Deductive Group Invariance Analysis of Boundary Layer Equations of a Special Non-Newtonian Fluid over a Stretching Sheet

R. M. Darji and M. G. Timol

***Abstract*—The general group-theoretic transformations are developed for the solution of highly non-linear partial differential equations governing the boundary layer flow of the special Non-Newtonian fluid so-called Sisko fluid past over stretching sheet. The application of a one-parameter group reduces the number of independent variables by one, and consequently the system of governing non-linear partial differential equations with boundary conditions reduces to a non-linear ordinary differential equation with appropriate boundary conditions. The numerical solution for the present flow situation is derived systematically from similarity requirement using Runge-Kutta scheme with shooting method.**

***Index Terms*—Group symmetry, sisko fluid, boundary-layer flow, similarity solution, skin-friction.**

***MSC 2010 Codes* – 76A05, 76M55, 54H15**

Nomenclature

a, b	flow parameters
f	dimensionless stream function
G	sub-group
n	power-law index
T_a	group transformation
U(x)	velocity of main stream
u	velocity in x direction
v	velocity in y direction
x, y	reference coordinate distance

Greek symbols

α, β	similarity conditional constant
η	similarity variable
∞	condition at infinity
ξ	invariant conditional function
τ_{yx}	shear stress

Subscript

w	condition at sheet wall

superscript

'	differentiation with respect to η

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I. Introduction

DEDUCTIVE group transformation analysis, also called symmetry analysis, is based on general group of transformation that was developed by Sophus Lie to find point transformations that map a given differential equation to itself. This method unifies almost all known exact integration techniques for both ordinary and partial differential equations [1]. Group analysis is the only rigorous mathematical method to find all symmetries of a given differential equation and no adhoc assumptions or a prior knowledge of the equation under investigation is needed. The boundary layer equations are especially interesting from a physical point of view because they have the capacity to admit a large number of invariant solutions, i.e. similarity solutions. In the present context, invariant solutions are meant to be a reduction to a simpler equation such as an ordinary differential equation. Prandtls boundary layer equations admit more and different symmetry groups. Symmetry groups or simply symmetries are invariant transformations, which do not alter the structural form of the equation under investigation [2].

Newtons law of viscosity states that shear stress is proportional to velocity gradient. Fluids that obey this law are known as Newtonian fluids. Amongst Newtonian fluids, we can cite water, benzene, ethyl alcohol, hexane and most solutions of simple molecules. Numerous fluids violate Newtons law of viscosity. On the other hand, fluids that do not obey Newtons law are known as Non-Newtonian fluids. Amongst Non-Newtonian fluids, we can cite some lubricants, whipped cream, some clays, some drilling mud, many paints, synovial fluid, suspensions of corn starch, sand in water, paper pulp in water, latex paint, ice, blood, syrup, molasses, blood plasma, custard etc. The characteristic of Non-Newtonian fluids are defined according to the non-linear stress-strain relationship. Some of the Non-Newtonian fluids are classify according to their stress-strain relationship are [3]; Power-law fluids, Eyring fluids, Sisko fluids, Prandtl - Eyring fluids, Sutterby fluids, Prandtl fluids, Ellis fluids, Williamson fluids, Reiner- Philippoff fluids, Powell-Eyring fluids etcetera. The non- linear character of the partial differential equations governing the motion of a fluid produces difficulties in solving the equations. In the field of fluid mechanics, most of the researchers try to obtain the similarity solutions in such cases. In case of deductive group of transformations, the group-invariant solutions are nothing but the well-known similarity solutions [4]. The most general form of Lie group of transformations, known as deductive group

transformation, is used in this paper to find out the full set of symmetries of the problem and then to study the appropriate group-invariant or more specifically similarity solutions.
Derivation of boundary layer equations of Newtonian fluids and their solutions using similarity transformations by Prandtl opened a new era in fluid mechanics in the beginning of previous century. After half a century, his method in deriving the equations and finding exact solutions is applied to Non-Newtonian fluids. Several boundary layer equations are derived for different Non-Newtonian fluids. The Newtonian fluids are those for which the shear stress is proportional to the shear rate and its model is given by

$$\tau_{yx} = \mu \frac{\partial u}{\partial y}$$

The fluids, for which shear stress is given by anything else except above relation is known as Non-Newtonian fluid. In particular, for two-dimensional power-law Non-Newtonian fluid the shear stress is given by

$$\tau_{yx} = K\left(\frac{\partial u}{\partial y}\right)^n$$

Here *n* is called the flow behavior index and K is called the flow consistency parameter.
The fluid having the property whose shear stress given by combining above two shear stresses is referred as special type of power-law Non-Newtonian fluid and for that the shear stress relation is given by

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} + K\left(\frac{\partial u}{\partial y}\right)^n$$

In general,

$$\tau_{yx} = a\frac{\partial u}{\partial y} + b\left(\frac{\partial u}{\partial y}\right)^n$$

Here a, b and n are constant defined differently for different fluid. It is interesting to observe that, for $a = 0$ $(b \neq 0, n \neq 1)$, the above equation will be equation of power-law fluids and for $b = 0$ $(a \neq 0)$ or for $n = 1$ it will be equation of Newtonian fluid.

In the literature, such types of fluids are found [5] and are recognize as third grade fluids or 'Sisko' fluids [6] . This type of fluid has been considered previously for flow of lubricants and it has great importance in lubricants theory. The multi grade engine oils are belonging to these categories of fluids. The analysis of boundary layer equations of such a fluid had been done earlier [7], but it is a special case of Sisko fluid. At present, we are analyzing the boundary layer equations of most general form of Sisko fluid, whose strain-stress relationship is given by above equation.

II. Governing Equations

The multiple deck boundary layer concepts has been applied to second and third grade fluids by Pakdemirli [8] and as a first approximation the assumption that second grade terms negligible compared to viscous third grade term leads the following dimensionless boundary layer equations [4], [7]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[a\frac{\partial u}{\partial y} + b\left(\frac{\partial u}{\partial y}\right)^n\right] + U\frac{dU}{dx} \tag{2}$$

The boundary conditions are

$$\left.\begin{array}{lll} u = U_w(x); & v = 0 \quad \text{at} & y = 0 \\ u = 0 & \quad\quad\;\; \text{at} & y \to \infty \end{array}\right\} \tag{3}$$

Introducing the dimensionless stream function $\psi(x, y)$ such that $u = \frac{\partial \psi}{\partial y},$, $v = -\frac{\partial \psi}{\partial x}$ which satisfies equation (1) identically, above equations (1) - (3) reduce to

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = a\frac{\partial^3 \psi}{\partial y^3} + bn\frac{\partial^3 \psi}{\partial y^3}\left(\frac{\partial^2 \psi}{\partial y^2}\right)^{n-1} + U\frac{dU}{dx} \tag{4}$$

$$\left.\begin{array}{lll} \frac{\partial \psi}{\partial y} = U_w(x); & \frac{\partial \psi}{\partial x} = 0 \quad \text{at} & y = 0 \\ \frac{\partial \psi}{\partial y} = 0; & \quad\quad\quad\;\; \text{at} & y \to \infty \end{array}\right\} \tag{5}$$

III. Deductive Group Symmetry analysis

Our method of solution depends on the application of a one-parameter deductive group of transformation to the partial differential equation (4). Under this transformation the two independent variables will be reduced by one and the differential equation (4) will transforms into the ordinary differential equations.

A. The group systematic formulation

The procedure is initiated with the group C_G, a class of transformations of one-parameter *a* of the form:

$$C_G: \quad T_a(Q) = \aleph^Q(a)\,Q + \Re^Q(a) = \bar{Q} \tag{6}$$

Where Q stands for x, y, ψ, U, $\aleph's$ and $\Re's$ are real-valued and at least differential in the real argument *a*.

B. The invariance analysis

To transform the differential equation, transformations of the derivatives are obtained from C_G via chain-rule operations:

$$\left.\begin{array}{l} \bar{Q}_{\bar{i}} = \left(\frac{\aleph^Q}{\aleph^i}\right)Q_i \\ \bar{Q}_{\bar{i}\bar{j}} = \left(\frac{\aleph^Q}{\aleph^i\aleph^j}\right)Q_{ij} \end{array}\right\} \quad Q = \psi, U; \quad i, j = x, y \tag{7}$$

Equation (4) is said to be invariantly transformed, for some functions $\xi(a)$ whenever,

$$\frac{\partial \bar{\psi}}{\partial \bar{y}}\frac{\partial^2 \bar{\psi}}{\partial \bar{y}\partial \bar{x}} - \frac{\partial \bar{\psi}}{\partial \bar{x}}\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} - a\frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3} - bn\left(\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2}\right)^{n-1}\frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3} - \bar{U}\frac{d\bar{U}}{d\bar{x}}$$
$$= \xi(a)\left[\begin{array}{l} \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} \\ -a\frac{\partial^3 \psi}{\partial y^3} - bn\left(\frac{\partial^2 \psi}{\partial y^2}\right)^{n-1}\frac{\partial^3 \psi}{\partial y^3} - U\frac{dU}{dx} \end{array}\right]$$

Substituting the values from the equation (7) in above equation, yields

$$\frac{(\aleph^{\psi})^2}{\aleph^x(\aleph^y)^2}\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right)-a\frac{\aleph^{\psi}}{(\aleph^y)^3}\frac{\partial^3\psi}{\partial y^3}$$
$$-bn\frac{(\aleph^{\psi})^n}{(\aleph^y)^{2n+1}}\left(\frac{\partial^2\psi}{\partial y^2}\right)^{n-1}\frac{\partial^3\psi}{\partial y^3}-\left\{\aleph^U U(x)+\Re^U\right\}\left(\frac{\aleph^U}{\aleph^x}\frac{dU}{dx}\right)$$
$$=\xi(a)\left[\begin{array}{l}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-a\frac{\partial^3\psi}{\partial y^3}\\ -bn\left(\frac{\partial^2\psi}{\partial y^2}\right)^{n-1}\frac{\partial^3\psi}{\partial y^3}-U\frac{dU}{dx}\end{array}\right]$$

The invariance of above equation together with boundary conditions (5), implies that

$$\left.\begin{array}{l}\frac{(\aleph^{\psi})^2}{\aleph^x(\aleph^y)^2}=\frac{\aleph^{\psi}}{(\aleph^y)^3}=\frac{(\aleph^{\psi})^n}{(\aleph^y)^{2n+1}}=\frac{(\aleph^U)^2}{\aleph^x}=\xi(a)\\ \Re^U=\Re^y=0\end{array}\right\}\quad(8)$$

These yields,

$$\aleph^x=(\aleph^y)^3,\quad \aleph^{\psi}=(\aleph^y)^2,\quad \aleph^U=\aleph^y$$

Finally, we get the one-parameter group G, which is subgroup of C_G and transforms invariantly the differential equation (4) and the auxiliary conditions (5).

$$G:\quad\left\{\begin{array}{l}G_H:\ \left\{\begin{array}{l}\bar{x}=(\aleph^y)^3x+\Re^x\\ \bar{y}=\aleph^y y\end{array}\right.\\ \bar{\psi}=(\aleph^y)^2\psi+\Re^{\psi}\\ \bar{U}=\aleph^y U\end{array}\right.\quad(9)$$

C. The complete set of absolute invariants

Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation. Now we have proceeded in our analysis to obtain a complete set of absolute invariants. If $\eta=\eta(x,y)$ is the absolute invariant of the independent variables then the absolute invariants for dependent variables ψ and U are given by, See [9]

$$\Gamma_j(x,y,\psi,U)=f_j(\eta),\quad j=1,2\quad(10)$$

The application of a basic theorem in group theory, Morgan [10], Moran and Gaggioli [11] states that: A function $\Gamma(x,y,\psi,U)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear partial differential equation,

$$\sum_{i=1}^{4}(\alpha_iQ_i+\beta_i)\frac{\partial\Gamma}{\partial Q_i}=0,\quad Q_i=x,y,\psi,U\quad(11)$$

Where

$$\alpha_i=\left.\frac{\partial\aleph^i}{\partial a}\right|_{a=a^0}\quad\text{and}\quad\beta_i=\left.\frac{\partial\Re^i}{\partial a}\right|_{a=a^0}\quad i=1,2,3,4\quad(12)$$

and $'a^0{}'$ denotes the value of $'a'$ which yields the identity element of the group G.
Since $\Re^U=\Re^y=0$ implies that $\beta_2=\beta_4=0$ and from (12) we get

$$\alpha_1=3\alpha_2=\frac{3}{2}\alpha_3=\alpha_4\quad(13)$$

Now the absolute invariant of independent variable owing the equation (11) is $\eta=\eta(x,y)$ if it satisfies the first order linear partial differential equation

$$(\alpha_1x+\beta_1)\frac{\partial\eta}{\partial x}+(\alpha_2y+\beta_2)\frac{\partial\eta}{\partial y}=0.\quad(14)$$

Substitute the values in (14) from (13)

$$3(x+\beta)\frac{\partial\eta}{\partial x}+y\frac{\partial\eta}{\partial y}=0,\quad\beta=\frac{\beta_1}{3\alpha_2}\quad(15)$$

Applying the variable separable method one can obtain

$$\eta(x,y)=y(x+\beta)^{-1/3}\quad(16)$$

Further, the absolute invariants of dependent variables owing the equation (11) are given by

$$\psi(x,y)+\beta'=(x+\beta)^{2/3}f(\eta);\qquad U=(x+\beta)^{1/3}$$
$$\text{where }\beta'=\beta_3/2\alpha_2$$

Hence, the final similarity transformations are:

$$\left.\begin{array}{l}\psi(x,y)+\beta'=(x+\beta)^{2/3}f(\eta)\\ U=(x+\beta)^{1/3}\\ \quad\text{where}\eta(x,y)=y(x+\beta)^{-1/3}\end{array}\right\}\quad(17)$$

D. The reduction to an ordinary differential equation

Using the similarity transformations (17) in equation (4) and using the fact for stretching sheet, outer velocity is zero [7], i.e. $U(x)=0$ yields to following non-linear ordinary differential equations

$$3\left[a+bn(f'')^{n-1}\right]f'''+2ff''-(f')^2=0\quad(18)$$

Further to transform the boundary conditions in to constant form the velocity near sheet surface must be proportional to $(x+\beta)^{1/3}$ i.e. of the form $U_w(x)=\lambda(x+\beta)^{1/3}$. Hence, the auxiliary conditions reduce to,

$$f(0)=0,\quad f'(0)=\lambda,\quad f'(\infty)=0\quad(19)$$

Where primes denote ordinary derivative with respect to the similarity variable η and λ is arbitrary constant.
It is interesting to observe that for a=1, b=1 and n=3 the differential equation (18) will reduce to that of obtained by [7] and thus the analysis of [7] will reduced to a particular case of the present study.

IV. Numerical solution

The interesting physical quantities are the shear stress at the surface boundary and is given by (See [7])

$$\tau_w=\tau_{yx}\quad\text{at}\quad y=0=\left[a\frac{\partial u}{\partial y}+b\left(\frac{\partial u}{\partial y}\right)^n\right]_{y=0}$$

In terms of similarity variable

$$\tau_w=a\,f''(0)+b[f''(0)]^n\quad(20)$$

And the horizontal component of velocity,

$$u=U(x)\,f'(\eta)\quad(21)$$

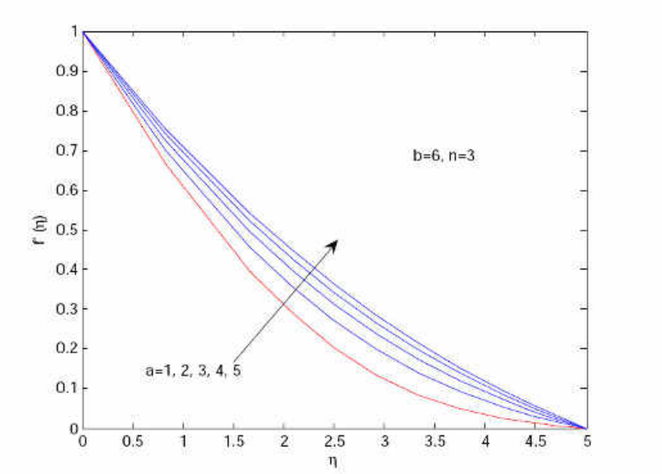

Fig. 1. Velocity profile for various *a*

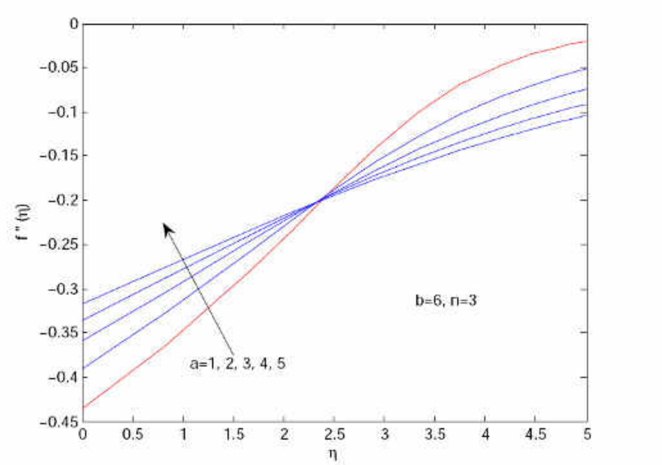

Fig. 2. Skin-friction profile for various *a*

Since the reduced similarity equation is highly non-linear, a numerical treatment is more appropriate. The numerical solution for various flow consistencies parameters is obtained using Runge-Kutta fourth order shooting method [12]. The numerical solution was done using Matlab computational software. A step size of $\Delta\eta$ = 0.001 was selected to be satisfactory for a convergence criterion of 10^{-7} in nearly all cases. The value of η_∞ was found to each iteration loop by assignment statement $\eta_\infty = \eta_\infty + \Delta\eta$. The maximum value of η_∞, to each group of parameter, determined when the values of unknown boundary conditions at $\eta = 0$ not change to successful loop with error less than 10^{-7}. In the following section, the results are discussed in detail.
Also the horizontal velocity is function of x, $x^{1/3}f'(\eta)$ is plotted from zero at the wall and levels off at its mainstream value for larger value say from 0 to 5.

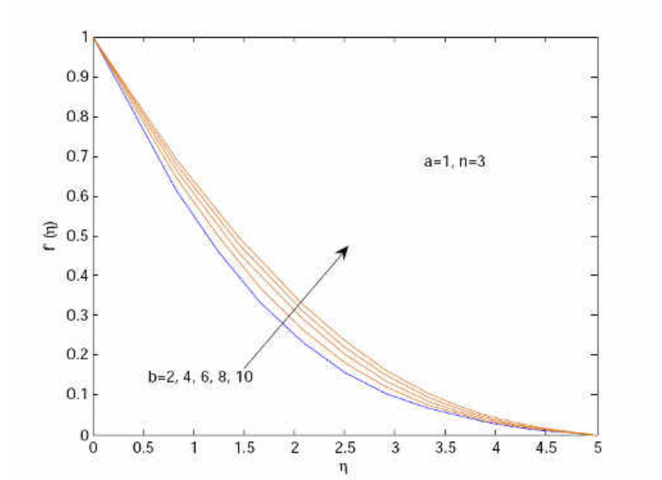

Fig. 3. Velocity profile for various *b*

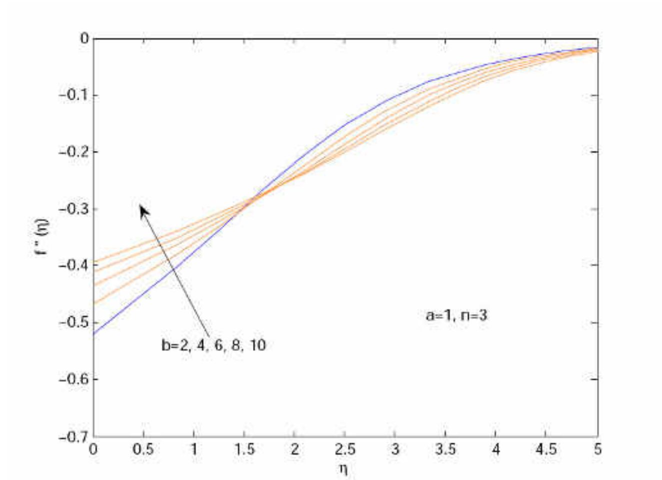

Fig. 4. Skin-friction profile for various *b*

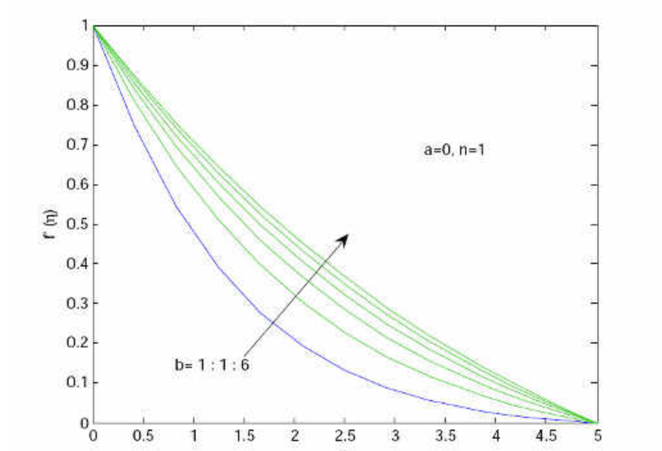

Fig. 5. Velocity profile for Newtonian fluids

V. Results and Discussion

- Using the algorithm discussed above (which is unconditionally stable), the numerical solutions are obtained for several sets of values of flow parameters a , b and

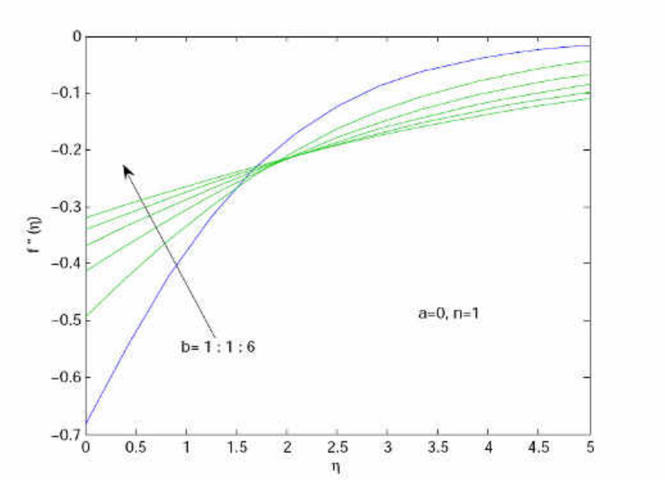

Fig. 6. Skin-friction profile for Newtonian fluids

n in order to bring out the salient features of the flow characteristics the numerical values are plotted.

- Figs. 1,3,5 are the graphical representation of the horizontal velocity profiles $f'(\eta)$ for different values of *n* [i. e. for Newtonian *n* =1 (Fig. 5) and for Non-Newtonian $n \neq 1$ Figs. 1, 3]. From these figures we notice that an increase in any flow parameter leads to a decrease in $f'(\eta)$.
- Fig 2, 4, 6 are the graphical representation of the slope of velocity $f''(\eta)$ from which we can observe the local skin friction coefficient in terms of $f''(0)$ at sheet wall. From these figures we notice that local skin friction increase with the increase of any one flow parameter.
- Further it is interesting to observe that for *a*=1, *b*=1 and *n*=3 equations (18) with boundary condition (19) will reduced to that of obtained by Yurusoy and Pakdemirli [7].
- It is worth to note that all solutions have derived for non-dimensional quantities and hence these results are applicable for all types of under considered Non-Newtonian fluids.

VI. Conclusion

In the present work we have transformed the highly non-linear PDEs governing the particular fluid flow of boundary layer theory into an ODE by searching the group of transformation subject to the similarity requirement. The reduced non-linear ODE-BVP is numerically solved by Runge-Kutta shooting method using Matlab computational algorithm.

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