A Fixed Point Theorem in Dislocated Fuzzy Quasi-Metric Spaces

G. Rano and T. Bag*

***Abstract*—In this paper the idea of Dislocated fuzzy quasi-metric space(d.f.q.m.s) is introduced and some basic properties are studied. Banach type fixed point theorem is established in such spaces with uniqueness.**

***Index Terms*— Dislocated fuzzy mertic, Dislocated fuzzy Quasi-metric, t-norm.**

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I. Introduction

THE concept of a fuzzy set was introduced by Zadeh[1] in 1965. After that, it has been developed in different direction viz. fuzzy real analysis, fuzzy topology, fuzzy functional analysis, decision making, fuzzy control theory etc.

One important development in fuzzy functional analysis is fuzzy metric space. Especially Zike-Deng[2], Kaleva and Seikkala[3], Kramosil and Michalek[4] have introduced the concept of fuzzy metric spaces in different ways. Many authors ([5], [6]) have studied fixed-point theorems in fuzzy metric spaces.

On the other hand, there have been a number of generalization and new concepts in metric spaces.

One such concept is dislocated quasi-metric spaces introduced by Zeyada et.al.[7] C.T.Aage and J.N.Salunke[8] have developed Some fixed point theorems and generalized the fixed point theorem established by Zeyada et.al.[7] in dislocated quasi-metric spaces.

In this paper, following the concept of Dislocated quasi-metric space, we introduce the idea - dislocated fuzzy quasi-metric space and study some basic properties. Banach fixed point type theorems are established in such space.

The organization of the paper is as follows: Section II comprises some preliminary results, which are used in this paper.

In Section III, a definition of dislocated fuzzy quasi-metric space is given.

In Section IV, we establish fixed point theorems in Dislocated fuzzy quasi-metric space. Throughout this paper, straightforward proofs are omitted.

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II. Some Preliminary Results

Definition 2.1[8] Let X be a nonempty set and let

$$d : X \times X \to [0, \infty)$$

be a function satisfying following conditions:

(i) $d(x, y) = d(y, x) = 0$ implies $x = y$;
(ii) $d(x, y) \leq d(x, z) + d(z, y) \;\; \forall\, x, y, z \in X$.

Then d is called a *dislocated quasi-metric* or *dq-metric* on X. If d satisfies $d(x, y) = d(y, x)$, then it is called *dislocated metric*. □

Definition 2.2[8] A sequence $\{x_n\}$ in dq-metric space (dislocated quasi-metric space) (X, d) is called *Cauchy sequence* if for given $\epsilon > 0$, $\exists\; n_0 \in N$, such that $\forall\; m,\; n \geq n_0$, implies $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$ i.e. $min\; \{d(x_m, x_n),\; d(x_n, x_m)\} < \epsilon$. □

In the above definition, if we replace $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$ by $max\; \{d(x_m, x_n),\; d(x_n, x_m)\} < \epsilon$, the sequence $\{x_n\}$ is called 'bi' Cauchy.

Definition 2.3 [8] A sequence $\{x_n\}$ dislocated quasi-converges to x if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$. In this case x is called a dq-limit of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$.

Definition 2.4[9] A binary operation

$$* : [0,\; 1] \times [0,\; 1] \to [0,\; 1]$$

is called t-norm if the following axioms are satisfied for all $a,\; b,\; d \in [0,\; 1]$:

(T1) $a * 1 = a$ (boundary condition).
(T2) $b \leq d$ implies $a * b \leq a * d$ (monotonicity).
(T3) $a * b = b * a$ (commutativity).
(T4) $a * (b * d) = (a * b) * d$ (associativity).

Definition 2.5[10] $*$ is said to be continuous if for any sequences $\{a_n\},\; \{b_n\}$ in $[0,\; 1]$ with

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b$$

implies $\lim_{n\to\infty} (a_n * b_n) = (a * b)$. □

Definition 2.6[4] (Kramosil and Michalek) The 3-tuple $(X,\; M,\; *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(M1) $M(x,\; y,\; 0) = 0$,
(M2)) $M(x,\; y,\; t) = 1 \quad \forall \;\; t > 0 \quad \text{iff} \quad x = y$,
(M3) $M(x,\; y,\; t) = M(y,\; x,\; t) \quad \forall x, y \in X$,
(M4) $M(x,\; y,\; t) * M(y,\; z,\; s) \leq M(x,\; z,\; t + s)$,
(M5) $M(x,\; y,\; .) : X^2 \times [0, \infty) \to [0, 1]$ is left continuous, where $x, y \in X \;\; and \;\; t\;, s > 0$.

M is called a fuzzy metric on X. □

Definition 2.7[4] Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent and converges to a point x in X if

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \ \ \forall t > 0. \ \square$$

Definition 2.8[4] In a fuzzy metric space $(X, M, *)$, a sequence $\{x_n\}$ is said to be a

(i) Cauchy sequence if

$$\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1 \ \ \forall t > 0, \quad \forall p = 1, 2, 3, ...$$

(ii) Convergent sequence if there exists a $x \in X$ such that

$$\lim_{n\to\infty} M(x_n, \ x \ , t) = 1 \ \ \forall t > 0$$

III. New Results

In this section we give the definition of dislocated fuzzy quasi-metric space(d.f.q.m.s), Cauchy sequence, Convergence of sequence, continuity of function and study some results which are used in this paper.

Definition 3.1 The 3-tuple $(X, \ M, \ *)$ is said to be a dislocated fuzzy quasi-metric space(d.f.q.m.s) if X is a nonempty arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(D1) $\quad M(x, \ y, \ t) = M(y, \ x, \ t) = 1 \ \forall \ t > 0 \ \Rightarrow \ x = y;$

(D2) $\quad M(x, \ y, \ t) * M(y, \ z, \ s) \leq M(x, \ z, \ t + s).$ $\square$

Note 3.1 We take the fuzzy metric in the sence of Kramosil and Michalek [4].

Example 3.1 Let (X, d) be a dislocated quasi-metric space. We define

$$M(x, \ y, \ t) = \begin{cases} \frac{t}{t + d(x, \ y)} & \text{for } t > \ 0 \\ 0 & \text{for } t = 0 \end{cases}$$

Then $(X, \ M, \ *)$ is a dislocated fuzzy quasi-metric space.

Proof Let $M(x, \ y, \ t) = 1 \ \forall \ t > 0$ then $d(x, \ y) = 0$.

If $M(y, \ x, \ t) = 1 \ \forall \ t > 0$ then $d(y, \ x) = 0$. Now $d(x, \ y) = 0$ and $d(y, \ x) = 0$ implies $x = y$.

Thus $M(x, \ y, \ t) = M(y, \ x, \ t) = 1 \ \forall \ t > 0 \ \Rightarrow \ x = y;$

So M satisfies (D1).

Now if $t = 0$ or $s = 0$ or $t = 0$ and $s = 0$ then $M(x, \ y, \ t) * M(y, \ z, \ s) = 0$, so M satisfies (D2).

Next suppose $t > 0$ and $s > 0$.

Then

$$\begin{aligned} M(x, z, t+s) &= \frac{t+s}{t+s+d(x, \ z)} \\ &\geq \frac{t+s}{t+s+d(x, \ y)+d(y, \ z)} \\ &\geq \min\{\frac{t}{t+d(x,y)}, \frac{s}{s+d(y,z)}\} \\ &\geq M(x,y,t) * M(y,z,s) \end{aligned}$$

Thus M satisfies (D2). So $(X, \ M, \ *)$ is a dislocated fuzzy quasi-metric space. $\square$

Definition 3.2 Let $(X, \ M, \ *)$ be a d.f.q.m.s.

(i) A sequence $\{x_n\}$ is said to be a Cauchy sequence if for any $t, \ \epsilon > 0, \ \exists \ n_0(t, \epsilon) \in \ N$ such that

$$M(x_m, \ x_n, \ t) > 1 - \epsilon, \ \ \forall m, \ n \geq \ n_0(t, \epsilon)$$

or

$$M(x_n, \ x_m, \ t) > 1 - \epsilon, \ \ \forall m, \ n \geq \ n_0(t, \epsilon)$$

i.e.

$$\max\{M(x_m, \ x_n, \ t), \ M(x_n, \ x_m, \ t)\} > 1 - \epsilon,$$

for all $m, \ n \geq \ n_0(t, \epsilon)$. That is,

$$\max\{\lim_{m,n\to\infty} M(x_m, \ x_n, \ t), \ \lim_{m,n\to\infty} M(x_n, \ x_m, \ t)\} = 1$$

for all $t > 0$.

In the above definition if we replace $M(x_m, \ x_n, \ t) > 1 - \epsilon$ or $M(x_n, \ x_m, \ t) > 1 - \epsilon$ by $\min\{M(x_m, \ x_n, \ t), \ M(x_n, \ x_m, \ t)\} > 1 - \epsilon$, the sequence $\{x_n\}$ is called 'bi' Cauchy. Note that every 'by' Cauchy sequence is Cauchy.

(ii) A sequence $\{x_n\}$ is said to converge to $x \in X$ if for any $t, \ \epsilon > 0, \ \exists \ n_0(t, \epsilon) \in \ N$ such that $M(x_n, \ x, \ t) > 1 - \epsilon$ and $M(x, \ x_n, \ t) > 1 - \epsilon$ for all $m, \ n \geq \ n_0(t, \epsilon)$

i.e. $\min\{M(x_n, \ x, \ t), \ M(x, \ x_n, \ t)\} > 1 - \epsilon$ for all $m, \ n \geq \ n_0(t, \epsilon)$

i.e.

$$\min\{\lim_{n\to\infty} M(x_n, \ x, \ t), \ \lim_{n\to\infty} M(x, \ x_n, \ t)\} = 1$$

for all $t > 0$.

In this case, x is called the limit of $\{x_n\}$ and we write $\lim_{n\to\infty} \ x_n = x$.

(iii) A subset $B \subset \ X$ is said to be complete if every Cauchy sequence in B converges in B. $\square$

Note 3.2 In a d.f.q.m.s $(X, \ M, \ *)$

(a) Every convergent sequence is a 'bi' Cauchy sequence but the converse is not true.

(b) The limit of a sequence if exists is unique.

(c) If $\{x_n\}$ is a convergent sequence which converges to x in X, then any subsequence $\{x_{n_k}\}$ also converges to x.

(d) Every subsequence of a Cauchy sequence is also a Cauchy sequence.

Definition 3.3 Let $(X, \ M_1, \ *)$ and $(Y, \ M_2, \ *)$ be two d.f.q.m.s and $f \ : \ X \to \ Y$ be a function. Then f is said to be continuous at $x \in \ X$, if for any sequence $\{x_n\}$ of X with $\lim_{n\to\infty} \ x_n = x$ implies $\lim_{n\to\infty} \ f(x_n) = f(x)$.

If f is continuous at every point in X, then it is called continuous function. $\square$

IV. Fixed Point Theorem

In this section we define a contraction function and prove Banach type fixed point theorem in fuzzy setting.

Definition 4.1 Let $(X, \ M, \ *)$ be a d.f.q.m.s and $T \ : \ X \to \ X$ be a function. Then T is called contraction if there exists $\lambda > 1$ such that

$$\min\{M(Tx, \ y, \ t), \ M(x, \ Ty, \ t)\} \geq \ M(x, \ y, \ \lambda \ t)$$

for all $x, y \in \ X, \ \forall \ t > 0$. $\square$

Lemma 4.1 Let $(X, \ M, \ *)$ be a d.f.q.m.s and $T \ : \ X \to \ X$ be a contraction . Assume that

$$\textbf{(D3)} \ \lim_{t\to\infty} M(x, \ y, \ t) = 1 \ \ \forall \ x, \ y \in \ X$$

Then for any $x_0 \in \ X, \ \ \{T^n(x_0)\}$ is a 'bi' Cauchy sequence in X.

Proof Let $(X,\ M,\ *)$ be a d.f.q.m.s and $T\ :\ X \to\ X$ be a contraction.

Let $x_0 \in\ X$ and $t > 0$. Define $x_k = T(x_{k-1})$ for all k. Then,

$$x_1 = T(x_0), \ldots\ldots\ldots, x_n = T(x_{n-1}) = T^n(x_0)$$

Then for any $x \in\ X$,

$$M(x_m,\ x_n,\ t) = M(T^m(x_0),\ T^n(x_0),\ t)$$

i.e.

$$M(x_m,\ x_n,\ t) \geq\ M(x_m,\ x,\ \frac{t}{2}) \star M(x,\ x_n,\ \frac{t}{2})$$

i.e.

$$M(x_m,\ x_n,\ t) \geq M(x_0,\ x,\ \lambda^m\ \frac{t}{2}) \star M(x,\ x_0,\ \lambda^n\ \frac{t}{2})$$

for some $\lambda\ > 1$

Thus,

$$\begin{aligned} \lim_{m,n\to\infty} M(x_m,\ x_n,\ t) &\geq \lim_{m\to\infty} M(x_0,\ x,\ \lambda^m\ \frac{t}{2}) \\ &\quad \star \lim_{n\to\infty} M(x,\ x_0,\ \lambda^n\ \frac{t}{2}) \\ &= 1 \end{aligned}$$

Thus,

$$\lim_{m,n\to\infty} M(x_m,\ x_n,\ t) = 1, \text{[by (D3)]}$$

Similarly we can be prove that

$$\lim_{m,n\to\infty} M(x_n,\ x_m,\ t) = 1$$

Hence $\{T^n(x_0)\}$ is a 'bi' Cauchy sequence in X. □

Theorem 4.1 Let $(X,\ M,\ *)$ be a complete d.f.q.m.s satisfying (D3) and $T\ :\ X \to\ X$ be a continuous contraction. Then T has a unique fixed point in X.

Proof Let $x_0 \in\ X$ and $t > 0$. Define $x_k = T(x_{k-1})$ for all k.

Then,

$$x_1 = T(x_0), \ldots\ldots\ldots, x_n = T(x_{n-1}) = T^n(x_0)$$

Then by Lemma 4.1, $\{x_n\}$ is a 'by'-Cauchy sequence in X.

So it is a Cauchy sequence in X. Since $(X,\ M,\ *)$ is complete, there exists $x \in\ X$ such that $\lim_{n\to\infty} x_n = x$.

Since T is continuous, $\lim_{n\to\infty} T(x_n) = Tx$.

Now $Tx = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x$.

Hence x is a fixed point of T.

Uniqueness: Let $x,\ y\ \in\ X$ be any two fixed points of T. Then for any $t > 0,\ n \in\ N,\ z \in\ X$

$$\begin{aligned} M(x,y,t) &= M(T^n x, T^n y, t) \\ &\geq M(T^n x, z, \tfrac{t}{2}) * M(z, T^n y, \tfrac{t}{2}) \\ &\geq M(x, z, \lambda^n \tfrac{t}{2}) * M(z, y, \lambda^n \tfrac{t}{2}) \end{aligned}$$

for some $\lambda\ > 1$. This implies that

$$\begin{aligned} M(x,y,t) &\geq \lim_{n\to\infty} M(x,z,\lambda^n \frac{t}{2}) * \lim_{n\to\infty} M(y,z,\lambda^n \frac{t}{2}) \\ &= 1 * 1 = 1 \end{aligned}$$

Similarly, $M(y,\ x,\ t) = 1\ \ \forall\ t > 0$.

Hence $x = y$. □

CONCLUSION

In this paper, the idea of dislocated fuzzy quasi-metric space (d.f.q.m.s) is introduced and some basic properties are studied. Banach type fixed point theorem is established in such spaces with uniqueness. There is a wide scope to study fuzzy metric spaces in this setting.

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