# Group Invariance for Non-Linear PDE's: 3-D MHD Stagnation Point Flow of Non-Newtonian Power-Law Fluids

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***Abstract*— A powerful group theoretic technique is applied to transform non linear PDE of steady three-dimensional MHD laminar boundary layer stagnation point flow of non-Newtonian fluid into non-linear ODE. Two different group transformations (Linear and Spiral) are discussed and applied to find similarity equations of above mentioned flow problem. Numerical solution of the problem is obtained and presented graphically, for both Newtonian and non-Newtonian fluids.**

***Index Terms*— boundary layer, Dilatant, Group invariance, stagnation point, MHD, Power-Law fluid, Pseudo plastic.**

***MSC 2010 Codes* —76W05, 76D10, 76M99.**

## NOMENCLATURE

$u$, v, w- Velocity components in X, Y, Z directions respectively
U, W- Main stream velocities in X and Z directions
$\overline{\tau}$ - stress component
$\overline{\Delta}$ - Strain rate component
$\tau_{yx}$ - stress tensor in the direction of X-axis perpendicular to Y-axis.
$\tau_{yz}$ - stress tensor in the direction of Z-axis perpendicular to Y-axis.
m- Physical constant,
n- Flow behaviour indices
MHD- Magneto hydro dynamics
S – Magnetic parameter
Re - Reynolds number
$\rho$ -field density
$\psi$ - Stream function
$\eta$ - Similarity variable
$\xi$ - Unknown function of x
F,G- Similarity Functions

## I. INTRODUCTION

In past couple of decades, there have been numerous attempts made in applying boundary layer theory to non-Newtonian fluids. The theory makes great simplification in equation of motion and as a consequence, the equations are much simple to solve. For various non-Newtonian fluid models the progress in such type of simplifications is bit slowly. Most work found in literature on the said topic is restricted to simple boundary layer flow of non-Newtonian power law fluids. Hansen and his co-worker [1, 2, 3] are rather first to derive the systematic similarity analysis for three-dimensional boundary layer flow of Newtonian fluids. Further they [4] have extended their work to non-Newtonian power- law fluids and have obtained class of similarity solutions including similarity solutions for small cross flow.

Recently Manisha et al [5] have derived family of similarity equations for three-dimensional unsteady boundary layer flow of non-Newtonian power-law fluids using free-parameter technique. They have derived seven sets of similarity equations for such flow, past various geometries with their possible practical applications. In another published paper [6] they have also derived the similarity equations for steady three-dimensional boundary layer flow of all viscoinelastic non-Newtonian fluids by new similarity formulism, namely, generalized dimensional analysis with group theory and matrix aspects.

Quite rare information is available in the literature about three dimensional magneto hydrodynamic boundary layer flows. This is because the governing partial differential equations for such flow are non-linear boundary value type and the presence of non-Newtonianality and the transverse magnetic field parameter poses extra difficulties while simplifying such flow equations. Timol and Timol [7] are probably first to drive basic equations and similarity solutions of three-dimensional magneto hydrodynamic boundary layer flow of Newtonian fluids. But in order to meet with similarity requirements they have assumed the specific form of outer flow and specific form of imposed magnetic field parameter in *priory* and hence sets of similarity equations they have derived are of limited practical applications.

In the present paper using group theoretic method, the systematic similarity method is proposed to find similarity equations for steady three dimensional incompressible boundary layer flow of electrically conducting non-Newtonian power-law fluids past external surface under the influence of transverse magnetic field. It is observed that flow situation under consideration is independent of z-coordinate. And hence it is essentially quasi three-dimensional flow. Such flows are

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characterized by the fact that streamline form a system of translates i.e. entire streamline pattern can be obtained by translating any particular streamline parallel to leading edge of the surface. It is hoped that by omitting dependence of flow quantities in one direction small qualitative information may be obtained on the characteristic of three-dimensional boundary layer flows of power-law fluids. It is observed that for some special cases present set of equations are well reduced to past well-known equations like Blasisus equation, Falkner, Skan equations etc..

## II. BASIC EQUATIONS

The power-law Ostwald-de Waele model has been found to be remarkably versatile and useful in representing flow behavior of many non-Newtonian fluids over quite a wide range of shearing rate. Mathematically it can be represented in the form [8]

$$\bar{\bar{\tau}} = -\left\{ m \left| \sqrt{\frac{1}{2}\bar{\bar{\Delta}}:\bar{\bar{\Delta}}} \right|^{n-1} \right\} \bar{\bar{\Delta}}$$

where $\bar{\bar{\tau}}$ and $\bar{\bar{\Delta}}$ are the stress tensor and the rate of deformation tensor, respectively; and $m$ and $n$ are physical constants different for different fluids which can be determined experimentally. Under the boundary layer assumptions, the only two non-vanishing components are [8]

$$\tau_{yx} = -m\left\{ \left[ \left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2 \right]^{\frac{n-1}{2}} \frac{\partial u'}{\partial y'} \right\}$$

$$\tau_{yz} = -m\left\{ \left[ \left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2 \right]^{\frac{n-1}{2}} \frac{\partial w'}{\partial y'} \right\}$$

Where the absolute sign has been dropped since both terms within the sign are positive. Following Manisha et al [5], for above "equation of state", the dimensionless equations governing the motion of three dimensional laminar incompressible magneto hydrodynamic boundary layer flow of non-Newtonian power-law fluids can be written as follow:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{ \left[ \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right]^{\frac{n-1}{2}} \frac{\partial u}{\partial y} \right\} + U\frac{dU}{dx} + S(x)\left(U - u\right) \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{ \left[ \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right]^{\frac{n-1}{2}} \frac{\partial w}{\partial y} \right\} + U\frac{dW}{dx} + S(x)\left(W - w\right) \tag{3}$$

with the boundary conditions

$$y = 0: \quad u = v = w = 0$$

$$y = \infty: \quad u = U(x), \quad w = W(x)$$

where the non-dimensional quantities used are,

$$u = \frac{u'}{U_0}, \quad v = \frac{v'}{U_0}\mathrm{Re}^{\frac{1}{n+1}}, \quad w = \frac{w'}{U_0}, \quad U = \frac{U'}{U_0}$$

$$W = \frac{W'}{U_0}, \quad x = \frac{x'}{L}, \quad y = \frac{y'}{L}\mathrm{Re}^{\frac{1}{n+1}}, \quad S(x) = \frac{L\,S'}{U_0}$$

Where $\mathrm{Re} = \frac{\rho\, U_0^{2-n}\, L^n}{m}$.

The flow problem is quasi-two-dimensional in nature since the velocity components are independent of the z-coordinates. This point is discussed in detail by Hansen and Herzig [9]. The equation of continuity can be satisfied identically by introducing a function, $\psi$, which gives

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

Equations (1), (2), and (3) then become

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left\{ \left[ \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right]^{\frac{n-1}{2}} \frac{\partial^2 \psi}{\partial y^2} \right\} + U\frac{dU}{dx} + S(x)\left(U - \frac{\partial \psi}{\partial y}\right) \tag{4}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left\{ \left[ \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right]^{\frac{n-1}{2}} \frac{\partial w}{\partial y} \right\} + U\frac{dW}{dx} + S(x)\left(W - w\right) \tag{5}$$

With the boundary conditions

$$y = 0: \qquad \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = \mathrm{w} = 0$$

$$\mathrm{y} = \infty: \qquad \frac{\partial \psi}{\partial y} = \mathrm{U(x)}, \quad \mathrm{w} = \mathrm{W(x)}$$

A group-theoretic analysis is employed in the next section to find the form of U(x) and W(x) for which similarity solutions will exist.

## III. GROUP-THEORETIC ANALYSIS

Similarity analysis by the group-theoretic method is based on concepts derived from the theory of transformation group. This method was first introduced by Birkoff [10] and Morgan [11] and is discussed in detail in [12] & [13]. Two different groups of one-parameter transformation are usually found to give adequate treatment of boundary layer equations. Each group gives rise two cases, case-I and case-II which will be separately discussed.

**Case I.** A one-parameter linear group of transformation is selected as

$$\mathrm{x} = e^{\alpha_1 \varepsilon} x^*, \qquad \mathrm{y} = e^{\alpha_2 \varepsilon} y^*, \qquad \psi = e^{\alpha_3 \varepsilon} \psi^*$$

$$\mathrm{w} = e^{\alpha_4 \varepsilon} w^*, \qquad \mathrm{U} = e^{\alpha_5 \varepsilon} U^*, \qquad \mathrm{W} = e^{\alpha_6 \varepsilon} W^*, \qquad \mathrm{S} = e^{\alpha_7 \varepsilon} S^*$$

Where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ and e are constants.

We now seek relations among α's such that the basic equations will be invariant under this group of transformation.

Substituting the above transformation in equation (4) and (5) we get,

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial \mathrm{y}^2} = \frac{\partial}{\partial \mathrm{y}}\left\{\left[\left(\frac{\partial^2 \psi}{\partial \mathrm{y}^2}\right)^2 + \left(\frac{\partial w}{\partial \mathrm{y}}\right)^2\right]^{\frac{n-1}{2}} \frac{\partial^2 \psi}{\partial \mathrm{y}^2}\right\} + \mathrm{U}\frac{dU}{dx} + \mathrm{S(x)}\left(\mathrm{U} - \frac{\partial \psi}{\partial y}\right)$$

$$e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial \psi^*}{\partial y^*} e^{(\alpha_3-\alpha_2-\alpha_1)\varepsilon}\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - e^{(\alpha_3-\alpha_1)\varepsilon}\frac{\partial \psi^*}{\partial x^*} e^{(\alpha_3-2\alpha_2)\varepsilon}\frac{\partial^2 \psi^*}{\partial y^{*2}} = \frac{\partial}{\partial y^*}\frac{\partial y^*}{\partial y} + e^{\alpha_5} U^* e^{(\alpha_5-\alpha_1)\varepsilon}\frac{dU^*}{dx^*} + e^{\alpha_7} S^*\left(e^{\alpha_5}U^* - e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial \psi^*}{\partial y^*}\right)$$

$$\Rightarrow e^{(2\alpha_3-2\alpha_2-\alpha_1)\varepsilon}\left[\frac{\partial \psi^*}{\partial y^*}\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial^2 \psi^*}{\partial y^{*2}}\right] =$$

$$\frac{\partial}{\partial y^*}\left\{\begin{array}{l}\left[e^{(\alpha_3(n-1)-2\alpha_2(n-1))\varepsilon}\left(\frac{\partial^2 \psi^*}{\partial y^{*2}}\right)^2 + e^{(\alpha_4(n-1)-\alpha_2(n-1))\varepsilon}\left(\frac{\partial w^*}{\partial y^*}\right)^2\right]^{\frac{n-1}{2}} \\ .\; e^{(\alpha_3-2\alpha_2)\varepsilon}\frac{\partial^2 \psi^*}{\partial y^{*2}}\end{array}\right\}$$

$$e^{-\alpha_2 \varepsilon} + e^{(2\alpha_5-\alpha_1)\varepsilon}U^*\frac{dU^*}{dx^*} + S^*\left(e^{(\alpha_7+\alpha_5)\varepsilon}U^* - e^{(\alpha_7+\alpha_3-\alpha_2)\varepsilon}\frac{\partial \psi^*}{\partial y^*}\right)$$

$$+e^{(2\alpha_5-\alpha_1)\varepsilon}U^*\frac{dU^*}{dx^*} + S^*\left(e^{(\alpha_7+\alpha_5)\varepsilon}U^* - e^{(\alpha_7+\alpha_3-\alpha_2)\varepsilon}\frac{\partial \psi^*}{\partial y^*}\right)$$

$$\Rightarrow e^{(2\alpha_3-2\alpha_2-\alpha_1)\varepsilon}\left[\frac{\partial \psi^*}{\partial y^*}\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial^2 \psi^*}{\partial y^{*2}}\right] =$$

$$\frac{\partial}{\partial y^*}\left\{\begin{array}{l}e^{(n\alpha_3-(2n+1)\alpha_2)\varepsilon}\left[\left(\frac{\partial^2 \psi^*}{\partial y^{*2}}\right)^2 + \left(\frac{\partial w^*}{\partial y^*}\right)^2\right]^{\frac{n-1}{2}} \\ .\, e^{(\alpha_4(n-1)-\alpha_2(n+2)+\alpha_3)\varepsilon}\frac{\partial^2 \psi^*}{\partial y^{*2}}\end{array}\right\}$$

$$+e^{(2\alpha_5-\alpha_1)\varepsilon}U^*\frac{dU^*}{dx^*} + S^*\left(e^{(\alpha_7+\alpha_5)\varepsilon}U^* - e^{(\alpha_7+\alpha_3-\alpha_2)\varepsilon}\frac{\partial \psi^*}{\partial y^*}\right) \tag{6}$$

Now for the equation (5),

$$\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial w}{\partial \mathrm{y}} = \frac{\partial}{\partial \mathrm{y}}\left\{\left[\left(\frac{\partial^2 \psi}{\partial \mathrm{y}^2}\right)^2 + \left(\frac{\partial w}{\partial \mathrm{y}}\right)^2\right]^{\frac{\mathrm{n}-1}{2}} \frac{\partial w}{\partial \mathrm{y}}\right\} + \mathrm{U}\frac{dW}{dx} + \mathrm{S(x)}\left(\mathrm{W} - \mathrm{w}\right)$$

$$\Rightarrow e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial\psi^*}{\partial y^*}e^{(\alpha_4-\alpha_1)\varepsilon}\frac{\partial w^*}{\partial x^*} - e^{(\alpha_3-\alpha_1)\varepsilon}\frac{\partial\psi^*}{\partial x^*}e^{(\alpha_4-\alpha_2)\varepsilon}\frac{\partial w^*}{\partial y^*} =$$

$$\frac{\partial}{\partial y^*}\left\{\left[\left(e^{(\alpha_3-2\alpha_2)\varepsilon}\frac{\partial^2\psi^*}{\partial y^{*2}}\right)^2 + \left(e^{(\alpha_4-\alpha_2)\varepsilon}\frac{\partial w^*}{\partial y^*}\right)^2\right]^{\frac{n-1}{2}} e^{(\alpha_4-\alpha_2)\varepsilon}\frac{\partial w^*}{\partial y^*}\right\}\frac{\partial y^*}{\partial y}$$

$$+e^{\alpha_5}U^* e^{(\alpha_6-\alpha_1)\varepsilon}\frac{dW^*}{dx^*} + e^{\alpha_7}S^*\left(e^{\alpha_6}W^* - e^{\alpha_4\varepsilon}w^*\right)$$

$$\Rightarrow e^{(\alpha_3+\alpha_4-\alpha_2-\alpha_1)\varepsilon}\left[\frac{\partial\psi^*}{\partial y^*}\frac{\partial w^*}{\partial x^*} - \frac{\partial\psi^*}{\partial x^*}\frac{\partial w^*}{\partial y^*}\right] =$$

$$\frac{\partial}{\partial y^*}\left\{\left[\left(e^{(\alpha_3(n-1)-2(n-1)\alpha_2)\varepsilon}\frac{\partial^2\psi^*}{\partial y^{*2}}\right)^2 + \left(e^{(\alpha_4(n-1)-\alpha_2(n-1))\varepsilon}\frac{\partial w^*}{\partial y^*}\right)^2\right]^{\frac{n-1}{2}} e^{(\alpha_4-\alpha_2)\varepsilon}\frac{\partial w^*}{\partial y^*}\right\} e^{-\alpha_2}$$

$$+e^{(\alpha_6+\alpha_5-\alpha_1)\varepsilon}U^*\frac{dW^*}{dx^*} + S^*\left(e^{(\alpha_6+\alpha_7)\varepsilon}W^* - e^{(\alpha_4+\alpha_7)\varepsilon}w^*\right)$$

$$\Rightarrow e^{(\alpha_3+\alpha_4-\alpha_2-\alpha_1)\varepsilon}\left[\frac{\partial\psi^*}{\partial y^*}\frac{\partial w^*}{\partial x^*} - \frac{\partial\psi^*}{\partial x^*}\frac{\partial w^*}{\partial y^*}\right] =$$

$$\frac{\partial}{\partial y^*}\left\{e^{(\alpha_3(n-1)-2n\alpha_2+\alpha_4)\varepsilon}\left[\left(\frac{\partial^2\psi^*}{\partial y^{*2}}\right)^2 + \left(\frac{\partial w^*}{\partial y^*}\right)^2\right]^{\frac{n-1}{2}} \cdot e^{(n\alpha_4-\alpha_2(n+1))\varepsilon}\frac{\partial w^*}{\partial y^*}\right\}$$

$$+e^{(\alpha_6+\alpha_5-\alpha_1)\varepsilon}U^*\frac{dW^*}{dx^*} + S^*\left(e^{(\alpha_6+\alpha_7)\varepsilon}W^* - e^{(\alpha_4+\alpha_7)\varepsilon}w^*\right) \quad (7)$$

With boundary conditions:

$$\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial x} = 0 \;;\; \frac{\partial\psi}{\partial y} = U(x)$$

$$;\, w = W(x) \quad e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial\psi^*}{\partial y^*} = e^{(\alpha_3-\alpha_1)\varepsilon}\frac{\partial\psi^*}{\partial x^*} = 0\,;$$

$$e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial\psi^*}{\partial y^*} = e^{\alpha_6\varepsilon}U^*; e^{\alpha_4\varepsilon}w^* = e^{\alpha_7\varepsilon}W^* \quad (8)$$

From equations (6), (7) and (8), it is seem that if the basic equations are to be invariant under this group of transformation, the power of e in each term should be equal. Therefore equations (6), (7) and (8) give

$$\alpha_3-\alpha_2 = \alpha_3-\alpha_1 = \alpha_4 \quad (9.1)$$

$$\alpha_3-\alpha_2 = \alpha_5 \quad (9.2)$$

$$\alpha_4 = \alpha_6 \quad (9.3)$$

$$(n-2)\alpha_3-(2n-1)\alpha_2+\alpha_1 = 0 \quad (9.4)$$

$$(n-1)\alpha_4-n\alpha_2-\alpha_3+\alpha_1 = 0 \quad (9.5)$$

$$\alpha_5-\alpha_3+\alpha_2 = 0 \quad (9.6)$$

$$\alpha_7+\alpha_5-2\alpha_3+2\alpha_2+\alpha_1 = 0 \quad (9.7)$$

$$\alpha_7-\alpha_3+\alpha_2+\alpha_1 = 0 \quad (9.8)$$

From equations (9.1) to (9.8),

Let $\alpha_4 = \alpha_5 = \alpha_6 = m \Rightarrow \frac{\alpha_4}{\alpha_1} = \frac{\alpha_5}{\alpha_1} = \frac{\alpha_6}{\alpha_1}$

$$\frac{\alpha_3}{\alpha_1} = \frac{(2n-1)m+1}{n+1} \quad (10)$$

$$\frac{\alpha_2}{\alpha_1} = \frac{(n-2)m+1}{n+1} \quad (11)$$

$$\frac{\alpha_7}{\alpha_1} = m-1 \quad (12)$$

The next step in this method is to find the so-called “absolute invariants” under this group of transformations. Absolute invariants are functions having the same form before and after the transformation.

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Expanding the exponential of transformation in terms of Taylor’s series and neglecting the terms with second and higher order of $\varepsilon$, we get following characteristic equation

$$\frac{dx}{\alpha_1 x}=\frac{dy}{\alpha_2 y}=\frac{d\psi}{\alpha_3\psi}=\frac{dw}{\alpha_4 w}=\frac{dU}{\alpha_5 U}=\frac{dW}{\alpha_6 W}=\frac{dS}{\alpha_7 S}$$

$$\Rightarrow$$

$$\frac{dx}{x}=\frac{dy}{\frac{\alpha_2}{\alpha_1}y}=\frac{d\psi}{\frac{\alpha_3}{\alpha_1}\psi}=\frac{dw}{\frac{\alpha_4}{\alpha_1}w}=\frac{dU}{\frac{\alpha_5}{\alpha_1}U}=\frac{dW}{\frac{\alpha_6}{\alpha_1}W}=\frac{dS}{\frac{\alpha_7}{\alpha_1}S}$$

$$\Rightarrow \frac{dx}{x}=\frac{dy}{\frac{(n-2)m+1}{n+1}y}=\frac{d\psi}{\frac{(2n-1)m+1}{n+1}\psi}=\frac{dw}{mw}=\frac{dU}{mU}=\frac{dW}{mW}=\frac{dS}{(m-1)S} \quad (13)$$

Solving (13) we get following similarity variables.

$$\eta = yx^{-\left[\frac{(n-2)m+1}{n+1}\right]} \quad (13.1)$$

$$F(\eta)=\psi x^{-\left[\frac{(2n-1)m+1}{n+1}\right]} \quad (13.2)$$

$$G(\eta)=wx^{-m} \quad (13.3)$$

$$U_0=Ux^{-m} \quad (13.4)$$

$$W_0=Wx^{-m} \quad (13.5)$$

$$S_0=Sx^{-(m-1)} \quad (13.6)$$

Now using this transformed similarity variables in equation (4) and (5), we obtain a set of ordinary differential equations.

$$mF'^2+FF''=\frac{\partial}{\partial\eta}\left\{\left[(F'')^2+(G')^2\right]^{\frac{n-1}{2}}F''\right\}+mU_0^2+S_0(U_0-F') \quad (14)$$

$$mF'G+\frac{(2n-1)m+1}{n+1}FG'=\frac{\partial}{\partial\eta}\left\{\left[(F'')^2+(G')^2\right]^{\frac{n-1}{2}}G'\right\}+mU_0W_0+S_0\left[W_0-G(\eta)\right] \quad (15)$$

With the transformed boundary conditions

$$\begin{aligned}\eta=0: &\quad F=F'=G=0\\ \eta=\infty: &\quad F'=U_0, G=W_0\end{aligned} \quad (16)$$

**Case II**: A one parameter spiral group of transformation is chosen in the form

$$x = x^*+\alpha_1\varepsilon,\ y=e^{\alpha_2\varepsilon}y^*,\ \psi=e^{\alpha_3\varepsilon}\psi^*$$
$$w=e^{\alpha_4\varepsilon}w^*,\ U=e^{\alpha_5\varepsilon}U^*,\ W=e^{\alpha_6\varepsilon}W^*, S=e^{\alpha_7\varepsilon}S^*$$

Where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7$ and e are constants.

We now seek relations among α's such that the basic equations will be invariant under this group of transformation. This can be achieved by substituting the transformation in to equations (4) and (5). Thus, we obtain

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial^2\psi}{\partial y^2}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial^2\psi}{\partial y^2}\right\}+U\frac{dU}{dx}+S(x)\left(U-\frac{\partial\psi}{\partial y}\right)$$

$$\Rightarrow e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial\psi^*}{\partial y^*}\left[e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial^2\psi^*}{\partial x^*\partial y^*}\right]-e^{\alpha_3\varepsilon}\frac{\partial\psi^*}{\partial x^*}\left[e^{(\alpha_3-2\alpha_2)\varepsilon}\frac{\partial^2\psi^*}{\partial y^{*2}}\right]=$$
$$\frac{\partial}{\partial y^*}\left\{\left[\left(e^{(\alpha_3-2\alpha_2)\varepsilon}\frac{\partial^2\psi^*}{\partial y^{*2}}\right)^2+\left(e^{(\alpha_4-\alpha_2)\varepsilon}\frac{\partial w^*}{\partial y^*}\right)^2\right]^{\frac{n-1}{2}}e^{(\alpha_3-2\alpha_2)\varepsilon}\frac{\partial^2\psi^*}{\partial y^{*2}}\right\}e^{-\alpha_2\varepsilon}$$
$$+e^{\alpha_5\varepsilon}U^*\left[e^{\alpha_5\varepsilon}\frac{dU^*}{dx^*}\right]+e^{\alpha_7\varepsilon}S^*\left[e^{\alpha_5\varepsilon}U^*-e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial\psi^*}{\partial y^*}\right]$$

$$\Rightarrow e^{(2\alpha_3-2\alpha_2)\varepsilon}\left[\frac{\partial\psi^*}{\partial y^*}\frac{\partial^2\psi^*}{\partial x^*\partial y^*}-\frac{\partial\psi^*}{\partial x^*}\frac{\partial^2\psi^*}{\partial y^{*2}}\right]=$$
$$\frac{\partial}{\partial y^*}\left\{\begin{array}{l}\left[e^{((n-1)\alpha_3-2(n-1)\alpha_2)\varepsilon}\left(\frac{\partial^2\psi^*}{\partial y^{*2}}\right)^2\right.\\ \left.+e^{((n-1)\alpha_4-(n-1)\alpha_2)\varepsilon}\left(\frac{\partial w^*}{\partial y^*}\right)^2\right]^{\frac{n-1}{2}}\\ .e^{(\alpha_3-3\alpha_2)\varepsilon}\frac{\partial^2\psi^*}{\partial y^{*2}}\end{array}\right\}$$
$$+e^{2\alpha_5\varepsilon}U^*\frac{dU^*}{dx^*}+S^*\left[e^{(\alpha_7+\alpha_5)\varepsilon}U^*-e^{(\alpha_3-\alpha_2+\alpha_7)\varepsilon}\frac{\partial\psi^*}{\partial y^*}\right]$$

$$\Rightarrow \frac{\partial \psi^*}{\partial y^*}\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial^2 \psi^*}{\partial y^{*2}} = \frac{\partial}{\partial y^*}\left\{ \begin{array}{l} \frac{e^{(n\alpha_3-(2n+1)\alpha_2)\varepsilon}}{e^{(2\alpha_3-2\alpha_2)\varepsilon}}\left[\left(\frac{\partial^2\psi^*}{\partial y^{*2}}\right)^2 + \left(\frac{\partial w^*}{\partial y^*}\right)^2\right]^{\frac{n-1}{2}} \\ \frac{e^{(\alpha_3-(n+2)\alpha_2+(n+1)\alpha_4)\varepsilon}}{e^{(2\alpha_3-2\alpha_2)\varepsilon}}\frac{\partial^2\psi^*}{\partial y^{*2}} \end{array}\right\} + \frac{e^{2\alpha_5\varepsilon}}{e^{(2\alpha_3-2\alpha_2)\varepsilon}}U^*\frac{dU^*}{dx^*} + S^*\left[\frac{e^{(\alpha_7+\alpha_5)\varepsilon}}{e^{(2\alpha_3-2\alpha_2)\varepsilon}}U^* - \frac{e^{(\alpha_3-\alpha_2+\alpha_7)\varepsilon}}{e^{(2\alpha_3-2\alpha_2)\varepsilon}}\frac{\partial\psi^*}{\partial y^*}\right] \tag{17}$$

Similarly,

$$\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial^2\psi}{\partial y^2}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial w}{\partial y}\right\} + U\frac{dW}{dx} + S(x)\left(W - w\right)$$

$$e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial\psi^*}{\partial y^*}\left[e^{\alpha_4\varepsilon}\frac{\partial w^*}{\partial x^*}\right] - e^{\alpha_3\varepsilon}\frac{\partial\psi^*}{\partial x^*}\left[e^{(\alpha_4-\alpha_2)\varepsilon}\frac{\partial w^*}{\partial y^*}\right] =$$

$$\frac{\partial}{\partial y^*}\left\{ \begin{array}{l} \left[\left(e^{(\alpha_3-2\alpha_2)\varepsilon}\frac{\partial^2\psi^*}{\partial y^{*2}}\right)^2 + \left(e^{(\alpha_4-\alpha_2)\varepsilon}\frac{\partial w^*}{\partial y^*}\right)^2\right]^{\frac{n-1}{2}} \\ e^{(\alpha_4-\alpha_2)\varepsilon}\frac{\partial w^*}{\partial y^*} \end{array}\right\} e^{-\alpha_2\varepsilon} + e^{(\alpha_5+\alpha_6)\varepsilon}U^*\frac{dW^*}{dx^*} + e^{\alpha_7\varepsilon}S^*\left[e^{\alpha_6\varepsilon}W^* - e^{\alpha_4\varepsilon}w^*\right]$$

$$\Rightarrow \left[\frac{\partial \psi^*}{\partial y^*}\frac{\partial w^*}{\partial x^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial w^*}{\partial y^*}\right] = \frac{\partial}{\partial y^*}\left\{ \begin{array}{l} \frac{e^{((n-1)\alpha_3-2n\alpha_2+\alpha_4)\varepsilon}}{e^{(\alpha_3-\alpha_2+\alpha_4)\varepsilon}}\left[\left(\frac{\partial^2\psi^*}{\partial y^{*2}}\right)^2 + \left(\frac{\partial w^*}{\partial y^*}\right)^2\right]^{\frac{n-1}{2}} \\ \frac{e^{((n-1)\alpha_4-(n-1)\alpha_2+\alpha_4-2\alpha_2)\varepsilon}}{e^{(\alpha_3-\alpha_2+\alpha_4)\varepsilon}}\frac{\partial w^*}{\partial y^*} \end{array}\right\} + \frac{e^{(\alpha_5+\alpha_6)\varepsilon}}{e^{(\alpha_3-\alpha_2+\alpha_4)\varepsilon}}U^*\frac{dW^*}{dx^*} + S^*\left[\frac{e^{(\alpha_6+\alpha_7)\varepsilon}}{e^{(\alpha_3-\alpha_2+\alpha_4)\varepsilon}}W^* - \frac{e^{(\alpha_4+\alpha_7)\varepsilon}}{e^{(\alpha_3-\alpha_2+\alpha_4)\varepsilon}}w^*\right] \tag{18}$$

With boundary conditions:

$$\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial x} = 0; \frac{\partial\psi}{\partial y} = U\left(x\right); w = W\left(x\right);$$

$$e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial\psi^*}{\partial y^*} = e^{\alpha_3\varepsilon}\frac{\partial\psi^*}{\partial x^*} = 0; e^{\alpha_4\varepsilon}w^* = e^{\alpha_7\varepsilon}W^*$$

$$e^{(\alpha_3-\alpha_2)\varepsilon}\frac{\partial\psi^*}{\partial y^*} = e^{\alpha_6\varepsilon}U^* \tag{19}$$

From equations (17), (18), and (19), it is seem that if the basic equations are to be invariant under this group of transformation, the power of e in each term should be equal. Therefore equations (17), (18) and (19) give

$$\alpha_3 - \alpha_2 = \alpha_3 = 0 \tag{20.1}$$

$$\alpha_3 - \alpha_2 = \alpha_6 \tag{20.2}$$

$$\alpha_4 = \alpha_7 \tag{20.3}$$

$$\left(n-2\right)\alpha_3 - \left(2n-1\right)\alpha_2 = 0 \tag{20.4}$$

$$\left(n-1\right)\alpha_4 - n\alpha_2 - \alpha_3 = 0 \tag{20.5}$$

$$\alpha_5 - \alpha_3 + \alpha_2 = 0 \tag{20.6}$$

$$\alpha_7 + \alpha_6 - \alpha_3 + \alpha_2 - \alpha_4 = 0 \tag{20.7}$$

$$\alpha_7 - \alpha_3 + \alpha_2 = 0 \tag{20.8}$$

From the above equations (20.1) to (20.8)

Let $\alpha_4 = \alpha_5 = \alpha_6 = m \Rightarrow \frac{\alpha_4}{\alpha_1} = \frac{\alpha_5}{\alpha_1} = \frac{\alpha_6}{\alpha_1}$

$$\frac{\alpha_3}{\alpha_1} = \frac{\left(2n-1\right)m}{n+1} \tag{21}$$

$$\frac{\alpha_2}{\alpha_1} = \frac{\left(n-2\right)m}{n+1} \tag{22}$$

$$\frac{\alpha_7}{\alpha_1} = m \tag{23}$$

The next step in this method is to find the so-called "absolute invariants" under this group of transformations. Absolute invariants are functions having the same form before and after the transformation.

Expanding the exponential of transformation in terms of Taylor's series and neglecting the terms with second and higher order of $\varepsilon$, we get following characteristic equation

$$\frac{dx}{x} = \frac{dy}{\alpha_2 y} = \frac{d\psi}{\alpha_3 \psi} = \frac{dw}{\alpha_4 w} = \frac{dU}{\alpha_5 U} = \frac{dW}{\alpha_6 W} = \frac{dS}{\alpha_7 S}$$

$$\Rightarrow dx = \frac{dy}{\frac{\alpha_2}{\alpha_1} y} = \frac{d\psi}{\frac{\alpha_3}{\alpha_1} \psi} = \frac{dw}{\frac{\alpha_4}{\alpha_1} w} = \frac{dU}{\frac{\alpha_5}{\alpha_1} U} = \frac{dW}{\frac{\alpha_6}{\alpha_1} W} = \frac{dS}{\frac{\alpha_7}{\alpha_1} S}$$

$$\Rightarrow dx = \frac{dy}{\frac{(n-2)m}{n+1} y} = \frac{d\psi}{\frac{(2n-1)m}{n+1} \psi} = \frac{dw}{mw} = \frac{dU}{mU} = \frac{dW}{mW} = \frac{dS}{mS} \tag{24}$$

Solving (24) we get following similarity variables,

$$\eta = y e^{-\left[\frac{(n-2)m}{n+1}\right]x} \tag{25.1}$$

$$F(\eta) = \psi e^{-\left[\frac{(2n-1)m}{n+1}\right]x} \tag{25.2}$$

$$G(\eta) = w e^{-mx} \tag{25.3}$$

$$U_0 = U e^{-mx} \tag{25.4}$$

$$W_0 = W e^{-mx} \tag{25.5}$$

$$S_0 = S e^{-mx} \tag{25.6}$$

Now using this transformed similarity variables in equation (4) and (5), we obtain a set of ordinary differential equations.

$$m\left(F'(\eta)\right)^2 - \left(\frac{(2n-1)m}{n+1}\right) F F'' = \frac{\partial}{\partial \eta}\left\{\left[\left(F''(\eta)\right)^2 + \left(G(\eta)\right)^2\right]^{\frac{n-1}{2}} F''(\eta)\right\} + m U_0^2 + S_0\left[U_0 - F'(\eta)\right] \tag{26}$$

$$\Rightarrow mF'G - \left(\frac{(2n-1)m}{n+1}\right) F G' = \frac{\partial}{\partial \eta}\left\{\left[\left(F''\right)^2 + \left(G'\right)^2\right]^{\frac{n-1}{2}} G'\right\} + m W_0 U_0 + S_0\left[W_0 - G(\eta)\right] \tag{27}$$

With the transformed boundary conditions

$$\begin{aligned} \eta = 0: &\quad F = F' = G = 0 \\ \eta = \infty: &\quad F' = U_0, G = W_0 \end{aligned} \tag{28}$$

## Results and discussion:

By applying linear transformation of group theoretic technique the governing equations reduce to a system of non-linear ODE's with the appropriate boundary conditions. Finally the system of similarity equations (14) and (15) along with boundary conditions (16) are solved numerically using Maple ODE solver. As, in the present paper, the stagnation point flow is considered, the value of physical constant *m* is taken 1.and the values of Stream velocities $W_0$ & $U_0$ are assumed to be 1. The numerical results of velocity components *F'* and *G* have been obtained for various values of the Power-Law viscosity index n taking values 0.5,1,1.2 and the various values of the magnetic parameter $S_0$ taking values 0.2,0.5,1,1.2.

Figures 1 and 2 shows that the velocity profiles *F'* and *G* increase rapidly with an increase in the magnetic parameter $S_0$ for Newtonian fluids (i.e. for $n=1$). Figures 3 and 4 concludes that the velocity profiles *F'* and *G* increase more rapidly than $n=1$ with an increase in the magnetic parameter $S_0$ for non-Newtonian fluids (i.e.Pseudoplastic ,$n<1$). Figures 5 and 6 shows that the velocity profiles *F'* and *G* increase less rapidly than $n=1$ with an increase in the magnatic parameter $S_0$ for Newtonian fluids (i.e. for $n=1$) Dilatant ($n>1$)

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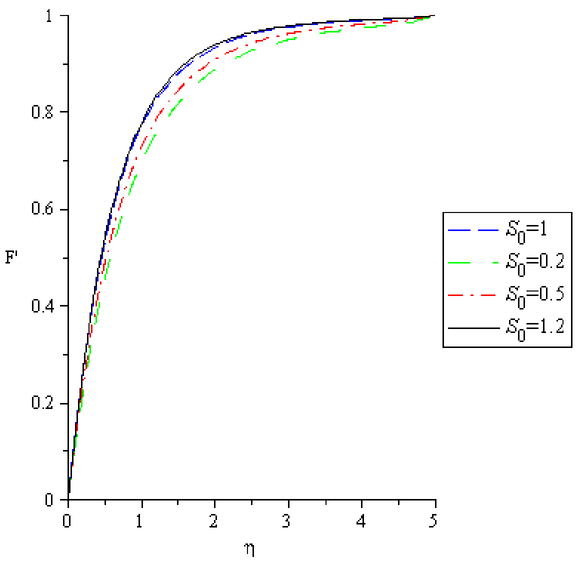


Figure 1: F' versus η; (n=1)

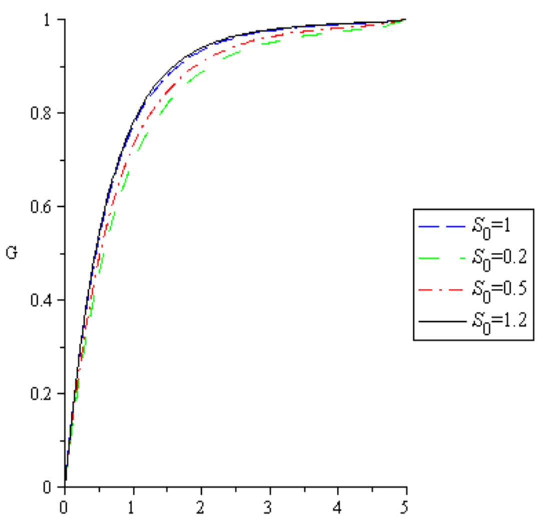


Figure 2: G versus η; (n=1)

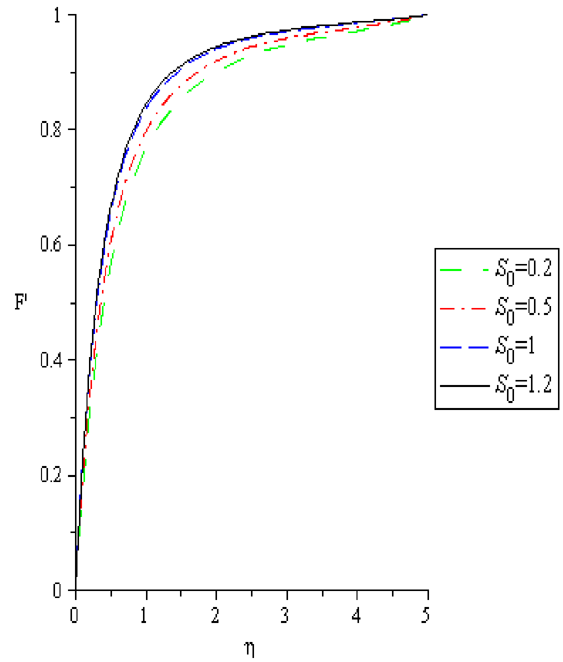


Figure 3: F' versus η; (n=0.5)

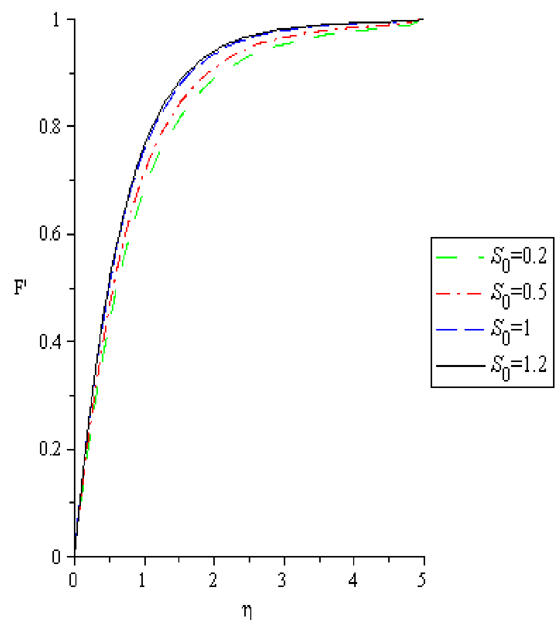


Figure 4: F' versus η; (n=1.2)

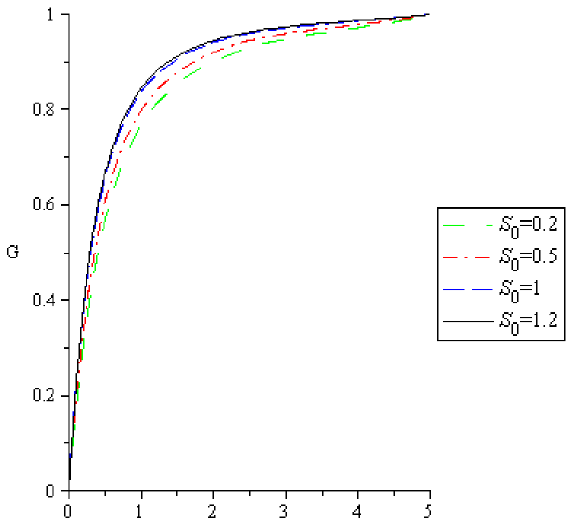


Figure 5: G versus η; (n=0.5)

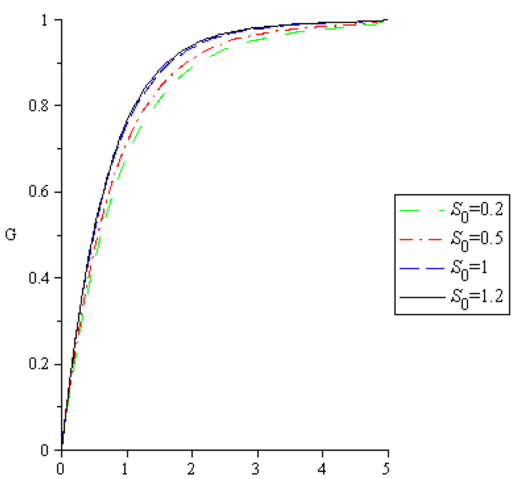


Figure 6: F versus η; ( n=1.2)