Fixed Point Theorems in Generating Spaces of Quasi-Metric Family

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***Abstract*—In this paper, we give the definition of contraction mapping in generating space of quasi-metric family (G.S.Q-M.F) and establish Banach type fixed point theorems in complete generating space of sub-strong quasi-metric family. Uniqueness of these theorems are studied. We also establish some fixed point theorems like Brower, Kannan, Caccioppoli, Edelstein and Caristi type in this setting and uniqueness are proved.**

***Index Terms*—Generating spaces of Quasi-metric family, Contraction mapping, Banach, Kannan, Caccioppoli, Edelstein, Caristi type fixed point theorems.**

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I. Introduction

In 1997, Chang et al.[1] first introduced a definition of generating space of quasi-metric family which is a most generalized structure unifying those of fuzzy metric space in the sense of Kaleva & Seikkala[2] and menger probabilistic metric spaces[3], and studied some properties and examples of the spaces. They also established some fixed point theorems and coincidence point theorems in generating space of quasi-metric family.

Jung et al.[4][5][6] established different types of minimization theorems and fixed point theorems in this space. They proved un-convex minimization theorems in complete generating space of quasi-metric family and Downing-Kirk's fixed point theorem in the same spaces. The versions of the work are studied in Probabilistic metric spaces. In 1999, they [7] introduced the concept of order relation in this space and established common fixed point theorems for single-valued and set-valued mappings. They also proved coincidence point theorems in 2000 in such spaces.

In this paper, we give the definition of contraction mapping in generating space of quasi-metric family (G.S.Q-M.F) and establish Banach type fixed point theorems in complete generating space of sub-strong quasi-metric family. Uniqueness of these theorems are studied. We also establish some fixed point theorems like Kannan, Caccioppoli, Edelstein and Caristi type in this setting and uniqueness are proved.

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The organization of the paper is as follows: In section II, comprises some preliminary results. We establish some fixed point theorems in G.S.Q-M.F in section III.

II. Some Preliminary Results

In this section, some preliminary results are given which are related to this paper.

Definition 2.1[1] Let X be a nonempty set and $\{d_\alpha :\ \alpha \in\ (0,1]\}$ be a family of mappings from $X\times\ X$ into R^+. Then $(X,\ d_\alpha :\ \alpha \in\ (0,1])$ is called a generating space of quasi-metric family(G.S.Q-M.F) if it satisfies the following conditions:

(QM-1) $d_\alpha(x,\ y) = 0 \quad \forall\alpha \in (0,\ 1]$ iff $x = y$;

(QM-2) $d_\alpha(x,\ y) = d_\alpha(y,\ x)\ \ \forall x,\ y \in\ X$ and $\forall\alpha \in (0,\ 1]$;

(QM-3) For any $\alpha \in (0,\ 1]$ there exists a $\beta \in (0,\ \alpha]$ such that

$$d_\alpha(x,\ y) \leq\ d_\beta(x,\ z)\ + d_\beta(z,\ y),\ \ \forall x,\ y,\ z \in X$$

(QM-4) For any $x,\ y \in\ X$, $d_\alpha(x,\ y)$ is non increasing and left continuous in α.

Remark 2.1 Clearly from (QM-3) and (QM-4) we get, for any $\alpha \in (0,\ 1]$ and $n \in Z^+$, there exists $\beta \in (0,\ \alpha]$ such that

$$d_\alpha(x_m,\ x_{m+p}) \leq \sum_{i=0}^{p-1} d_\beta(x_{m+i},\ x_{m+i+1})$$

$\forall\ x_{m+i} \in\ X(i = 1,\ 2, \ldots,\ n)$.

Definition 2.2 Let $(X,\ d_\alpha :\ \alpha \in\ (0,1])$ be a generating space of quasi-metric family(G.S.Q-M.F), then it is called a generating space of sub-strong quasi-metric family, strong quasi-metric family and semi-metric family respectively, if (QM-3) is strengthened to (QM-3u), (QM-3t) and (QM-3e), where

(QN-3u) for any $\alpha \in (0,\ 1]$ there exists $\beta \in (0,\ \alpha]$ such that $d_\alpha(x_m,\ x_{m+p}) \leq \sum_{i=0}^{p-1} d_\beta(x_{m+i},\ x_{m+i+1})$ for any $p \in Z^+$, $x_{m+i} \in X(i = 1, 2, ..., p-1)$;

(QM-3t) for any $\alpha \in (0,\ 1]$ there exists a $\beta \in (0, \alpha]$ such that $d_\alpha(x,\ z)\ \leq\ \ d_\alpha(x,\ y) + d_\beta(y,\ z)$ for $x,\ y,\ z \in X$;

(QM-3e) for any $\alpha \in (0,\ 1]$, it holds that $d_\alpha(x,\ z) \leq d_\alpha(x,\ y) + d_\alpha(y,\ z)$ for $x,\ y,\ z \in X$.

Definition 2.3 Let $(X,\ d_\alpha\ :\ \alpha\ \in\ (0,1])$ be a generating space of semi-metric family(G.S.S-M.F), where $(d_\alpha :\ \alpha \in\ (0,1])$ satisfies

(QM-5): if $x \neq\ y$ in X then $d_\alpha(x,\ y)\ > 0\ \forall\alpha \in (0,1]$.

Then $(X,\ d_\alpha :\ \alpha \in\ (0,1])$ is called a generating space of metric family(G.S.M.F) and $(d_\alpha : \alpha \in (0,1])$ is called a family of metric on X.

Definition 2.4[1] Let $(X,\ d_\alpha : \alpha \in (0,1])$ be a generating space of quasi-metric family(G.S.Q-M.F).

(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said

(a) to converge to $x \in X$ denoted by $\lim\limits_{n\to\infty} x_n = x$ if $\lim\limits_{n\to\infty} d_\alpha(x_n,\ x) = 0$ for each $\alpha \in (0,\ 1]$;

(b) to be a Cauchy sequence if $\lim\limits_{m,\ n\to\infty} d_\alpha(x_n,\ x_m) = 0$ for each $\alpha \in (0,\ 1]$.

(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B.

Definition 2.5 Let $(X_1,\ d_\alpha^1 : \alpha \in (0,1])$ and $(X_2,\ d_\alpha^2 : \alpha \in (0,1])$ be two G.S.Q-M.F and $T : X_1 \to X_2$ be an operator. Then T is said to be continuous at $x \in X_1$ if for any sequence $\{x_n\}$ of X_1 with $x_n \to x$ i.e. with $\lim\limits_{n\to\infty} d_\alpha^1(x_n,\ x) = 0\ \ \forall \alpha \in (0,\ 1]$ implies $T(x_n) \to T(x)$. i.e. $\lim\limits_{n\to\infty} d_\alpha^2(T(x_n),\ T(x)) = 0\ \ \forall \alpha \in (0,\ 1]$. If T is continuous at each point of X_1, then T is said to be continuous on X_1.

Lemma 2.1 Let $(X,\ d_\alpha : \alpha \in (0,1])$ be a generating space of strong quasi-metric family. Then for each $\alpha \in (0,1]$, $d_\alpha(x,\ y)$ is a continuous function on $X \times X$ i.e. if the sequences $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y in X then the sequence $\{d_\alpha(x_n,\ y_n)\}$ converges to $d_\alpha(x,\ y)$ for all $\alpha \in (0,\ 1]$.

Proof Since $(X,\ d_\alpha : \alpha \in (0,1])$ be a generating space of strong quasi-metric family, for each $\alpha \in (0,1]$ there exists $\beta \in (0,\ \alpha]$ such that

$d_\alpha(x_n,\ y_n) \le d_\alpha(x_n,\ y) + d_\beta(y,\ y_n) \le d_\alpha(x,\ y) + d_\beta(x_n,\ x) + d_\beta(y,\ y_n)$

$\Rightarrow d_\alpha(x_n,\ y_n) - d_\alpha(x,\ y) \le d_\beta(x_n,\ x) + d_\beta(y,\ y_n)$.

Similarly we can prove

$d_\alpha(x,\ y) - d_\alpha(x_n,\ y_n) \le d_\beta(x_n,\ x) + d_\beta(y_n,\ y)$

$\Rightarrow |d_\alpha(x_n,\ y_n) - d_\alpha(x,\ y)| \le d_\beta(x_n,\ x) + d_\beta(y_n,\ y)$

$\Rightarrow \lim\limits_{n\to\infty} |d_\alpha(x_n,\ y_n) - d_\alpha(x,\ y)| \le \lim\limits_{n\to\infty} d_\beta(x_n,\ x) + \lim\limits_{n\to\infty} d_\beta(y_n,\ y) = 0$.

Hence the lemma.

III. Fixed Point Theorems

In this section, we establish some fixed point theorems like Banach, Brower, Kannan, Caccioppoli, Edelstein and Caristi type and uniqueness of these theorems are proved.

Definition 3.1 Let $(X,\ d_\alpha : \alpha \in (0,1])$ be a generating space of quasi-metric family and $T : X \to X$. The operator T is said to satisfy Lipschitz Condition with Lipschitz constant δ if

$d_\alpha(Tx,\ Ty) \le \delta\ d_\alpha(x,\ y)\ \ \forall \alpha \in (0,\ 1],\ \forall\ x,\ y \in X$.

If the above condition is satisfied when the Lipschitz constant $\delta \in (0,\ 1)$ then T is called a *Contraction mapping.*

Remark 3.1 Contraction mapping is continuous.

Lemma 3.1 Let $(X,\ d_\alpha : \alpha \in (0,1])$ be a generating space of sub-strong quasi-metric family and $T : X \to X$ be a contraction mapping. Then for any $x_0 \in X$, $\{T^n(x_0)\}$ is a Cauchy sequence in X.

Proof Let $x_0 \in X$ and $x_1 = T(x_0),\ x_2 = T(x_1) = T^2(x_0)$,........., $x_n = T(x_{n-1}) = T^n(x_0)$ and so on.

If $m < n$, let $p = n - m$ and $\alpha \in (0,\ 1]$

$d_\alpha(T^m(x_0),\ T^n(x_0)) \le \delta\ d_\alpha(T^{m-1}(x_0),\ T^{n-1}(x_0))$
$\le \delta^m\ d_\alpha(x_0,\ x_p)$,

for some $\delta \in (0,\ 1)$. So by (QM-3u), there exists $\beta \in (0,\ \alpha]$ such that

$d_\alpha(T^m(x_0),\ T^n(x_0)) \le \delta^m\ \{d_\beta(x_0,\ x_1) + d_\beta(x_1,\ x_2) + \ldots\ldots + d_\beta(x_{p-1},\ x_p)\}$

$\Rightarrow d_\alpha(T^m(x_0),\ T^n(x_0))$
$\le \delta^m\ \{d_\beta(x_0,\ x_1) + \delta\ d_\beta(x_0,\ x_1) + \delta^2\ d_\beta(x_0,\ x_1) + \ldots\ldots + \delta^{p-1}\ d_\beta(x_0,\ x_1)\}$

$\Rightarrow d_\alpha(T^m(x_0),\ T^n(x_0)) \le \frac{\delta^m(1-\delta^p)}{(1-\delta)}\ d_\beta(x_0,\ x_1)$.

$\Rightarrow \lim\limits_{m,n\to\infty} |T^m(x_0) - T^n(x_0)|_\alpha$

$\le \lim\limits_{m\to\infty} \dfrac{\delta^m(1-\delta^p)}{(1-\delta)}\ d_\beta(x_0,\ x_1)$

$\Rightarrow \lim\limits_{m,\ n\to\infty} |T^m(x_0) - T^n(x_0)|_\alpha = 0$.

Since $\alpha \in (0,\ 1]$ is arbitrary

$\lim\limits_{m,n\to\infty} |T^m(x_0) - T^n(x_0)|_\alpha = 0\ \ \ \forall \alpha \in (0,\ 1])$.

Hence $\{T^n(x_0)\}$ is a Cauchy sequence in $(X,\ d_\alpha : \alpha \in (0,1])$.

Theorem 3.1(Banach) Let $(X,\ d_\alpha : \alpha \in (0,1])$ be a complete generating space of sub-strong quasi-metric family and $T : X \to X$ be a contraction mapping. Then the operator T has a Unique fixed point.

Proof Let $d_\alpha(Tx,\ Ty) \le \delta\ d_\alpha(x,\ y)\ \ \forall \alpha \in (0,\ 1],\ \forall\ x,\ y \in X$, where the Lipschitz constant $\delta \in (0,\ 1)$. We have to prove T has a unique fixed point.

Let $x_0 \in X$ and $x_1 = T(x_0),\ x_2 = T(x_1) = T^2(x_0), \ldots\ldots\ldots,\ x_n = T(x_{n-1}) = T^n(x_0)$ and so on.

Then by Lemma 3.1, $\{x_n\}$ is a Cauchy sequence in $(X,\ d_\alpha : \alpha \in (0,1])$. Since $(X,\ d_\alpha : \alpha \in (0,1])$ is complete, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$.

Since T is a contraction mapping it is continuous, so $\lim\limits_{n\to\infty} T(x_n) = Tx$.

Now $Tx = \lim\limits_{n\to\infty} T(x_n) = \lim\limits_{n\to\infty} x_{n+1} = x$.

Hence x is a fixed point of T.

Uniqueness: Let $x,\ y \in X$ be any two fixed points of T. Then for any $\alpha \in (0,\ 1]$

$d_\alpha(x,\ y) = d_\alpha(T^n x,\ T^n y) \le \delta^n d_\alpha(x,\ y)$.

$\Rightarrow d_\alpha(x,\ y) \le \lim\limits_{n\to\infty} \delta^n d_\alpha(x,\ y) = 0$.

Hence $x = y$.

Theorem 3.2(Brower) Let $(X,\ d_\alpha : \alpha \in (0,1])$ be a complete generating space of sub-strong quasi-metric family and $T : X \to X$ be a contraction mapping i.e.

$$d_\alpha(Tx,\ Ty) \le \delta\ d_\alpha(x,\ y)\ \ \forall \alpha \in (0,\ 1],\ \forall\ x,\ y \in X,$$

for some $\delta \in (0,\ 1)$. Assume that

$$d_\alpha(x_0,\ Tx_0) < (1-\delta)r\ \ \forall \alpha \in (0,\ 1].$$

Then the iterative sequence starting from x_0 converges to an $x \in \overline{S} = \{y \in X : \bigvee\limits_{\alpha\in(0,\ 1]} d_\alpha(x_0,\ y) \le r\}$ and the operator T has a Unique fixed point in $\overline{S}$.

Proof We verify by induction that

$$x_1 = Tx_0, x_2 = Tx_1, \ldots, x_n = Tx_{n-1}, \ldots$$

are in $\overline{S}$.

Clearly $x_1 \in \overline{S}$. Assume that $x_1,\ x_2, ...x_{n-1}$ are in $\overline{S}$. We have to show that $x_n \in \overline{S}$. Let $\alpha \in (0,\ 1]$ then we have
$d_\alpha(x_2,\ x_1) \leq \delta\ d_\alpha(x_1,\ x_0) \leq \delta(1-\delta)r$,
$d_\alpha(x_3,\ x_2) \leq \delta^2\ d_\alpha(x_1,\ x_0) \leq \delta^2(1-\delta)r$,
.........................,
.........................,
$d_\alpha(x_n,\ x_{n-1}) \leq \delta^{n-1} d_\alpha(x_1,\ x_0) \leq \delta^{n-1}(1-\delta)r$.

Therefore by (QM-3u), there exists $\beta \in (0,\ \alpha]$ such that
$d_\alpha(x_0,\ x_n)$
$\leq d_\beta(x_0,\ x_1) + d_\beta(x_1,\ x_2) + ... + d_\beta(x_{n-1},\ x_n)$
$\Rightarrow\ d_\alpha(x_0,\ x_n)$
$< (1-\delta)r + \delta(1-\delta)r + \delta^2(1-\delta)r+, ... + \delta^{n-1}(1-\delta)r$
$\Rightarrow\ d_\alpha(x_0,\ x_n) < (1-\delta^n)r <\ r$.

So $x_n \in \overline{S}$ and the every member of the sequence $\{x_n\}$ lies in $\overline{S}$. By Theorem 3.1, the sequence $\{x_n\}$ converges and converges to x(say) and this x is the unique fixed point of T.

Now we shall prove $x \in \overline{S}$.

If possible let $x \notin \overline{S}$, then there exists an $\alpha \in (0,\ 1]$ such that $d_\alpha(x_0,\ x) >\ r$.

Now there exists $\beta \in (0,\ \alpha]$ such that
$r < d_\alpha(x_0,\ x) \leq\ d_\beta(x_0,\ x_n) + d_\beta(x_n,\ x)$
$\Rightarrow\ r < \lim_{n\to\infty} d_\beta(x_0,\ x_n)$.

So there exists a natural number N such that
$d_\beta(x_0,\ x_N) > r$ which contradict the fact that $x_n \in \overline{S}\ \ \forall\ n \in\ N$.

Theorem 3.3 Let $(X,\ d_\alpha:\ \alpha \in (0,1])$ be a complete generating space of sub-strong quasi-metric family and $T : X \to\ X$ be a continuous mapping. If for some positive integer m, T^m is a contraction mapping i.e.
$d_\alpha(T^m x,\ \ T^m y) \leq \delta\ d_\alpha(x,\ y)\ \ \forall\alpha \in (0,\ 1],\ \forall\ x,\ y \in\ X$, where the Lipschitz constant $\delta \in (0,\ 1)$.
Then T has a unique fixed point.

Proof Let $B = T^m$. If n is a positive integer, then for $x_0 \in\ X$ and $\alpha \in (0,\ 1]$,
$d_\alpha(B^n Tx_0,\ B^n x_0) =\ d_\alpha(B(B^{n-1}Tx_0),\ B(B^{n-1}x_0))$
$......... \leq\ \delta\ d_\alpha(B^{n-1}Tx_0,\ B^{n-1}x_0)$
$......... \leq\ \delta^2\ d_\alpha(B^{n-2}Tx_0,\ B^{n-2}x_0)$
....
$......... \leq\ \delta^n\ d_\alpha(Tx_0,\ x_0) \to\ 0$ as $n \to \infty$...........(1).

By Theorem 3.1, B has a unique fixed point x(say) and

$$\lim_{n\to\infty} B^n x_0 = x \tag{2}$$

As the mapping T is continuous, we get that

$$B^n Tx_0 = TB^n x_0 \to\ Tx, \quad \text{as } n \to \infty \tag{3}$$

Now there exists $\beta \in (0,\ \alpha]$ such that
$d_\alpha(Tx,\ x) \leq\ d_\beta(Tx,\ B^n Tx_0)$
$+ d_\beta(B^n Tx_0,\ B^n x_0) +\ d_\beta(B^n x_0,\ x)$
$\Rightarrow \lim_{n\to\infty}\ d_\alpha(Tx,\ x) = \lim_{n\to\infty} d_\beta(Tx,\ B^n Tx_0)$
$+ \lim_{n\to\infty} d_\beta(B^n Tx_0,\ B^n x_0) + \lim_{n\to\infty} d_\beta(B^n x_0,\ x) = 0$.
Now
$\lim_{n\to\infty}\ d_\alpha(Tx,\ x) = 0\ \ \forall\alpha \in\ (0,\ 1]$ gives $Tx = x$. Next we shall prove x is the unique fixed point of T. If y is a fixed point of T then $Ty = y$ and $By = T^m y = T^{m-1}(Ty) = T^{m-1}y = ... = y$ which contradict the fact that B has a unique fixed point.

Theorem 3.4(Caccioppoli) Let $(X,\ d_\alpha:\ \alpha \in\ (0,1])$ be a complete generating space of sub-strong quasi-metric family and $T : X \to\ X$. Suppose that for each positive integer n
$d_\alpha(T^n x,\ T^n y)\ \leq\ a_n\ d_\alpha(x,\ y)\ \ \forall\alpha \in (0,\ 1],\ \forall\ x,\ y \in\ X$
Where $a_n > 0$ is independent of $x,\ y$. If the series $\sum_{n=1}^{\infty} a_n$ is convergent , then T has a unique fixed point in X.

Proof Since the series $\sum_{n=1}^{\infty} a_n$ is convergent, there exists n_0 such that $a_n < \frac{1}{2}\ \ \forall\ n \geq\ n_0$. Let m be one of such integers, i.e. $a_m < \frac{1}{2}$. Then T^m becomes a contraction mapping. Also, T is clearly continuous. So, by Theorem 3.3, T has a unique fixed point in X.

Theorem 3.5(Edelstein) Let $(X,\ d_\alpha:\ \alpha \in\ (0,1])$ be a generating space of strong quasi-metric family satisfying (QM-5) and $T : X \to\ X$ be a continuous mapping such that
$d_\alpha(Tx,\ Ty)\ <\ d_\alpha(x,\ y)\quad \forall\alpha \in (0,\ 1],\ \ \forall\ x \neq\ y$.
Suppose that there exists a point $x \in\ X$ such that the sequence $\{T^n(x)\}$ has a subsequence converging to $\xi \in\ X$. Then ξ is the unique fixed point of T.

Proof Let $\{T^{n_k}(x)\}$ be that subsequence of $\{T^n(x)\}$ which converges to ξ. If $T^r(x) = T^{r+1}(x)$ for some r then clearly ξ is a fixed point of T.
If $T^r(x) \neq\ T^{r+1}(x)$ for any r and $T\xi \neq \xi$ then
$d_\alpha(T\xi,\ T^2\xi)\ <\ d_\alpha(\xi,\ T\xi)\quad \forall\alpha \in (0,\ 1]$.......(1).
Now $\{T^{n_k}(x)\}$ converges to ξ implies $\{T^{n_k+1}(x)\}$ converges to $T\xi$. Since $(X,\ d_\alpha:\ \alpha \in\ (0,1])$ is generating space of strong quasi-metric family, by Lemma 2.1
$\{d_\alpha(T^{n_k+1}(x),\ \ T^{n_k}(x))\}$ converges to $d_\alpha(T\xi,\ \xi)\ \ \forall\alpha \in\ (0,\ 1]$. Since $\{d_\alpha(T^{n+1}(x),\ T^n(x))\}$ is a strictly decreasing sequence and has a subsequence converging to $d_\alpha(T\xi,\ \xi)\ \ \forall\alpha \in\ (0,\ 1]$, so the sequence $\{d_\alpha(T^{n+1}(x),\ T^n(x))\}$ converges to $d_\alpha(T\xi,\ \xi)\ \ \forall\alpha \in\ (0,\ 1]$ and hence
$d_\alpha(T\xi,\ \xi)\ \leq\ d_\alpha(T^{n_k+1}(x),\ T^{n_k+2}(x))\quad \forall\alpha \in\ (0,\ 1]$.
Taking $k \to \infty$ we get
$d_\alpha(\xi,\ T\xi) \leq\ d_\alpha(T\xi,\ T^2\xi)\ \ \forall\alpha \in (0,\ 1]$ which contradict with (1). So ξ is a fixed point of T.

Uniqueness: Let $\xi,\ \eta\ \in\ X$ be any two fixed points of T. Then for any $\alpha \in\ (0,\ 1)$
$d_\alpha(\xi,\ \eta) = d_\alpha(T\xi,\ T\eta) <\ d_\alpha(\xi,\ \eta)$ which is a contradiction.

Theorem 3.6(Kannan) Let $(X,\ d_\alpha:\ \alpha \in\ (0,1])$ be a complete generating space of sub-strong quasi-metric family and $T : X \to\ X$ be a continuous mapping such that
$d_\alpha(Tx,\ Ty)\ \leq\ \delta\ [d_\alpha(x,\ Tx)\ +\ d_\alpha(y,\ Ty)]\quad \forall\alpha \in (0,\ 1],\ \ \forall\ x,\ y \in\ X$,
where $0 < \delta < \frac{1}{2}$. Then the operator T has a unique fixed point.

Proof Let $x_0 \in\ X$ and $x_1 = T(x_0),\ x_2 = T(x_1) = T^2(x_0), ..., x_n = T(x_{n-1}) = T^n(x_0)$ and so on. Then
$d_\alpha(x_1,\ x_2) = d_\alpha(T(x_0),\ T(x_1))\ \leq\ \delta\ [d_\alpha(x_0,\ T(x_0)) + d_\alpha(x_1,\ T(x_1))]$
$= \delta\ [d_\alpha(x_0,\ T(x_0)) + d_\alpha(x_1,\ x_2)]$
$\Rightarrow\ d_\alpha(x_1,\ x_2) \leq \frac{\delta}{1-\delta} d_\alpha(x_0,\ T(x_0))$.
Similarly we can show that
$d_\alpha(x_2,\ x_3) \leq (\frac{\delta}{1-\delta})^2 d_\alpha(x_0,\ T(x_0))$,

$d_\alpha(x_3,\ x_4) \leq (\frac{\delta}{1-\delta})^3 d_\alpha(x_0,\ T(x_0)),$
....
$d_\alpha(x_n,\ x_{n+1}) \leq (\frac{\delta}{1-\delta})^n d_\alpha(x_0,\ T(x_0)).$
By (QM-3u), there exists $\beta \in (0,\ \alpha]$ such that
$d_\alpha(x_n,\ x_{n+p})\ \leq\ d_\beta(x_n,\ x_{n+1}) + d_\beta(x_{n+1},\ x_{n+2}) + ...$
$.... + d_\beta(x_{n+p-1},\ x_{n+p})$
$\leq\ (r^n + r^{n+1} + r^{n+2} + ...$
$............. + r^{n+p-1})\ d_\beta(x_0,\ T(x_0)),$
where $r = \frac{\delta}{1-\delta}$. Hence
$d_\alpha(x_n,\ x_{n+p}) \leq\ \frac{r^n(1-r^p)}{1-r}\ d_\beta((x_0),\ T(x_0)).$
Since $0 < \delta < \frac{1}{2}$, we have $0 < r < 1$ and
$\lim_{n\to\infty}\ d_\alpha(x_n,\ x_{n+p}) = 0$.
Since $\alpha \in (0,\ 1]$ is arbitrary
$\lim_{m,n\to\infty}\ d_\alpha(x_n,\ x_{n+p}) = 0 \quad \forall\alpha \in (0,\ 1].$
Hence $\{x_n\}$ is a Cauchy sequence in $(X,\ d_\alpha :\ \alpha \in\ (0,1])$. Since $(X,\ d_\alpha :\ \alpha \in\ (0,1])$ is complete, there exists $x \in\ X$ such that $\lim_{n\to\infty}\ x_n = x$.
Since T is continuous, so $\lim_{n\to\infty}\ T(x_n) = Tx$.
Now $Tx = \lim_{n\to\infty}\ T(x_n) = \lim_{n\to\infty}\ x_{n+1} = x$.
Hence x is a fixed point of T.

Uniqueness: Let $x,\ y\ \in\ X$ be any two fixed points of T. Then for any $\alpha \in\ (0,\ 1]$
$d_\alpha(x,\ y) = d_\alpha(Tx,\ Ty) \leq\ \delta[d_\alpha(x,\ Tx) + d_\alpha(y,\ Ty)] = 0.$
Hence $x = y$.

Theorem 3.7(Caristi) Let $(X,\ d_\alpha :\ \alpha \in\ (0,1])$ be a complete generating space of sub-strong quasi-metric family and $T : X \to\ X$ be a continuous mapping. Suppose that for each $\alpha \in\ (0,\ 1]$ there exists a mapping $P_\alpha : X \to\ (0,\ \infty)$ such that
$d_\alpha(x,\ Tx)\ \leq\ P_\alpha(x) - P_\alpha(Tx) \quad \forall\alpha \in (0,\ 1],\ \ \forall\ x \in\ X,$
then T has a Unique fixed point.

Proof Let $x_0 \in\ X,\ \alpha \in (0,\ 1]$ and $x_1 = T(x_0),\ x_2 = T(x_1) = T^2(x_0), ..., x_n = T(x_{n-1}) = T^n(x_0)$ and so on. For any positive integer v we have

$$d_\alpha(x_v,\ x_{v+1})\ = d_\alpha(x_v,\ Tx_v) \leq\ P_\alpha(x_v) - P_\alpha(Tx_v)$$
$$\Rightarrow\ d_\alpha(x_v,\ x_{v+1}) \leq\ P_\alpha(x_v) - P_\alpha(x_{v+1})$$
$$\Rightarrow \sum_{v=0}^{n-1}\ d_\alpha(x_v,\ x_{v+1}) \leq\ \sum_{v=0}^{n-1}[P_\alpha(x_v) - P_\alpha(x_{v+1})]$$
$$= P_\alpha(x_0) - P_\alpha(x_n)$$
$$\leq\ P_\alpha(x_0).$$

So, the series
$$\sum_{v=0}^{\infty}\ d_\alpha(x_v,\ x_{v+1})$$
is convergent $\forall\alpha \in (0,\ 1]$. If $n,\ m = n + p$ are two positive integer then, by (QM-3u), there exists $\beta \in\ (0,\ \alpha]$ such that
$$d_\alpha(x_n,\ x_m) \leq\ \sum_{v=n}^{m-1}\ d_\beta(x_v,\ x_{v+1}).$$
Since the series $\sum_{v=0}^{\infty}\ d_\beta(x_v,\ x_{v+1})$ is convergent, for arbitrary $\epsilon > 0$ there exists n_0 such that
$$\sum_{v=n}^{m-1}\ d_\beta(x_v,\ x_{v+1}) < \epsilon \text{ whenever } m >\ n \geq\ n_0$$
$$\Rightarrow\ d_\alpha(x_n,\ x_m) < \epsilon \text{ whenever } m >\ n \geq\ n_0.$$
Hence $\{x_n\}$ is a Cauchy sequence in $(X,\ d_\alpha\ :\ \alpha \in\ (0,1])$. Since $(X,\ d_\alpha\ :\ \alpha \in\ (0,1])$ is complete, there exists $x \in\ X$ such that $\lim_{n\to\infty}\ x_n = x$.
Since T is a continuous mapping, so $\lim_{n\to\infty}\ T(x_n) = Tx$.
Now $Tx = \lim_{n\to\infty}\ T(x_n) = \lim_{n\to\infty}\ x_{n+1} = x$.
Hence x is a fixed point of T and clearly it is unique.

IV. Conclusion

Fixed points and fixed point theorems have always been a major tool in theoretical Mathematics. It has various applications in differential equation, game theory, dynamics, optimal control and functional analysis. Keeping this in mind and using contraction mapping, in this paper, we have established several fixed point theorems like Brower, Kannan, Caccioppoli, Edelstein and Caristi type in Generating spaces of quasi-metric family.

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