S}_n$. As $A_j \in \mathbb{S}_n$, $\sum_{k=0}^{n} a_k A_j^{*k} A_j^{n-k} = 0$, for each j. Further

$$\| \sum_{k=0}^{n} a_k A_j^{*k} A_j^{n-k} - \sum_{k=0}^{n} a_k A^{*k} A^{n-k} \|$$
$$\leq \sum_{k=0}^{n} a_k \| A_j^{*k} A_j^{n-k} - A^{*k} A^{n-k} \|$$

Since $A_j \to A$ and adjoint operation is continuous, $A_j^* \to A^*$ in $BL(H)$. Also, as multiplication is jointly continuous, $A_j^{*k} A_j^{n-k} \to A^{*k} A^{n-k}$ in $BL(H)$. So, we get $\| \sum_{k=0}^{n} a_k A^{*k} A^{n-k} \| = 0$ or $\sum_{k=0}^{n} a_k A^{*k} A^{n-k} = 0$. Hence $A \in \mathbb{S}_n$. Consequently, $\mathbb{S}_n$ is closed in $BL(H)$.

To prove the main result, we shall need the following basic result [6].

Proposition 2.3 Let $A \in BL(H)$ be a normal operator. Then, $\| A^2 \| = \| A \|^2$. Consequently, $\| A^{2k} \| = \| A \|^{2k}$ for each $k \in \mathbb{N}$.

Let N_R denote the set of all normal operators on H.

Theorem 2.4. $\mathbb{S}_n \cap N_R = \mathbb{S}_1$.

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Proof. Let $A \in \mathbb{S}_n \cap N_R$. Then $AA^* = A^*A$ and $\sum_{k=0}^{n} a_k A^{*k} A^{n-k} = 0$. Since the operators A and A^* commutes, $\sum_{k=0}^{n} a_k A^{*k} A^{n-k} = (A - A^*)^n$. So $(A - A^*)^n = 0$. Taking $B = A - A^*$, we get B normal and also $B^n = 0$. So $B^{2n} = 0$. Using Proposition 2.3 for B, we get $\| B \| = 0$ or $B = 0$, i. e., $A = A^*$. Thus $A \in \mathbb{S}_1$. Therefore $\mathbb{S}_n \cap N \subset \mathbb{S}_1$. It is clear that $\mathbb{S}_1 \subset \mathbb{S}_n \cap N_R$. Consequently, $\mathbb{S}_n \cap N_R = \mathbb{S}_1$.

Note that the set $\mathbb{S}_n$ is not algebraically closed, even if $dimH$ is finite, which we shall discuss in Section 3.

The following result can be easily verified from the definition.

Theorem 2.5. If $A \in \mathbb{S}_n$ and $\alpha \in \mathbb{R}$, then $\alpha A \in \mathbb{S}_n$.

With some extra condition, we get the following result.

Theorem 2.6. Let $A \in \mathbb{S}_n,\ B \in \mathbb{S}_1$ and A commutes with B. Then $AB \in \mathbb{S}_n$.

Proof. As $A \in \mathbb{S}_n$, $\sum_{k=0}^{n} a_k A^{*k} A^{n-k} = 0$. Also $B = B^*$ and $AB = BA$. So $(AB)^{*k} = A^{*k} B^k$ and $(AB)^{n-k} = A^{n-k} B^{n-k} = B^{n-k} A^{n-k}$. Thus

$$\begin{aligned} \sum_{k=0}^{n} a_k (AB)^{*k} (AB)^{n-k} &= \sum_{k=0}^{n} a_k (A^{*k} B^k)(B^{n-k} A^{n-k}) \\ &= [\sum_{k=0}^{n} a_k A^{*k} A^{n-k}] B^n \\ &= 0 \end{aligned}$$

Hence $AB \in \mathbb{S}_n$.

III. Algebraic Properties

The theory of n-self adjoint operators on an infinite dimensional Hilbert space differs significantly from that of n-self adjoint matrices. For example, the main characterization (Theorem 3.1) fails for infinite dimensional space.

Throughout this section, we assume that H is a Hilbert space with $dimH = p < \infty$. So, an operator $A \in BL(H)$ is considered as $p \times p$ matrix $A = [a_{ij}]$ and I denote the identity operator or matrix. We discussed the algebraic properties of $\mathbb{S}_n$.

We start with the following very useful characterization of $\mathbb{S}_n$.

Theorem 3.1.[5] An operator $A \in BL(H)$, with $dimH < \infty$, is in $\mathbb{S}_n$ if and only if $A = T + N$, where T is self adjoint, $TN = NT$ and $N^{\left[\frac{n+1}{2}\right]} = 0$ ($[r]$ denote the integer part of r).

Remarks 3.2.

1) It is clear that $\mathbb{S}_{2n} = \mathbb{S}_{2n-1}$.
2) Let $A \in \mathbb{S}_n$ with $A = T + N$. Then, $N^{\left[\frac{n+1}{2}\right]} = 0$ and so, for $m \geq n$, $N^{\left[\frac{n+1}{2}\right]} = 0$. Thus $A \in \mathbb{S}_m, \forall m \geq n$.
3) $\mathbb{S}_1 = \mathbb{S}_2 \subset \mathbb{S}_3 = \mathbb{S}_4 \subset ... \subset \mathbb{S}_{2n-1} = \mathbb{S}_{2n}$.

First, we show that certain matrices are always in $\mathbb{S}_n$.

Theorem 3.3. Let $T = \alpha I$ with $\alpha \in \mathbb{R}$ and N be an upper (lower) triangular matrix. Then $A = T + N$ is in $\mathbb{S}_n$, $\forall n \geq 2p - 1$.

Proof. Take $A = T + N$. It is clear that T is self adjoint and $TN = (\alpha I)N = NT$. Further, if $n \geq 2p - 1$, then $\left[\frac{n+1}{2}\right] \geq \left[\frac{2p-1+1}{2}\right] = p$. Since N is an upper (lower) triangular matrix, we get, $N^{\left[\frac{n+1}{2}\right]} = 0$. Therefore, by Theorem 3.1, $A \in \mathbb{S}_n$, if $n \geq 2p - 1$.

Since addition, scalar multiplication and multiplication of upper triangular matrices are upper triangular, we get the following result.

Theorem 3.4. Let $A, B \in \mathbb{S}_n$ be as in Theorem 3.3. Then $A + B$ and AB are in $\mathbb{S}_n$.

Unfortunately, $\mathbb{S}_n$ is not closed under addition, even for $n = 3$.

Example 3.5. Let $A = \begin{pmatrix} 3 & i \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 0 \\ 2i & 4 \end{pmatrix}$.

Then $A, B \in \mathbb{S}_3$ but $C = A + B \notin \mathbb{S}_3$, as

$\sum_{k=0}^{3} (-1)^k \binom{3}{k} C^{*k} C^{3-k} = \begin{pmatrix} 0 & -24i \\ -24i & 0 \end{pmatrix} \neq 0.$

The next example shows that even with $B = B^*, A + B$ may not be in $\mathbb{S}_3$.

Example 3.6. Let $A = \begin{pmatrix} 2 & i \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

Then $B = B^*$ and $A \in \mathbb{S}_3$. But, $A + B$ is not in $\mathbb{S}_3$.

However, for special types of A and B in $\mathbb{S}_n$, we prove that $A + B \in \mathbb{S}_n$.

Theorem 3.7. Let $A \in \mathbb{S}_n$ with $A = T + N, B = T_0 + (\overline{N})^t$ with T_0 self adjoint and $(\overline{N})^t$ the conjugate transpose of N. Then, $A + B$ is self adjoint. Consequently, $A + B \in \mathbb{S}_n$.

Proof. Let $A = T + N$ and $B = T_0 + (\overline{N})^t$ with T and T_0 self adjoint, $TN = NT$ and $N^{\left[\frac{n+1}{2}\right]} = 0$. Then, $A + B = (T + T_0) + (N + (\overline{N})^t)$. But, for any matrix N, $(N + (\overline{N})^t)$ is self adjoint. So, $A + B$ is self adjoint. Hence $A + B \in \mathbb{S}_n$.

Theorem 3.8. Let $A \in \mathbb{S}_n$. Then $A^k \in \mathbb{S}_n$, for each $k \in \mathbb{N}$.

Proof. Let $A \in \mathbb{S}_n$. Then $A = T + N$ with $T = T^*$, $TN = NT$ and $N^{\left[\frac{n+1}{2}\right]} = 0$. Now, $A^2 = T^2 + 2NT + N^2 = T^2 + N(2T + N) = T^2 + NP_2(T, N)$. And, $[NP_2(T, N)]^{\left[\frac{n+1}{2}\right]} = N^{\left[\frac{n+1}{2}\right]} P_2(T, N)^{\left[\frac{n+1}{2}\right]}$, as T and N commutes. So, $[NP_2(T, N)]^{\left[\frac{n+1}{2}\right]} = 0$. Also, $T^2[NP_2(T, N)] = [NP_2(T, N)]T^2$. Further, it is clear that T^2 is self adjoint and So, by Theorem 3.1, $A^2 \in \mathbb{S}_n$. Now $A^3 = (T + N)A^2 = (T + N)(T^2 + NP_2(t, N)) = T^3 + NP_3(T, N)$, where $P_3(T, N)$ is a polynomial in T and N. In general, for each k, it can be checked that $A^k = T^k + NP_k(T, N)$. So, as we have seen earlier, $T^k[NP_k(T,N)] = [NP_k(T,N)]T^k$. Also, T^k is self adjoint and $[NP_k(T,N)]^{\left[\frac{n+1}{2}\right]} = 0$. So, by Theorem 3.1, $A^k \in \mathbb{S}_n$.

In general, $\mathbb{S}_n$ is not closed under multiplication (Example 3.6).

The next result shows that $\mathbb{S}_n$ is closed under adjoint.

Theorem 3.9. If $A \in \mathbb{S}_n$, then $A^* \in \mathbb{S}_n$.

Proof. Let $A \in \mathbb{S}_n$. Then $A = T + N$ with $T = T^*, TN = NT$ and $N^{\left[\frac{n+1}{2}\right]} = 0$. Now, $A^* = T^* + N^* = T + N^*$. Also, $(N^*)^{\left[\frac{n+1}{2}\right]} = 0$ and $TN = NT \Rightarrow N^*T = TN^*$. Thus, by Theorem 3.1, $A^* \in \mathbb{S}_n$.

IV. Conclusion

We discussed the algebraic properties of the n-self adjoint operators, $\mathbb{S}_n$. We obtained two results similar to self adjoint operators (Theorems 2.2 and 2.5). But, the sum or product of two n-self adjoint operators may not be n-self adjoint operator, even if $dimH < \infty$. However, with special form, we get partial results (Theorems 3.3, 3.5 and 3.7). Also, we proved that with normality, n-self adjoint operators become self adjoint. It will be interesting to find conditions under which $\mathbb{S}_n$ is closed under addition and multiplication.

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