

Numerical Solution of Third Order Singularly Perturbed ODE of Convection-Diffusion Type using Spline Collocation

J. U. Pandya¹ and H. D. Doctor²

Abstract—Two point boundary value problems for third order singularly perturbed ordinary differential equations in which the highest order derivative is multiplied by a small parameter are solved by the quartic spline collocation method. First third order linear equation is solved and then semi-linear equation is solved. The quasi-linearization technique is used to solve semi-linear equation. The method is tested on examples and results are compared with the available results obtained by Boundary Value Technique.

Index Terms— Quartic spline collocation, singular perturbation, third order ODE, quasilinearization technique.

MSC 2010 Codes — 65L10, 65L11.

I. INTRODUCTION

SINGULAR perturbation problems appear in many branches of applied mathematics, and an extensive research has been made on numerical methods for more than two decades. Quite a good amount of research work is reported on the qualitative and quantitative analysis for singularly perturbed for ordinary differential equations (ODEs)[1,2,3,4,5].

Most of the papers concerning computational aspects are confined to second order equations. Only a few results are available for higher order equations. Singularly perturbed higher-order problems are classified on the basis of how the order of the original differential equation is affected if one sets $\mathcal{E} = 0$ by Roos, Stynes & Tobiska [6], where \mathcal{E} is a small positive parameter assigned with order of the differential equation.

We say that if the order of the differential equation is reduced by one, then the Singular Perturbation Problem (SPP) is said to be of convection diffusion type and if the order is reduced by two, then it is called reaction diffusion type. There are a variety of techniques for solving singularly perturbed boundary value problems. O'Malley [7], Howes [8], and Zhao [9] derived analytical results of third order nonlinear SPPs.

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Feckan [10] considered higher order problems and his approach is based on the nonlinear analysis involving fixed-point theory, Leray-Schauder theory, etc.

A FEM for convection-reaction type problems is described by Sun & Stynes[11,12] and Semper [13]. Roberts [14] suggested a numerical method of finding an approximate solution for third order ODEs. He introduced a technique called Boundary Value Technique (BVT).

In Section II, we suggest spline collocation method to obtain solutions of the Boundary Value Problems (BVPs) of higher order SPODEs. Aziz & Arshad [15] has solved second order singularly perturbed boundary value problem using cubic spline method. The boundary value technique to find numerical solution for third order SPODEs subject to certain types of boundary conditions is suggested by Valarmathi & Ramanujam [16]. In the present paper, we solve the third order singularly perturbed boundary value problem using quartic spline collocation method.

II. SINGULARLY PERTURBED THIRD ORDER ORDINARY DIFFERENTIAL EQUATION

We consider the following problem

$$-\varepsilon y'''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x), \quad x \in D \quad (1)$$

$$y(0) = p, \quad y'(0) = q, \quad y'(1) = r \quad (2)$$

where $\varepsilon > 0$ is a small parameter, $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the following conditions:

$$a(x) \geq -\alpha, \quad \alpha > 0$$

$$b(x) \geq 0$$

$$0 \geq c(x) \geq -\gamma, \quad \gamma > 0$$

$$\delta = \alpha - \gamma(1 + 3\eta) \geq \eta' > 0 \text{ for some } \eta \text{ and } \eta' \text{ with}$$

$$D = (0, 1), \quad D_o = [0, 1], \quad \bar{D} = [0, 1]$$

$$\text{and } y \in c^{(3)}(D) \cap c^{(1)}(\bar{D})$$

III. QUARTIC SPLINE ALLOCATION METHOD

The fourth degree spline is used to find numerical solutions to the boundary value problems discussed in equation (1) together with equation (2). A detailed description of spline functions generated by subdivision is given by De Boor [17], Micula & Micula[18]. Consider equally spaced knots of a partition π : $a = x_0 < x_1 < x_2 < \dots < x_n = b$ on $[a, b]$. Let $S_4[\pi]$ be the space of continuously differentiable, piecewise, Quartic polynomials on π . That is, $S_4[\pi]$ is the space of Quartic polynomials on π . The Quartic spline is given by Bickley [19].

$$s(x) = a_0 + b_0(x - x_0) + \frac{1}{2}c_0(x - x_0)^2 + \frac{1}{6}d_0(x - x_0)^3 + \frac{1}{24} \sum_{k=0}^{n-1} e_k(x - x_k)_+^4 \tag{3}$$

where the power function $(x - x_k)_+$ defined as

$$(x - x_k)_+ = \begin{cases} x - x_k, & \text{if } x > x_k \\ 0, & \text{if } x \leq x_k \end{cases}$$

Consider a third order linear boundary value problem of the form

$$y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) = m(x), \quad a \leq x \leq b \tag{4}$$

with the boundary conditions

$$\left. \begin{aligned} \alpha_0 y_0 + \beta_0 y_n' + \gamma_0 y_n'' &= \delta_0 \\ \alpha_1 y_0' + \beta_1 y_n + \gamma_1 y_n'' &= \delta_1 \\ \alpha_2 y_0'' + \beta_2 y_n + \gamma_2 y_n' &= \delta_2 \end{aligned} \right\} \tag{5}$$

where $y(x), p(x), q(x), r(x), m(x)$ are continuous functions defined in the interval $x \in [a, b]$; $\delta_0, \delta_1, \delta_2$ are finite real constants.

Let equation (3) be an approximate solution of equation (4), where $a_0, b_0, c_0, d_0, e_0, e_1, \dots, e_{n-1}$ are real coefficients to be determined. Let x_0, x_1, \dots, x_n be $n+1$ grid points in the interval $[a, b]$, so that

$$x_i = a + ih, \quad i=0, 1, \dots, n; \quad x_0 = a, \quad x_n = b \quad \& \quad h = (b-a)/n \tag{6}$$

It is required that the approximate solution (3) satisfies the differential equation at the point $x=x_i$. Putting (3) together with its successive derivatives in (4), we obtain the collocation equations as follows:

$$\left. \begin{aligned} \sum_{k=0}^{n-1} e_k \left\{ (x_i - x_k)_+ + \frac{1}{2} p(x_i)(x_i - x_k)_+^2 + \frac{1}{6} q(x_i)(x_i - x_k)_+^3 \right\} \\ + \frac{1}{24} r(x_i)(x_i - x_k)_+^4 \end{aligned} \right\} + d_0 \left\{ 1 + p(x_i)(x_i - x_0) + \frac{1}{2} q(x_i)(x_i - x_0)^2 + \frac{1}{6} r(x_i)(x_i - x_0)^3 \right\} + c_0 \left\{ p(x_i) + q(x_i)(x_i - x_0) + \frac{1}{2} r(x_i)(x_i - x_0)^2 \right\} + b_0 \{ p(x_i) + r(x_i)(x_i - x_0) \} + a_0 \{ r(x_i) \} = m(x_i), \quad \text{where } i = 0, 1, 2, \dots \tag{7}$$

From boundary conditions, we get

$$\sum_{k=0}^{n-1} e_k \left(\frac{\beta_0}{6} (b - x_k)_+^3 + \frac{\gamma_0}{2} (b - x_k)_+^2 \right) + d_0 \left(\frac{\beta_0}{2} (b - a)^2 + \gamma_0 (b - a) \right) + c_0 (\beta_0 (b - a) + \gamma_0) + b_0 (\beta_0) + a_0 (\alpha_0) = \delta_0 \tag{8}$$

$$\sum_{k=0}^{n-1} e_k \left(\frac{\beta_1}{24} (b - x_k)_+^4 + \frac{\gamma_1}{2} (b - x_k)_+^2 \right) + d_0 \left(\frac{\beta_1}{6} (b - a)^3 + \gamma_1 (b - a) \right) + c_0 \left(\frac{\beta_1}{2} (b - a)^2 + \gamma_1 \right) + b_0 (\beta_1 (b - a) + \alpha_1) + a_0 (\beta_1) = \delta_1 \tag{9}$$

and

$$\sum_{k=0}^{n-1} e_k \left(\frac{\beta_2}{24} (b - x_k)_+^4 + \frac{\gamma_2}{6} (b - x_k)_+^3 \right) + d_0 \left(\frac{\beta_2}{6} (b - a)^3 + \frac{\gamma_2}{2} (b - a)^2 \right) + c_0 \left(\frac{\beta_2}{2} (b - a)^2 + \gamma_2 (b - a) + \alpha_2 \right) + b_0 (\beta_2 (b - a) + \gamma_2) + a_0 (\beta_2) = \delta_2 \tag{10}$$

Using the power function $(x - x_k)_+$ in the above equations a system of $n+4$ linear equations in $n+4$ unknowns $a_0, b_0, c_0, d_0, e_0, e_1, \dots, e_{n-1}$ is thus obtained. This system can be written in matrix-vector form as follows

$$AX = B \tag{11}$$

where

$$X = [e_{n-1}, e_{n-2}, \dots, e_2, e_1, e_0, d_0, c_0, b_0, a_0]^T$$

$$B = [\delta_2, \delta_1, \delta_0, m(x_n), m(x_{n-1}), \dots, m(x_1), m(x_0)]^T$$

The coefficient matrix A is an upper triangular Hessenberg matrix with a single lower sub diagonal, principal and upper diagonal having non-zero elements. Because of this nature of matrix A, the determination of the required quantities becomes simple and consumes less time. The values of these constants ultimately yield the Quartic spline $s(x)$ in equation (3).

In case of nonlinear boundary value problem, the equations can be converted into linear form by quasilinearization method [Bellman et al (20)] and hence this method can be used as iterative method. The procedure to obtain a spline approximation of y_i ($i=0, 1, 2, \dots, j$; where j denotes the number of iteration) by an iterative method starts with fitting a curve satisfying the end conditions and this curve is designated as y_i . We obtain the successive iterations y_i 's with the help of an algorithm described as above till desired accuracy.

IV. QUARTIC SPLINE SOLUTION FOR THE THIRD ORDER SINGULARLY PERTURBED ODE

Consider the boundary value problem

$$\begin{aligned}
 &-\varepsilon y'''(x) - 2y''(x) + 4y'(x) - y(x) = \\
 &4 \left[1 + \left(1 - e^{-2x/\varepsilon} \right) / 2 \left(1 - e^{-2/\varepsilon} \right) - \frac{x}{2} \right] \\
 &+ \varepsilon \left(1 - e^{-2x/\varepsilon} \right) / 4 \left(1 - e^{-2/\varepsilon} \right) + \frac{x^2}{4} - \\
 &x \left(1 + \frac{1}{2 \left(1 - e^{-2/\varepsilon} \right)} \right) \tag{12}
 \end{aligned}$$

$$y(0) = 1, y'(0)=1, y'(1) = 1$$

where

Let equation (3) be an approximate solution of above equation (12), where real constants are to be determined. It is required that the approximate solution (3) satisfies the differential equation at the point $x=x_i$. Putting (3) in (12), we obtain the collocation equations as follows:

$$\begin{aligned}
 &\sum_{k=0}^{n-1} e_k \left\{ \frac{-\varepsilon (x_i - x_k)_+ - (x_i - x_k)_+^2}{3} + \frac{2}{3} (x_i - x_k)_+^3 - \frac{1}{24} (x_i - x_k)_+^4 \right\} \\
 &+ d_0 \left\{ \frac{-\varepsilon - 2(x_i - x_0) + 2(x_i - x_0)^2}{6} - \frac{1}{6} (x_i - x_0)^3 \right\} \\
 &+ c_0 \left\{ -2p(x_i) + 4(x_i - x_0) - \frac{1}{2} (x_i - x_0)^2 \right\} \\
 &+ b_0 \{ 4 - (x_i - x_0) \} + a_0 \{ -1 \} = f(x_i) \tag{13}
 \end{aligned}$$

where

$$\begin{aligned}
 f(x_i) &= 4 \left[1 + \left(1 - e^{-2x_i/\varepsilon} \right) / 2 \left(1 - e^{-2/\varepsilon} \right) - x_i / 2 \right] \\
 &+ \varepsilon \left(1 - e^{-2x_i/\varepsilon} \right) / 4 \left(1 - e^{-2/\varepsilon} \right) + x_i^2 / 4 - \\
 &x_i \left(1 + 1/2 \left(1 - e^{-2/\varepsilon} \right) \right)
 \end{aligned}$$

From the boundary conditions we get

$$a_0 = 1 \tag{14}$$

$$b_0 = 1 \tag{15}$$

$$\frac{1}{6} \sum_{k=0}^{n-1} e_k (1 - x_k)_+^3 + \frac{1}{2} d_0 + c_0 = 0 \tag{16}$$

Solving above system of equations and proceeding as explained in section-3, we obtain the results of equation (12) using the quartic spline collocation method. These results are compared with the available results obtained by Valarmathi & Ramanujam [16] through Boundary Value Technique (BVT). Table-1 represents above said. Results are also given in Figure-1.

TABLE I
NUMERICAL SOLUTIONS OF THE BVP FOR THE THIRD ORDER SPODE
($h = 0.1, \varepsilon = 0.0001$)

x	y(x)	
	Quartic spline Solution	Solution by BVT
0.0	1.0000000	1.0000000
0.1	1.1486024	1.150154
0.2	1.3001891	1.292340
0.3	1.4414960	1.429561
0.4	1.5853251	1.561814
0.5	1.7202127	1.689101
0.6	1.8574433	1.811420
0.7	1.9871635	1.928770
0.8	2.1193909	2.041150
0.9	2.2457522	2.148559
1.0	2.3752337	2.250994

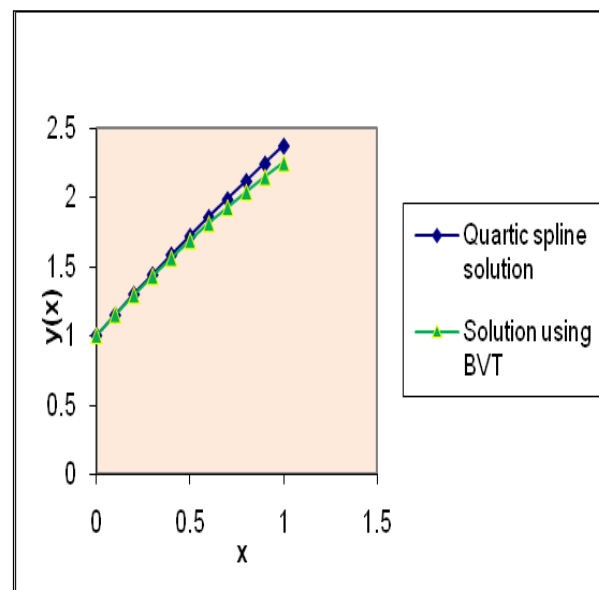


Fig. 1. Numerical Solution of the third order SPODE

V. BOUNDARY VALUE PROBLEMS FOR THE THIRD ORDER NONLINEAR SPODE

Consider the nonlinear BVP

$$\varepsilon y'''(x) = F(x, y, y', y''), \quad x \in D, \tag{17}$$

$$y(0) = p, \quad y'(0) = q, \quad y'(1) = r \tag{18}$$

where $F(x, y, y', y'')$ is a smooth function such that

$$F_y''(x, y, y', y'') \leq -\alpha, \quad \alpha > 0,$$

$$F_{y'}(x, y, y', y'') \geq 0,$$

$$0 \geq F_{y''}(x, y, y', y'') \geq -\gamma, \quad \gamma > 0,$$

$$\delta = \alpha - \gamma(1 + 3\eta) \geq \eta' > 0 \quad \text{for some } \eta \text{ and } \eta'$$

In order to obtain a numerical solution of (17), with boundary conditions (18), Newton's method of quasilinearization is applied. Consequently, we get a sequence $\{y^{[m]}\}_0^\infty$ of successive approximations with a proper choice of initial guess $y^{[0]}$. Then define $y^{[m+1]}$ for each fixed non-negative integer m , to be the solution of the following linear problem:

$$-\varepsilon (Y''')^{[m+1]} + a^m(x)(y'')^{[m+1]} + b^m(x)(y')^{[m+1]} + c^m(x)(y)^{[m+1]} = F^m(x)$$

where

$$a^m(x) = F_{y''}(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}),$$

$$b^m(x) = F_{y'}(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}),$$

$$c^m(x) = F_y(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}),$$

$$F^m(x) = F(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}) -$$

$$y^{[m]} F_y(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]})$$

$$- (y')^{[m]} F_{y'}(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]})$$

$$- (y'')^{[m]} F_{y''}(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]})$$

$$y^{[m+1]}(0) = p, \quad (y')^{[m+1]}(0) = q, \quad (y'')^{[m+1]}(1) = r$$

VI. QUATIC SPLINE SOLUTION OF THIRD ORDER NONLINEAR SPODE

Consider the nonlinear boundary value problem

$$-\varepsilon y'''(x) - 2y''(x) + 4y'(x) - y^2(x) = 1 + 4 \left[1 - \frac{x}{2} + \left(1 - e^{-2x/\varepsilon} \right) / 2 \left(1 - e^{-2/\varepsilon} \right) \right] \left[1 - \varepsilon \left(1 - e^{-2x/\varepsilon} \right) / 4 \left(1 - e^{-2/\varepsilon} \right) - \frac{x^2}{4} + x \left(1 + \frac{1}{2 \left(1 - e^{-2/\varepsilon} \right)} \right) \right] \tag{19}$$

$$y(0) = 1, \quad y'(0) = 1, \quad y'(1) = 1$$

The above nonlinear problem is reduced to a linear one by applying quasilinearization technique as discussed in section-5. A linear form of the problem is given by

$$-\varepsilon (y''')^{[n+1]} - 2(y'')^{[n+1]} + 4(y')^{[n+1]} - 2(y)^{[n]}(y')^{[n+1]} = f^n(x) - (y^{[n]})^2$$

The curve $y = x + 1$ satisfies the boundary conditions of the problem from which initial values of y are obtained.

Table 2 shows the progress of iterative scheme obtained by using the quartic spline collocation method in form of successive iterations. The iterative progress is not continued further as the results agree well up to three decimal places. The obtained results are compared with the results obtained by Valarmathi & Ramanujam [16].

TABLE II
QUARTIC SPLINE SOLUTION OF THE THIRD ORDER NONLINEAR SPODE
($h = 0.1, \varepsilon = 0.0001$)

X	y(x)				Solution by BVT
	i=0	i=1	i=2	i=3	
0.0	1.0000000	X	1.0000000	1.0000000	1.0000000
0.1	1.1000000	1.1489649	1.1504450	1.1504483	1.150140
0.2	1.2000000	1.3016267	1.3050836	1.3050914	1.292318
0.3	1.3000000	1.4444965	1.4500703	1.4500828	1.429536
0.4	1.4000000	1.5898698	1.5980006	1.5980182	1.561794
0.5	1.5000000	1.7267282	1.7374762	1.7374983	1.689093
0.6	1.6000000	1.8658398	1.8794485	1.8794746	1.811431
0.7	1.7000000	1.9976165	2.0139053	2.0139389	1.928804
0.8	1.8000000	2.1314347	2.1502820	2.1503220	2.041208
0.9	1.9000000	2.2588782	2.2796753	2.2797318	2.148636
1.0	2.0000000	2.3880161	2.4100813	2.4101028	2.251081

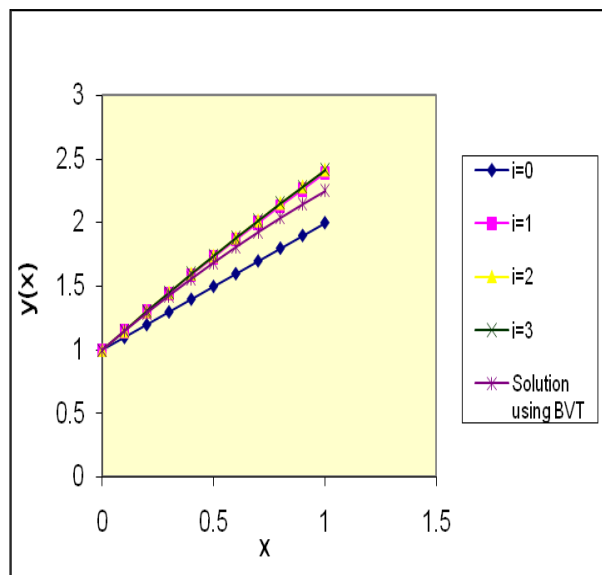


Fig. 2 Numerical Solution of the non-linear third order SPODE

VII. CONCLUSION

In this paper, we have implemented the quartic spline collocation method to solve such type of singularly perturbed boundary value problems. The cubic spline collocation method is applied by the authors to solve second order singularly perturbed boundary value problems. This work is an extension of this method from second-order to third order boundary value problems. The method presented here is easy to apply for third order singularly perturbed ordinary differential equations. The nonlinear boundary value problems can be solved by linearizing them by the Newton's method of quasi linearization as done. The problems are also solved by the Finite difference method. Sufficient accuracy in the approximate results is regarded as the agreement of the spline solutions with the available solutions. More accurate results can be obtained by selecting proper size for subinterval of the domain associated with the problem.

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