On The Existence Of Solutions Of Impulsive Fractional Integro-Differential Equations

K. Malar, P. Karthikeyan and R. Arul

***Abstract*—This paper is devoted to study the existence and uniqueness of solution for nonlocal Impulsive fractional Integro-Differential equations involving the Caputo fractional derivative in a Banach Space. The arguments are based upon contraction mapping principle and Krasnoselskii's fixed point theorem.**

***Index Terms*—Fractional integro-differential equations, contraction principle, Krasnoselskii's fixed point theorem, Impulsive condition.**

***MSC 2010 Codes* – 26A33, 34A37, 34K05.**

I. Introduction

DIFFERENTIAL equations of fractional order have proved to be valuable tools in the modelling of many phenomena in various fields of science and engineering.

Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc.(see [5, 15, 16, 18, 23, 24, 28]).

There has been a significant development in fractional differential and partial differential equations in recent years; see the monographs of Kilbas *et al* [21], Miller and Ross [25], Samko *et al* [32] and the papers of Agarwal *et al* [1], Babakhani and Daftardar-Gejji [3, 4], Belmekki *et al* [8], Benchohra *et al* [7, 9, 10, 12], Daftardar-Gejji and Jafari [13], Delbosco and Rodino [14], Kaufmann and Mboumi [19], Kilbas and Marzan [20], Mainardi [23], Momani and Hadid [26], Momani *et al* [27],Podlubny *et al* [31], Yu and Gao [34] and Zhang [35] and the references therein.

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences.

There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Benchohra *et al* [11], Lakshmikantham *et al* [22], and Samoilenko and Perestyuk [33] and the references therein.

Archana Chauhan *et al* [2] studied the existence of mild solutions for impulsive fractional-order semilinear evoluation equations with nonlocal conditions and Bashir Ahmad & S. Sivasundaram [6] investigated the some existence results for fractional integro-differential equations with nonlinear conditions in a Banach space.

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The impulsive conditions are combinations of traditional initial value problems and short term perturbations whose duration can be negligible in comparison with the duration of the process. They have advantages over traditional initial value problems because they can be used to model phenomena that cannot be modeled by traditional initial value problems.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions and boundary conditions. Caputo's fractional derivative satisfies these demands.

More details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types can be obtained from [17, 30].

We reconsider the impulsive differential equations with Caputo fractional derivative and seek a correct formula of the solution for this problem.

In this paper, we study the existence and uniqueness results in a Banach space for an impulsive fractional integro-differential equation with nonlocal conditions given by

$$\begin{cases} {}^cD_t^q x(t) = f(t,x(t)) + \int_0^t k(t,s,x(s))ds, \\ \qquad t \in J^{'} = J/\{t_1,...,t_m\}, J := [0,T] \\ x(t_k^+) = x(t_k^-) + y_k, \quad k = 1,2,...,m \\ x(0) = x_0 - g(x) \quad y_k \in X \end{cases} \tag{1.1}$$

where ${}^cD_t^q$ is the Caputo fractional derivative of order q.

The mappings $f : J \times X \to X,\ k : J \times J \times X \to X$ are jointly continuous.

The mapping $g : C \to X$ is continuous, t_k satisfy

$$0 = t_0 < t_1 < ... < t_m < t_{m+1} = T,$$

$$x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$$

and

$$x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)$$

represent the right and left limits of $x(t)$ at $t = t_k$.

The formula of solutions for equation (1.1) should be

$$x(t)=\begin{cases} x_0-g(x)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}[f(t,x(t)) \\ \quad +\int_0^t k(\sigma,s,x(s))d\sigma]ds \ for \ t\in[0,t_1) \\ x_0-g(x)+y_1 \\ \quad +\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}[f(t,x(t)) \\ \quad +\int_0^t k(\sigma,s,x(s))d\sigma]ds \ for \ t\in(t_1,t_2) \\ x_0-g(x)+y_1+y_2 \\ \quad +\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}[f(t,x(t)) \\ \quad +\int_0^t k(\sigma,s,x(s))d\sigma]ds \ for \ t\in(t_2,t_3) \\ \vdots \\ x_0-g(x)+\sum_{i=0}^m y_i \\ \quad +\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}[f(t,x(t)) \\ \quad +\int_0^t k(\sigma,s,x(s))d\sigma]ds \ for \ t\in(t_m,T] \end{cases} \tag{1.2}$$

In passing, we remark that the application of nonlinear condition $x(0)=x_0-g(x)$ in physical problems yields better effect than the initial condition $x(0)=x_0$ [8].

In Section 2, we give some notations, recall some concepts and preparation results, and introduce a concept of a piecewise continuous solution for our problem. In section 3, we give two main results, the first result based on Banach contraction principle, the second result based on Schaefer$'$s fixed point theorem. Some example is given in section 4 to demonstrate the application of our main results.

II. Preliminaries

In this section, we introduce notations, definition and preliminary facts. Throughout this paper, Let $C(J,X)$ be the Banach space of all continuous function from J into X with th norm $||x||_c := sup\{||x(t)|| : t\in J\}$ for $x\in C(J,X)$. We also introduce the Banach space

$$PC(J,X)=\{x:J\to X : x\in C(t_k,t_{k+1}],X\},$$

$$k=0,1,2,...,m$$

and their exist $x(t_k^-)$ and $x(t_k^+), k = 0,1,2,..,m$ with $x(t_k^-) = x(t_k)$ with the norm $||x||_{PC} := sup\{||x(t)|| : t\in J\}$.

Denote

$$PC^1(J,X)\equiv\{x\in PC(J,X)|\dot{x}\in PC(J,X)\}.$$

Set $||x||_{PC}=||x||_{PC}+||\dot{x}||_{PC}$. It can be seen that endowed with the norm $||.||_{PC^1}, PC^1(J,X)$ is also a Banach space.

Let us recall the following known definitions.

Definition 2.1: The fractionl integral of order q with the lower limit zero for a funtion

$$f:[0,\infty)\to X$$

is defined as

$$I_t^q f(t)=\frac{1}{\Gamma(q)}\int_0^t\frac{f(s)}{(t-s)^{1-q}}ds, \ t>0, \ q>0$$

provied the right side is point-wise defined on $[0,\infty)$, where $\Gamma(.)$ is the gamma funtion.

Definition 2.2: The Riemann-Liouville derivative of order q with the lower limit zero for a function $f:[0,\infty)\to X$ can be written as

$$^LD_t^q f(t)=\frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{q+1-n}}ds, \quad t>0, \ n-1<q<n.$$

Definition 2.3: The Caputo derivative of order q for function $f:[0,\infty)\to X$ can be written as

$$^cD_t^q f(t)={}^L D^q\left[f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right], \quad t>0, \ n-1<q<n.$$

Definition 2.4: A function $x\in PC^1(J,X)$ is said to be a solution of the problem (1.1) if x satisfies the equation $^cD_t^q x(t) = f(t,x(t))+\int_0^t k(t,s,x(s))ds, \ t\in J^{'}$ a.e on $J^{'}, \ g:C\to X$ is continuous,and the conditions $x(t_k^+) = x(t_k^-)+y_k, \ k=1,2,...,m$ and $x(0)=x_0-g(t)$.

Lemma 2.1: Let $q\in(0,1)$ and $h:J\to X$ be continuous,$g:C\to X$ is continuous. A function $x\in C(J,X)$ is a solution of the fractional integral equation

$$x(t)=x_0-g(t)-\frac{1}{\Gamma(q)}\int_0^a(a-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}h(s)ds, \tag{2.1}$$

if and only if u is a solution of the following fractional Cauchy problems

$$\begin{cases} ^cD_t^q x(t)=h(t), \ t\in J, \\ x(0)=x_0-g(t), a>0. \end{cases} \tag{2.2}$$

As a consequence of Lemma 2.1, we have the following result which is useful in what follows.

Lemma 2.2: Let $q\in(0,1)$ and $h:J\to X$ be continuous,$g:C\to X$ is continuous. A function x is a solution of the fractional integral equation

$$x(t)=\begin{cases} x_0-g(t)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}h(s)ds \\ \qquad for \ t\in[0,t_1) \\ x_0-g(t)+y_1 \\ \quad +\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}h(s)ds \ for \ t\in(t_1,t_2) \\ x_0-g(t)+y_1+y_2 \\ \quad +\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}h(s)ds \ for \ t\in(t_2,t_3) \\ \vdots \\ x_0-g(t)+\sum_{i=0}^m y_i \\ \quad +\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}h(s)ds \ for \ t\in(t_m,T] \end{cases} \tag{2.3}$$

if and only if u is a solution of the following impulsive problem

$$\begin{cases} {}^cD_t^q x(t) = h(t), \ t \in (0,T], \\ x(t_k^+) = x(t_k^-) + y_k, \quad k = 1,2,...,m \\ x(0) = x_0 - g(x) \end{cases} \tag{2.4}$$

Proof: Assume x satisfies (2.4). If $t \in [0, t_1]$ then

$$\begin{cases} {}^cD_t^q x(t) = h(t), \ t \in (0,t_1], \\ x(0) = x_0 - g(t) \end{cases} \tag{2.5}$$

Integrating the expression (2.5) from 0 to t by virtue of Definition 2.1, one can obtain

$$u(t) = x_0 - g(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds$$

If $t \in (t_1, t_2]$ then

$$\begin{cases} {}^cD_t^q x(t) = h(t), \ t \in (t_1,t_2], \\ x(t_k^+) = x(t_k^-) + y_k \end{cases} \tag{2.6}$$

By Lemma 2.1, one obtain

$$\begin{aligned} x(t) &= x(t_1^+) - g(t) \\ &- \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} h(s) ds \\ &+ \frac{1}{\Gamma(q)} \int_0^{t} (t-s)^{q-1} h(s) ds, \\ &= x(t_1^-) - g(t) + y_1 \\ &+ -\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} h(s) ds \\ &+ \frac{1}{\Gamma(q)} \int_0^{t} (t-s)^{q-1} h(s) ds, \\ &= x_0 + y_1 \\ &+ \frac{1}{\Gamma(q)} \int_0^{t} (t-s)^{q-1} h(s) ds \end{aligned}$$

If $t \in (t_2, t_3]$ then again from Lemma 2.2 we get

$$\begin{aligned} x(t) &= x(t_2^+) - g(t) \\ &- \frac{1}{\Gamma(q)} \int_0^{t_2} (t_1-s)^{q-1} h(s) ds \\ &+ \frac{1}{\Gamma(q)} \int_0^{t} (t-s)^{q-1} h(s) ds, \\ &= x(t_2^-) - g(t) + y_2 \\ &+ \frac{1}{\Gamma(q)} \int_0^{t_2} (t_1-s)^{q-1} h(s) ds \\ &+ \frac{1}{\Gamma(q)} \int_0^{t} (t-s)^{q-1} h(s) ds, \\ &= x_0 + y_1 + y_2 \\ &+ \frac{1}{\Gamma(q)} \int_0^{t} (t-s)^{q-1} h(s) ds \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 2.2 we get

$$\begin{aligned} x(t) &= x_0 - g(t) + \sum_{i=0}^{m} y_i \\ &+ \frac{1}{\Gamma(q)} \int_0^{t} (t-s)^{q-1} h(s) ds \end{aligned}$$

Conversely, assume that x satisfies (2.3). If $t \in (0, t_1]$ then $x(0) = x_0$ and using the fact that ${}^cD_t^q$ is the left inverse of I_t^q we get (2.5). If $t \in (t_k, t_{k+1}], \ k = 1,2,...,m$ and using the fact of the Caputo derivative of a constant is equal to zero, we obtain ${}^cD_t^q x(t) = h(t), \ t \in (t_k, t_{k+1}]$, and $x(t_k^+) = x(t_k^-) + y_k$. This completes the proof. $\diamond$

Now, we state a known result due to Krasnoselskii which is needed to prove the existence of at least one solution of (1.1).

Theorem 2.1 (Krasnoselskii Theorem) Let M be a closed convex and nonempty subset of a Banach space X. Let A,B be the operators such that

(i) $Ax + By \in M$ whenever $x, y \in M$

(ii) A is compact and continuous

(iii) B is a contraction mapping.

Then there exists $Z \in M$ such that

$$Z = Az + Bz.$$

III. Main Results

This section deals with the existence and uniqueness of solutions for the problem (1.1). Before stating and proving the main result, We introduce the following hypotheses.

(H_1) There exists a constant $L_1 > 0$ such that $||f(t,x_1) - f(t,x_2)|| \leq L_1||x_1 - x_2||, \forall \ t \in J' \ \ x_1, x_2 \in X$

(H_2) $k : \Delta \times X \to X$ is continuous.

$$||k(t,s,x_1) - k(t,s,x_2)|| \leq L_2||x_1 - x_2||,$$

$$\forall \ t,s \in J' \quad x_1, x_2 \in X$$

(H_3) g is continuous, and there exists a constant $b < 1$ such that

$$||g(x_1 - g(x_2)|| \leq b||x_1 - x_2||$$

$$\forall \ x_1, x_2 \in X,$$

(H_4) There exists function $\mu, \sigma \in L^1_{Loc}(I, R^+)$ such that $||f(t,x)|| \leq \mu(t), \ (t,x) \in [0,T] \times X$

$$||k(t,s,x)|| \leq \sigma(t),$$

$$(t,s,x) \in [0,T] \times [0,T] \times X$$

Theorem 3.1: Assume that $(H_1), (H_2), (H_3)$ hold. If

$$b + ||\sum_{i=0}^{m} y_i|| + \frac{L_1 T^{q_i}}{\Gamma(q_i+1)} + \frac{L_2 q_i T^{q_i+1}}{\Gamma(q_i+2)} < 1$$

Then the cauchy problem (1.1) has a unique solution provided $b < \frac{1}{2}, \ L_1 \leq \frac{\Gamma(q_i+1)}{4T^{q_i}}$, $L_2 \leq \frac{\Gamma(q_i+2)}{4q_i T^{q_i+1}}, \ i = 0,1,...,m.$

Proof : Define $\theta : PC \to PC$ by

$$(\theta x)(t) = \begin{cases} x_0 - g(x) \\ \quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(t, x(t)) \\ \quad + \int_0^t k(\sigma, s, x(s)) d\sigma] ds \\ \qquad \text{for } \ t \in [0, t_1) \\ x_0 - g(x) + y_1 \\ \quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(t, x(t)) \\ \quad + \int_0^t k(\sigma, s, x(s)) d\sigma] ds \\ \qquad \text{for } t \in (t_1, t_2) \\ x_0 - g(x) + y_1 + y_2 \\ \quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(t, x(t)) \\ \quad + \int_0^t k(\sigma, s, x(s)) d\sigma] ds \\ \qquad \text{for } t \in (t_2, t_3) \\ \vdots \\ x_0 - g(x) + \sum_{i=0}^{m} y_i \\ \quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(t, x(t)) \\ \quad + \int_0^t k(\sigma, s, x(s)) d\sigma] ds \\ \qquad \text{for } t \in (t_m, T] \end{cases}$$

Let us set

$$\sup_{t \in J'} ||f(t, 0)|| = M_1,$$

$$\sup_{t,s \in J'} ||k(t, s, 0)|| = M_2$$

and

$$\sup_{x \in X} ||g(x)|| = G,$$

it can be shown that $\theta B_r \subset B_r$, where $B_r = \{x \in X; ||x|| \leq r\}$.

Choose,
$r_i \geq 2\left(||x_0|| + G + ||y_r|| + \frac{M_1 T^{q_i}}{\Gamma(q_i+1)} + \frac{q_i M_2 T^{q_i+1}}{\Gamma(q_i+2)}\right)$
$i = 0, 1, 2, ..., m$ and $r = 1, 2, ..., m$.

Step 1: For $t \in (0, t_1]$ we have

$$\begin{aligned} ||(\theta x)(t)|| &\leq ||x_0|| + ||g(x)|| \\ &\quad + \frac{1}{\Gamma(q)} \int_0^{t_1} (t-s)^{q-1} [||f(t, x(t))|| \\ &\quad + \int_0^{t_1} ||k(\sigma, s, x(s))|| d\sigma] ds \\ &\leq ||x_0|| + G + \frac{1}{\Gamma(q)} \int_0^{t_1} (t-s)^{q-1} \\ &\quad \left[||f(t, x(t))|| + \int_0^{t_1} ||k(\sigma, s, x(s))|| d\sigma \right] ds \\ &\leq ||x_0|| + G + \frac{1}{\Gamma(q)} \int_0^{t_1} (t-s)^{q-1} \Big[||f(t, x(t)) \\ &\quad - f(s, 0) + f(s, 0)|| \\ &\quad + \int_0^{t_1} (||k(\sigma, s, x(s)) - k(\sigma, s, 0) + k(\sigma, s, 0)||) \\ &\quad d\sigma \Big] ds \\ &\leq ||x_0|| + G + \frac{1}{\Gamma(q)} \int_0^{t_1} (t-s)^{q-1} \\ &\quad \Big[||f(t, x(t)) - f(s, 0)|| + ||f(s, 0)|| \\ &\quad + \int_0^{t_1} (||k(\sigma, s, x(s)) - k(\sigma, s, 0)|| + ||k(\sigma, s, 0)||) \\ &\quad d\sigma \Big] ds \\ &\leq ||x_0|| + G + \frac{1}{\Gamma(q)} \int_0^{t_1} (t-s)^{q-1} \\ &\quad (L_1 ||x(s)|| + M_1) \, ds \\ &\quad + \int_0^{t_1} (t-s)^{q} (L_2 ||x(s)|| + M_2) \, ds \\ &\leq ||x_0|| + G + \frac{L_1 r + M_1}{\Gamma(q)} \int_0^{t_1} (t-s)^{q-1} ds \\ &\quad + \frac{L_2 r + M_2}{\Gamma(q)} \int_0^{t_1} (t-s)^{q} ds \\ &\leq ||x_0|| + G + \frac{(L_1 r + M_1) T^{q_0}}{\Gamma(q_0 + 1)} \\ &\quad + \frac{(L_2 r + M_2) T^{q_0 + 1}}{\Gamma(q_0 + 2)} \\ &\leq r_1 \end{aligned}$$

Now, for $x_1, x_2 \in X$, We obtain

$$(\theta x_1)(t) = g(x_1) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(t, x_1(t)) + \int_0^{t_1} k(\sigma, s, x_1(s)) d\sigma] ds \quad (3.1)$$

$$(\theta x_2)(t) = g(x_2) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(t, x_2(t)) + \int_0^{t_1} k(\sigma, s, x_2(s)) d\sigma] ds \quad (3.2)$$

Subtracting Eq. (3.2) from Eq. (3.1) and taking norm on both sides, we get

$$\begin{aligned}
&||(\theta x_1)(t) - (\theta x_2)(t)|| \\
&\leq ||g(x_1) - g(x_2)|| + \frac{1}{\Gamma(q)} \int_0^{t_1} (t-s)^{q-1} \\
&\qquad [||f(t, x_1(t)) - f(t, x_2(t))|| \\
&\qquad + \int_0^{t_1} ||k(\sigma, s, x_1(s)) - k(\sigma, s, x_2(s))|| d\sigma] ds \\
&\leq b||x_1 - x_2|| + \frac{1}{\Gamma(q)} \int_0^{t_1} (t-s)^{q-1} [L_1 ||x_1 - x_2|| \\
&\qquad + \int_0^{t_1} L_2 ||x_1 - x_2|| d\sigma] ds \\
&\leq b||x_1 - x_2|| + \frac{L_1 ||x_1 - x_2|| T^{q_0}}{\Gamma(q_0+1)} \\
&\qquad + \frac{L_2 ||x_1 - x_2|| q_0 T^{q_0+1}}{\Gamma(q_0+2)} \\
&\leq \left(b + \frac{L_1 T^{q_0}}{\Gamma(q_0+1)} + \frac{L_2 q_0 T^{q_0+1}}{\Gamma(q_0+2)} \right) ||x_1 - x_2|| \\
&\leq \Lambda_{b, L_1, L_2, T, q_0} ||x_1 - x_2||
\end{aligned}$$

where

$$\Lambda_{b, L_1, L_2, T, q_0} = b + \frac{L_1 T^{q_0}}{\Gamma(q_0+1)} + \frac{L_2 q_0 T^{q_0+1}}{\Gamma(q_0+2)}$$

depends only on the parameters involved in the problem as $\Lambda_{b, L_1, L_2, T, q_0} < 1$. So the conclusion of the theorem follows by the contraction mapping principle for the interval $t \in (0, t_1]$.

Step 2: For $t \in (t_1, t_2]$, we have

$$\begin{aligned}
&||(\theta x)(t)|| \leq ||x_0|| + G + ||y_1|| \\
&\qquad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t-s)^{q-1} [||f(t, x(t))|| \\
&\qquad + \int_{t_1}^{t_2} ||k(\sigma, s, x(s))|| d\sigma] ds \\
&\leq ||x_0|| + G + ||y_1|| + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t-s)^{q-1} \\
&\qquad \Big[||f(t, x(t)) - f(s, 0)|| + ||f(s, 0)|| \\
&\qquad + \int_{t_1}^{t_2} (||k(\sigma, s, x(s)) - k(\sigma, s, 0)|| + ||k(\sigma, s, 0)||) \\
&\qquad d\sigma \Big] ds \\
&\leq ||x_0|| + G + ||y_1|| \\
&\qquad + \frac{L_1 r + M_1}{\Gamma(q)} \int_{t_1}^{t_2} (t-s)^{q-1} ds \\
&\qquad + \frac{L_2 r + M_2}{\Gamma(q)} \int_{t_1}^{t_2} (t-s)^{q} ds \\
&\leq ||x_0|| + G + ||y_1|| + \frac{(L_1 r + M_1) T^{q_1}}{\Gamma(q_1+1)} \\
&\qquad + \frac{(L_2 r + M_2) T^{q_1+1}}{\Gamma(q_1+2)} \\
&\leq r_2
\end{aligned}$$

Now, for $x_1, x_2 \in X$, We obtain

$$\begin{aligned}
&||(\theta x_1)(t) - (\theta x_2)(t)|| \\
&\leq ||g(x_1) - g(x_2)|| + ||y_1|| \\
&\qquad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t-s)^{q-1} \\
&\qquad \Big[||f(t, x_1(t)) - f(t, x_2(t))|| \\
&\qquad + \int_{t_1}^{t_2} ||k(\sigma, s, x_1(s)) - k(\sigma, s, x_2(s))|| d\sigma \Big] ds \\
&\leq \Lambda_{b, L_1, L_2, T, q_1} ||x_1 - x_2||
\end{aligned}$$

where

$$\Lambda_{b, L_1, L_2, T, q_1} = b + ||y_1|| + \frac{L_1 T^{q_1}}{\Gamma(q_1+1)} + \frac{L_2 q_1 T^{q_1+1}}{\Gamma(q_1+2)}$$

depends only on the parameters involved in the problem as $\Lambda_{b, L_1, L_2, T, q_1} < 1$. So the conclusion of the theorem follows by the contraction mapping principle for the interval $t \in (t_1, t_2]$. Similarly $(t_2, t_3), (t_3, t_4), \ldots$

Step 3: For $t \in (t_m, T]$, we have

$$\begin{aligned}
&||(\theta x)(t)|| \\
&\leq ||x_0|| + G + ||\sum_{i=0}^{m} y_i|| + \frac{1}{\Gamma(q)} \int_{t_m}^{T} (t-s)^{q-1} \\
&\qquad \Big[||f(t, x(t))|| + \int_{t_m}^{T} ||k(\sigma, s, x(s))|| d\sigma \Big] ds \\
&\leq ||x_0|| + G + ||\sum_{i=0}^{m} y_i|| + \frac{1}{\Gamma(q)} \int_{t_m}^{T} (t-s)^{q-1} \\
&\qquad \Big[||f(t, x(t)) - f(s, 0)|| + ||f(s, 0)|| \\
&\qquad + \int_{t_m}^{T} (||k(\sigma, s, x(s)) - k(\sigma, s, 0)|| + ||k(\sigma, s, 0)||) \\
&\qquad d\sigma \Big] ds \\
&\leq ||x_0|| + G + ||\sum_{i=0}^{m} y_i|| \\
&\qquad + \frac{L_1 r + M_1}{\Gamma(q)} \int_{t_m}^{T} (t-s)^{q-1} ds \\
&\qquad + \frac{L_2 r + M_2}{\Gamma(q)} \int_{t_m}^{T} (t-s)^{q} ds \\
&\leq ||x_0|| + G + ||\sum_{i=0}^{m} y_i|| + \frac{(L_1 r + M_1) T^{q_m}}{\Gamma(q_m+1)} \\
&\qquad + \frac{(L_2 r + M_2) T^{q_m+1}}{\Gamma(q_m+2)} \\
&\leq r_m
\end{aligned}$$

Now, for $x_1, x_2 \in X$, we obtain

$$\begin{aligned}
&||(\theta x_1)(t) - (\theta x_2)(t)|| \\
&\leq ||g(x_1) - g(x_2)|| + ||\sum_{i=0}^{m} y_i|| \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_m}^{T} (t-s)^{q-1} \\
&\quad \Big[||f(t, x_1(t)) - f(t, x_2(t))|| \\
&+ \int_{t_m}^{T} ||k(\sigma, s, x_1(s)) - k(\sigma, s, x_2(s))|| d\sigma\Big] ds \\
&\leq \Lambda_{b,L_1,L_2,T,q_m} ||x_1 - x_2||
\end{aligned}$$

where

$$\Lambda_{b,L_1,L_2,T,q_m} = b + ||\sum_{i=0}^{m} y_i|| + \frac{L_1 T^{q_m}}{\Gamma(q_m+1)}$$

$$+ \frac{L_2 q_m T^{q_m+1}}{\Gamma(q_m+2)}$$

depends only on the parameters involved in the problem as $\Lambda_{b,L_1,L_2,T,q_m} < 1$. So the conclusion of the theorem follows by the contraction mapping principle. $\diamond$

Theorem 3.2: Assume that $(H_4), (H_5)$ hold. Then the Cauchy problem (1.1) has atleast one solution on $[0, T]$.

Proof :

$$\begin{aligned}
\text{Fix } r &\geq ||x_0|| + G + ||\sum_{i=0}^{m} y_i|| \\
&\quad + \frac{||\mu||_{L'} T^{q_m}}{\Gamma(q_m+1)} + \frac{q||\sigma||_{L'} T^{q_m+1}}{\Gamma(q_m+2)},
\end{aligned}$$

Consider $B_r = \{x \in X : ||x|| < r\}$.
We define the operators Φ and Ψ on B_r as

$$\begin{aligned}
(\Phi x)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \\
&\Big[f(s, x(s)) + \int_0^t k(\sigma, s, x(s)) d\sigma\Big] ds \\
(\Psi x)(t) &= x_0 - g(x) + \sum_{i=1}^{m} y_i
\end{aligned}$$

For $x, y \in B_r$. We find that

$$\begin{aligned}
&||\Phi x + \Psi y|| \\
&\quad = ||x_0 - g(x) + \sum_{i=1}^{m} y_i + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \\
&\quad\quad \Big[f(s, x(s)) + \int_0^t k(\sigma, s, x(s)) d\sigma\Big] ds|| \\
&\quad \leq ||x_0|| - ||g(x)|| + ||\sum_{i=1}^{m} y_i|| \\
&\quad\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [||f(s, x(s))|| \\
&\quad\quad + \int_0^t ||k(\sigma, s, x(s))|| d\sigma] ds \\
&\quad \leq ||x_0|| + G + ||\sum_{i=0}^{m} y_i|| + \frac{||\mu||_{L'} T^{q_m}}{\Gamma(q_m+1)} \\
&\quad\quad + \frac{q||\sigma||_{L'} T^{q_m+1}}{\Gamma(q_m+2)} \\
&\quad \leq r
\end{aligned}$$

Thus, $\Phi x + \Psi y \in B_r$.

If follows that the assumption (H_5) that Ψ is a contraction mapping. continuity of f and k demanded in (1.1) implies that the operator Φ is continuous.

Also Φ is uniformly bounded on B_r as

$$||\Phi|| \leq \left(\frac{||\mu||_{L'} T^{q_m}}{\Gamma(q_m+1)} + \frac{q||\sigma||_{L'} T^{q_m+1}}{\Gamma(q_m+2)} \right)$$

Now, we prove the compactness of the operator Φ. Since f and k are respectively bounded on the compact sets $\Omega_1 = [0, T] \times X$ and $\Omega_2 = [0, T] \times [0, T] \times X$, therefore, we define

$$\sup_{(t,x) \in \Omega_1} ||f(t, x)|| = C_1, \quad \sup_{(t,s,x) \in \Omega_2} ||k(t, s, x)|| = C_2$$

.

For $t_2, t_1 \in [0, T],\ x \in B_r$,

Now we see $(\Phi x)(t_2)$ and $(\Phi x)(t_1)$ equations are

$$\begin{aligned}
(\Phi x)(t_2) &= \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} [f(s, x(s)) \\
&\quad + \int_0^{t_2} k(\sigma, s, x(s)) d\sigma] ds \\
(\Phi x)(t_1) &= \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} [f(s, x(s)) \\
&\quad + \int_0^{t_1} k(\sigma, s, x(s)) d\sigma] ds
\end{aligned}$$

Taking norm on both sides, we get

$$\begin{aligned}
&||(\Phi x)(t_2) - (\Phi x)(t_1)|| \\
&= \frac{1}{\Gamma(q)}|| \int_0^{t_2} (t_2 - s)^{q-1}[f(s, x(s)) \\
&\qquad + \int_0^{t_2} k(\sigma, s, x(s))d\sigma]ds \\
&\quad - \int_0^{t_1} (t_1 - s)^{q-1}[f(s, x(s)) \\
&\quad + \int_0^{t_1} k(\sigma, s, x(s))d\sigma]ds|| \\
&= \frac{1}{\Gamma(q)}|| \int_{t_1}^{t_2} (t_2 - s)^{q-1}[f(s, x(s)) \\
&\quad + \int_0^{t_2} k(\sigma, s, x(s))d\sigma]ds \\
&- \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}]f(s, x(s))ds \\
&- \int_0^{t_1} [(t_1 - s)^{q-1} \int_{t_1}^0 k(\sigma, s, x(s))d\sigma \\
&\quad - (t_2 - s)^{q-1} \int_{t_2}^0 k(\sigma, s, x(s))d\sigma]ds|| \\
&\leq \frac{C_1}{\Gamma(q+1)} \left|2(t_2 - t_1)^q + t_1^q - t_2^q\right| \\
&\quad + \frac{qC_2}{\Gamma(q+2)} \left|2(t_2 - t_1)^{q+1} + t_1^{q+2} - t_2^{q+1}\right| \\
&\leq \frac{2C_1}{\Gamma(q+1)} \left|t_2 - t_1\right|^q + \frac{2qC_2}{\Gamma(q+2)} \left|t_2 - t_1\right|^{q+1}
\end{aligned}$$

which is independent of x. so Φ is relatively compact on B_r. Hence, By Arzela Ascoli Theorem, Φ is compact on B_r. Thus all the assumption of Theorem(2.1) are satisfied. Consequently, the conclusion of Theorem(2.1) applied and the Cauchy problem (1.1) has atleast one solution. $\diamond$

IV. Example

Consider the following fractional integro-differential equation with nonlocal impulsive condition of the form

$$^cD_t^q x(t) = \frac{1}{(t+2)^2}\frac{|x|}{1+|x|} + \int_0^t e^{\frac{1}{8}x(s)}ds \tag{4.1}$$

$$x(t_k^+) = x(t_k^-) + \frac{1}{4}, \tag{4.2}$$

$$x(0) = x_0 - \sum_{i=1}^m c_i x(t_i) \tag{4.3}$$

Take $J = [0, 1]$ and so $T = 1$.
Set

$$f(t, x(t)) = \frac{1}{(t+2)^2}\frac{|x|}{1+|x|}, t \in J^{'}.x \in X$$
$$k(t, s, x(s)) = \int_0^t e^{\frac{1}{8}x(s)}ds$$

Let $x_1, x_2 \in X$ and $t \in J^{'}$. Then we have

$$\begin{aligned}
&||f(t, x_1(t)) - f(t, x_2(t))|| \\
&= \left|\frac{1}{(t+2)^2}\frac{|x_1|}{1+|x_1|} - \frac{1}{(t+2)^2}\frac{|x_2|}{1+|x_2|}\right| \\
&= \left|\frac{1}{(t+2)^2}\left(\frac{|x_1|}{1+|x_1|} - \frac{|x_2|}{1+|x_2|}\right)\right| \\
&\leq \frac{1}{(t+2)^2}|x_1 - x_2| \\
&\leq \frac{1}{4}|x_1 - x_2|
\end{aligned}$$

$$\begin{aligned}
&||k(t, s, x_1(s)) - k(t, s, x_2(s))|| \\
&= \left|\int_0^t e^{\frac{1}{8}x_1(s)}ds - \int_0^t e^{\frac{1}{8}x_2(s)}ds\right| \\
&\leq \frac{1}{8}|x_1 - x_2|
\end{aligned}$$

Hence the conditions $(H_1) - (H_2)$ hold.
Note that $L_1 = \frac{1}{4}$ and $L_2 = \frac{1}{8}$.
Choose $b = \frac{1}{4}$.
We shall check that condition

$$b + ||\sum_{i=0}^m y_i|| + \frac{L_1 T^{q_i}}{\Gamma(q_i+1)} + \frac{L_2 q_i T^{q_i+1}}{\Gamma(q_i+2)} < 1$$

is satisfied. Indeed for i=0.

$$b + ||\sum_{i=0}^m y_i|| + \frac{L_1 T^{q_i}}{\Gamma(q_i+1)} + \frac{L_2 q_i T^{q_i+1}}{\Gamma(q_i+2)} < 1 \Leftrightarrow \frac{7}{8}, \tag{4.4}$$

which is satisfied for some $q_0 \in [0, t_1)$.

Then by Theorem 3.1, the problem equations $(4.1) - (4.3)$ has an unique solution on $[0, 1]$ for the values of q satisfying equation (4.4). $\diamond$

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