A Note on Spectral Norms of Circulant Matrices with Binomial Coefficients

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***Abstract*—In [1] Rabago introduced the concept of circulant determinant sequences with binomial coefficients and derived a formula for the $n-$th term of the given sequence. He also fomulated an expression to find the n-th partial sum of the sequence. In this note, we will study the spectral properties of circulant matrices with binomial coefficients.**

***Index Terms*—Circulant matrices, spectral norms, binomial coefficients.**

***MSC 2010 Codes* – 15B36**

I. Introduction

In [1], Rabago defined a right-circulant and left-circulant determinant sequences with binomial coefficients as follows:

Definition I.1. *The right-circulant determinant sequence with binomial coefficients, denoted by $\{R_n\}$, is the sequence of the form*

$$\{R_n\} = \left\{ |1|, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \\ 3 & 1 & 1 & 3 \\ 3 & 3 & 1 & 1 \end{vmatrix}, \dots \right\}.$$

Definition I.2. *The left-circulant determinant sequence with binomial coefficients, denoted by $\{L_n\}$, is the sequence of the form*

$$\{L_n\} = \left\{ |1|, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 3 & 3 & 1 \\ 3 & 3 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 1 & 1 & 3 & 3 \end{vmatrix}, \dots \right\}.$$

Clearly, from the above definitions, the right-circulant matrix with binomail coefficients, denoted by R_n, and the left-circulant matrix with binomial coefficients, denoted by L_n, has the form

$$R_n = circ(C_0^{n-1}, C_1^{n-1}, C_2^{n-1}, \dots, C_{n-1}^{n-1})$$

and

$$L_n = (-1)^{\lfloor \frac{n-1}{2} \rfloor} circ(C_0^{n-1}, C_1^{n-1}, C_2^{n-1}, \dots, C_{n-1}^{n-1})$$

where $C_b^a = \begin{pmatrix} a \\ b \end{pmatrix}$, respectively.

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Rabago [1] have determined the respective formula for the $n-$th term of the two sequences which is, in fact, the determinant R_n and L_n. That is, he have shown that

$$R_n = (1 + (-1)^{n-1})2^{n-2}$$

and

$$L_n = (-1)^{\lfloor \frac{n-1}{2} \rfloor} \left(1 + (-1)^{n-1}\right) 2^{n-2}.$$

Many authors have studied the properties of different examples of circulant matrices. Some of these are Bahsi and Solak [2] and Bueno [3]. Bahsi and Solak have defined a circulant matrix with arithmetic sequence $C_{a,r} = [c_{ij}]$ as an $n \times n$ matrix, where $c_{ij} \equiv a + (j - i \pmod n))r$ and a and r are real numbers. They studied the eigenvalues, determinant , spectral norm, Euclidean norm of the matrix $C_{a,r}$ and investigated the spectral norm, Euclidean norm of inverse of the matrix $C_{a,r}$. As an analogue, Bueno have defined a right-circulant matrix with geometric progression as an $n \times n$ matrix with circulant vector $\vec{c} = (a, ar, ar^2, \dots, ar^{n-1})$ with real numbers $a \neq 0$ and $r \neq 0, 1$ and do the same investigation done by Solak and Bahsi.

In this note, we study some of the properties of a right-circulant matrix with binomial coefficients using the results found by Rabago in [1]. In particular, we will study the eigenvalues, determinant, spectral norm and Euclidean norm of circulant matrices with binomial coefficients. We also state some properties of left-circulant matrices with binomial coefficients.

II. Main Results

We start this section by stating some of the results found by Rabago in [1].

Theorem II.1. *The eigenvalues of a right-circulant matrix with binomial coefficients are given by*

$$\lambda_0 = 2^{n-1},\ \lambda_m = \left(1 + e^{\frac{2\pi i m}{n}}\right)^{n-1},$$

for $m = 1, 2, \dots, n-1$.

Corollary II.2. *The eigenvalues of the inverse of a right-circulant matrix with binomial coefficients satisfy the following equalities*

$$\psi_0 = \lambda_0^{-1} = 2^{1-n},\ \psi_m = \lambda_m^{-1} = \left(1 + e^{\frac{2\pi i m}{n}}\right)^{1-n},$$

for $m = 1, 2, \dots, n-1$.

Theorem II.3. *The determinant of an $n \times n$ right-circulant matrix with binomial coefficients, denoted by R_n, is given by*

$$R_n = \left(1 + (-1)^{n-1}\right) 2^{n-2}.$$

Remark II.4. *From the previous theorem, it is obvious that for any odd integer n, the matrix R_n is nonsingular and $R_n^{-1} = 2^{1-n}$. Furthermore, if n is odd, we have $\left(R_n^{-1}\right)^T = \left(R_n^T\right)^{-1}$.*

Theorem II.5. *The determinant of an $n \times n$ left-circulant matrix with binomial coefficients, denoted by L_n, is given by*

$$L_n = (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(1 + (-1)^{n-1}\right) 2^{n-2}.$$

Remark II.6. *Similar observations from Remark 2.4 can be seen easily from the above theorem. Take note that L_n is nonsingular and $L_n^{-1} = -2^{1-n}$ if and only if n is odd. Moreover, the matrix L_n is symmetric and so $L_n = L_n^T$. Clearly, for odd integer n, $\left(L_n^{-1}\right)^T = \left(L_n^T\right)^{-1}$ and L_n is diagonalizable.*

See Lemma 2.3, Theorem 2.5, and Theorem 2.8 in [1] for the proof of Theorem 2.1, 2.3 and 2.5, respectively.

Theorem II.7. *The spectral norm of a right-circulant matrix with binomial coefficients is*

$$\|R_n\|_2 = max \left\{2^{n-1}, \left|2\cos\frac{\pi}{n}\right|^{n-1}\right\}.$$

Proof: Because a circulant matrix is a normal matrix, then

$$\|R_n\|_2 = \max\left\{2^{n-1}, \max_{m=1:n-1}(|\lambda_m|)\right\}.$$

But, for every $m = 1, 2, \ldots, n-1$,

$$\begin{aligned}
|\lambda_m| &= \left|\left(1 + e^{\frac{2\pi i m}{n}}\right)^{n-1}\right| \\
&= \left|1 + \cos\frac{2\pi m}{n} + i\sin\frac{2\pi m}{n}\right|^{n-1} \\
&= \left(\sqrt{2\left(1 + \cos\frac{2\pi m}{n}\right)}\right)^{n-1} \\
&= \left(\sqrt{4\left(\cos\frac{\pi m}{n}\right)^2}\right)^{n-1} \\
&= \left|2\cos\frac{\pi m}{n}\right|^{n-1}.
\end{aligned}$$

Hence, $\{|\lambda_m|\}$ is maximum if $m = 1$. It follows that

$$\max_{m=0:n-1}(|\lambda_m|) = \left(2\cos\frac{\pi}{n}\right)^{n-1}.$$

Therefore, conclusion follows. ∎

Corollary II.8. $\|R_n\|_1 = \|R_n\|_\infty = 2^{n-1}$.

Corollary II.9. $\|L_n\|_1 = \|L_n\|_\infty = 2^{n-1}$.

Remark II.10. *Based from the above corollaries, it is clear that $\lambda_{max} \leq max_i \sum_{j=1}^{n} a_{ij} = 2^{n-1}$.*

Lemma II.11. *For any positive integer n and k,*

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Proof: Since $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, then

$$\begin{aligned}
(1+x)^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} x^k \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k \cdot \sum_{k=0}^{n} \binom{n}{k} x^k \\
&= (1+x)^n (1+x)^n.
\end{aligned}$$

By performing the operations and comparing the coefficients of x^n on both sides, we can see that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

But note that $\binom{n}{k} = \binom{n}{n-k}$. Thus,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$$

∎

Theorem II.12. *The euclidean norm of a right-circulant matrix with binomial coefficients is*

$$\|R_n\|_E = \sqrt{n\binom{2n-2}{n-1}}$$

Proof: From the definition of the Euclidean norm,

$$\|R_n\|_E^2 = n\sum_{k=0}^{n-1} \binom{n-1}{k}^2$$

and by virtue of Lemma 2.2, we have

$$\|R_n\|_E = \sqrt{n\binom{2n-2}{n-1}}.$$

∎

Theorem II.13. *The spectral norm of R_n^{-1}, n is odd, is given by*

$$\left\|R_n^{-1}\right\|_2 = max\left\{2^{1-n}, \left(2\cos\left(\frac{(n-1)\pi}{2n}\right)\right)^{1-n}\right\}.$$

The proof of Theorem 2.13 is similar to Theorem 2.7 but with the asumption that $|\psi_m| = \left(2\cos\left(\frac{m\pi}{n}\right)\right)^{1-n}$ is maximum at $m = \frac{n-1}{2}$.

From the previous results, the norms for left circulant matrices with binomial coefficients can be easily derived using the same arguments.

III. Conclusion

The spectral properties for circulant matrices with binomial coefficients have been studied and found similar results for right circulant matrices with binomial coefficients and left circulant matrices with binomial coefficients.

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