On Supercovering Spaces

Z. Vaziry , S.B. Nimse and D. Leseberg

***Abstract*—In this paper we consider supercovering which is natural generalization of uniform cover spaces and supertopologies as defined by Doitchinov in 1964. Supernearness which is natural generalization of nearness spaces and supertopologies defined by D. Leseberg in 2002, here we found relations between supercovering, supernearness, superfarness and supersmallness, then we defined some properties of supercovering . At the end we extend the term of completeness and construct simple completion of closed graded C_1 -paracovering spaces. Its simple completion offer us in a special case the simple completion of a separated N_1- space in the sense of Herrlich.**

***Index Terms*—Covering, Supercovering, Superfarness, Supersmallness, Paracovering, graded, Compact, Precompact simple completion.**

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I. Introduction

It is well known that uniform spaces are a nice framework for studying metric properties. In the past two equivalent concepts were defined by A. Weil in 1937, [1] named *diagonal uniformities* and by J. W. Tukey in 1940, [2] called *covering structures*. In 1974 Herrlich generalized both to *nearness* respectively *uniform cover spaces* in order to get a common generalization of uniformity, proximity and topology [3], [4]. Hence, completeness and completion were intensely studied for the corresponding spaces.

In 2002 D. Leseberg defined supernearness [5], [6] which is a natural generalization of nearness spaces and supertopologies as defined by Doitchinov in 1964 [7].

In our paper, we have found *supercovering spaces* which is a natural generalization of uniform cover spaces and supertopologies , then we found relations between supercovering, supernearness, superfarness and supersmallness. Also we defined compactness, paracompactness and totally bounded on supernear spaces and supercovering spaces.

At the end, we extend the term of completeness and construct simple completion of closed graded C_1 -paracovering spaces. Then its simple completion offer us in a special case the simple completion of a separated N_1- space in the sense of Herrlich.

II. Background

Definition 1: Let X be a set and let ξ be a subset of P^2X. Consider the following axioms:

Dr. S.B. Nimse is Vice-Chancellor of SRTM University, Nanded-431606, India. E-mail: dr.sbnimse@rediffmail.com

Zohreh Vaziry is with the Department of Mathematics, University of Pune, Pune-411007, India. E-mail: z_m_vaziry@yahoo.co.in

PD Dr. Dieter Leseberg is a Professor at the Department of Mathematics and Informatics, Free University of Berlin, Germany. And head of Chemistry library, Technical University of Braunschweig, Germany, a.D. (retired). E-mail: d.leseberg@tu-bs.de

(N1) If $\mathcal{A} << \mathcal{B}$ and $\mathcal{B} \in \xi$ then $\mathcal{A} \in \xi$, where $\mathcal{A} << \mathcal{B}$ iff $\forall A \in \mathcal{A} \exists B \in \mathcal{B}, A \supseteq B$;
(N2) If $\bigcap \mathcal{A} \neq \emptyset$ then $\mathcal{A} \in \xi$;
(N3) $\emptyset \neq \xi \neq P^2X$;
(N4) If $(\mathcal{A} \bigvee \mathcal{B}) \in \xi$ then $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$, where $\mathcal{A} \bigvee \mathcal{B} := \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$;
(N5) If $\{cl_\xi A | A \in \mathcal{A}\} \in \xi$ then $\mathcal{A} \in \xi$, where

$$cl_\xi A := \{x \in X | \{A, \{x\}\} \in \xi\}$$

ξ is called *prenearness structure on* X iff ξ satisfying to (N1)-(N3).

ξ is called *seminearness structure on* X iff ξ satisfying to (N1)-(N4).

ξ is called *nearness structure on* X iff ξ satisfying to (N1)-(N5).

The pair (X, ξ) is called a (pre-, semi-) nearness space - shortly a (pre-, semi-) N-space- iff ξ is a (pre-, semi-) nearness structure on X.

If (X, ξ) and (Y, η) are N-spaces then a function $f : X \longrightarrow Y$ is called a nearness preserving map from (X, ξ) to (Y, η) iff $\mathcal{A} \in \xi$ implies $f\mathcal{A} \in \eta$, where $f\mathcal{A} := \{f[A] : A \in \mathcal{A}\}$.

Let (X, ξ) be a N-space, then a subset $\mathcal{A}$ of PX is called a $\xi - cluster$ iff $\mathcal{A}$ is maximal element of the set $\xi \setminus \{\emptyset\}$, ordered by inclusion. We call (X, ξ) complete iff every $\xi - cluster$ contains an element $\{x\}$ for some $x \in X$. ■

Definition 2: Let X be a set then the relation δ on PX which is satisfying in the following conditions is a *EF-proximity* on X,

(P0) $A\delta B$ implies $B\delta A$;
(P1) $A \subset B$ and $A\delta C$ imply $B\delta C$;
(P2) $A \cap B \neq \emptyset$ implies $A\delta B$;
(P3) $A\delta B$ implies $A \neq \emptyset$;
(P4) $A\delta(B \cup C)$ implies $A\delta B$ or $A\delta C$;
(P5) $A\overline{\delta}B$ implies there exists a subset C of X such that $A\overline{\delta}C$ and $(X \setminus C)\overline{\delta}B$. ($A\overline{\delta}B$ means that A is not in relation to B.)

The pair (X, δ) is a *EF-proximal space*.

If we replace (P5) to (P5′), then δ is a *Lodato-proximity* (or shortly *LO -proximity*) and the pair (X, δ) is a *Lodato proximal space* (or shortly *LO -proximal space*).

(P5′) $A\delta B$ and $B \subset cl_\delta C$ imply $A\delta C$, where $cl_\delta C := \{x \in X | \{x\} \delta C\}$.

Easily we can see each EF- proximal space is LO-proximal space.

Let (X_1, δ_1) and (X_2, δ_2) be two EF-proximal spaces (respectively LO-proximal spaces), a function $f : X_1 \to X_2$ is δ-map if, for every $A, B \subset X$, $A\delta_1 B$ implies $f(A)\delta_2 f(B)$.

The corresponding category is called **EF-PROX** (respectively, **LO-PROX**). ■

Remark 1: If (X, ξ) is a nearness space then the relation δ on PX defined by

$$A\delta B \text{ iff } \{A, B\} \in \xi$$

is a Lodato -proximity. ■

Definition 3: Let X be a set and let $\overline{\xi}$ be a subset of P^2X. Consider the following axioms:

(F1) If $\mathcal{A} << \mathcal{B}$ and $\mathcal{A} \in \overline{\xi}$ then $\mathcal{B} \in \overline{\xi}$;
(F2) If $\mathcal{A} \in \overline{\xi}$ then $\cap\mathcal{A} = \emptyset$;
(F3) $\emptyset \neq \overline{\xi} \neq P^2X$;
(F4) If $\mathcal{A} \in \overline{\xi}$ and $\mathcal{B} \in \overline{\xi}$ then $(\mathcal{A} \bigvee \mathcal{B}) \in \overline{\xi}$;
(F5) If $\mathcal{A} \in \overline{\xi}$ then $\left\{cl_{\overline{\xi}}A | A \in \mathcal{A}\right\} \in \overline{\xi}$, where

$$cl_{\overline{\xi}}A := \left\{x \in X | \{A, \{x\}\} \notin \overline{\xi}\right\}$$

$\overline{\xi}$ is called *farness structure on* X iff $\overline{\xi}$ satisfying to above conditions, and the pair $(X, \overline{\xi})$ is called a *farness space* iff $\overline{\xi}$ is a farness structure on X. ■

Definition 4: Let X be a set and let γ be a subset of P^2X. Consider the following axioms:

(S1) If $\mathcal{A} << \mathcal{B}$ and $\mathcal{A} \in \gamma$ then $\mathcal{B} \in \gamma$;
(S2) $\forall x \in X, \{\{x\}\} \in \gamma$;
(S3) $\emptyset \neq \gamma \neq P^2X$;
(S4) If $(\mathcal{A} \bigcup \mathcal{B}) \in \gamma$ then $\mathcal{A} \in \gamma$ or $\mathcal{B} \in \gamma$;
(S5) If $sec\{cl_\gamma A | A \in \mathcal{A}\} \in \gamma$ then $sec\mathcal{A} \in \gamma$, where $cl_\gamma A = \{x \in X | sec\{A, \{x\}\} \in \gamma\}$.

γ is called smallness structure on X iff γ satisfying to above conditions, and the pair (X, γ) is called a smallness space iff γ is a smallness structure on X. ■

Definition 5: Let X be a set and let μ be a subset of P^2X. Consider the following axioms:

(C1) If $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{A} \in \mu$ then $\mathcal{B} \in \mu$. Where $\mathcal{A} \prec \mathcal{B}$ iff $\forall\, A \in \mathcal{A}\, \exists\, B \in \mathcal{B},\ A \subset B$;
(C2) If $\mathcal{A} \in \mu$ then $\cup\mathcal{A} = X$;
(C3) $\emptyset \neq \mu \neq P^2X$;
(C4) If $\mathcal{A} \in \mu$ and $\mathcal{B} \in \mu$ then $(\mathcal{A} \bigwedge \mathcal{B}) \in \mu$, where

$$\mathcal{A} \bigwedge \mathcal{B} := \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$$

(C5) If $\mathcal{A} \in \mu$ then $\{int_\mu A | A \in \mathcal{A}\} \in \mu$, where

$$int_\mu \mathcal{A} := \{x \in X | \{A, X \setminus \{x\}\} \in \mu\}$$

μ is called *covering structure* (resp. *cover structure*) on X iff μ satisfying to above conditions, and the pair (X, μ) is called a *cover space* iff μ is a cover structure on X.

If (X, μ_1) and (Y, μ_2) are cover spaces then a function $f : X \longrightarrow Y$ is called a *uniform continuous map* from (X, μ_1) to (Y, μ_2) iff $\mathcal{A} \in \mu_2$ implies $f^{-1}\mathcal{A} \in \mu_1$, where

$$f^{-1}\mathcal{A} := \left\{f^{-1}[A] : A \in \mathcal{B}\right\} ■$$

Recall 1 : Let $\mathcal{A}, \mathcal{B} \subset PX$ then

(1) $stack\mathcal{A} = \{B \subset X | \exists A \in \mathcal{A}, A \subset B\}$;
(2) $sec\mathcal{A} = \{B \subset X | \forall A \in \mathcal{A}, A \cap B \neq \emptyset\}$;
(3) $stack\mathcal{A} << \mathcal{A}$;
(4) $stack\mathcal{A} = sec^2\mathcal{A}$;
(5) $sec^3\mathcal{A} = sec\mathcal{A}$;
(6) $stack(\mathcal{A} \cup \mathcal{B}) = stack\mathcal{A} \cup stack\mathcal{B}$. ■

Next, we define the following.

Definition 6: Let ξ be a nearness structure on X. Then

(i) $\overline{\xi} = P^2X - \xi$ is called the farness structure induced on X by ξ;
(ii) $\mu = \mu_\xi = \left\{\mathcal{A} \subset PX | \{X - A | A \in \mathcal{A}\} \in \overline{\xi}\right\}$ is called the covering structure induced on X by ξ.
(iii) $\gamma_\xi = \{A \subset PX |\ \forall\ \mathcal{B} \in \mu\ \ \mathcal{B} \cap stack\mathcal{A} \neq \emptyset\}$ is called the smallness structure induced on X by ξ. ■

Proposition 1: Let ξ, $\overline{\xi}$, μ and γ be respectively nearness, farness, covering and smallness structures induced by each other on X. Then following relations hold:

(1) $\mathcal{A} \in \xi$ iff $\mathcal{A} \notin \overline{\xi}$;
(2) $\mathcal{A} \in \overline{\xi}$ iff $\mathcal{A} \notin \xi$;
(3) $\mathcal{A} \in \overline{\xi}$ iff $\{X - A | A \in \mathcal{A}\} \in \mu$;
(4) $\mathcal{A} \in \mu$ iff $\{X - A | A \in \mathcal{A}\} \in \overline{\xi}$;
(5) $\mathcal{A} \in \mu$ iff $\forall \mathcal{B} \in \xi\ \ \mathcal{A} \cap sec\mathcal{B} \neq \emptyset$;
(6) $\mathcal{A} \in \xi$ iff $\forall \mathcal{B} \in \mu\ \ \mathcal{B} \cap sec\mathcal{A} \neq \emptyset$;
(7) $\mathcal{A} \in \gamma$ iff $\forall \mathcal{B} \in \mu$, $\mathcal{B} \cap stack\mathcal{A} \neq \emptyset$;
(8) $\mathcal{A} \in \mu$ iff $\forall \mathcal{B} \in \gamma$, $\mathcal{A} \cap stack\mathcal{B} \neq \emptyset$;
(9) $\mathcal{A} \in \xi$ iff $sec\mathcal{A} \in \gamma$;
(10) $\mathcal{A} \in \gamma$ iff $sec\mathcal{A} \in \xi$;
(11) $\mathcal{A} \in \gamma$ iff $\forall\ \mathcal{B} \in \overline{\xi}\quad \exists A \in \mathcal{A}\ \ \exists B \in \mathcal{B}$, $A \cap B = \emptyset$;
(12) $\mathcal{A} \in \overline{\xi}$ iff $\forall\ \mathcal{B} \in \gamma\quad \exists A \in \mathcal{A}\ \ \exists B \in \mathcal{B}$, $A \cap B = \emptyset$. ■

Definition 7: An N-space (X, ξ) is called *topological* iff it satisfies in the following equivalent condition :

(T) If $\mathcal{A} \in \xi$ then $\bigcap\{cl_\xi A | A \in \mathcal{A}\} \neq \emptyset$;

(T′) If $X = \bigcup\{int_\mu A | A \in \mathcal{A}\}$ then $\mathcal{A} \in \mu$.

It is called *contigual* iff it satisfies in the following equivalent condition :

(C) If every finite subset of $\mathcal{A}$ belongs to ξ then $\mathcal{A}$ belongs to ξ;

(C′) If $\mathcal{A} \in \overline{\xi}$ then there exists a finite subset $\mathcal{B}$ of $\mathcal{A}$ with $\mathcal{B} \in \overline{\xi}$.

(C″) If $\mathcal{A} \in \mu$ then there exists a finite subset $\mathcal{B}$ of $\mathcal{A}$ with $\mathcal{B} \in \mu$.

It is called *uniform* iff it satisfies in the following equivalent condition :

(U) If $\mathcal{A} \in \overline{\xi}$ then there exists $\mathcal{B} \in \overline{\xi}$ such that for each $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ with $A \subset \bigcap\{C \in \mathcal{B} | C \cup B \neq X\}$.

(U′) If $\mathcal{A} \in \overline{\xi}$ then there exists $\mathcal{B} \in \overline{\xi}$ such that for each $x \in X$ there exists $A \in \mathcal{A}$ with $A \subset \bigcap\{B \notin \mathcal{B} | x \in B\}$.

(U″) If $\mathcal{A} \in \mu$ then there exists $\mathcal{B} \in \mu$ such that for each $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ with $\bigcup\{C \in \mathcal{B} | B \cap C \neq \emptyset\} \subset A$, i.e. every uniform cover has a uniform star-refinement. ■

Definition 8: A nearness space (X, ξ) is called

(1) *Compact* iff it is topological and contigual;
(2) *Paracompact* iff it is topological and uniform $N1-space$ (See Definition 9.);
(3) *Proximal* iff it is contigual and uniform.
(4) *totally bounded* iff $\mathcal{F} \in FIL(X) \backslash \{PX\} \Rightarrow \mathcal{F} \in \xi$. (Where $FIL(X)$ is set of filters on X.)

Note that the category of all totally bounded uniform spaces and uniformly continuous maps is isomorphic to the category *EF-PROX*. (See Definition 2.) ■

Remark 2: For all these definitions see Herrlich. H . A concept of nearness. ■

Definition 9: A N-space is called $N1-space$ iff it satisfies in the following equivalent conditions.

(1) If $\{\{x\},\{y\}\} \in \xi$ then $x = y$.
(2) If $\{\{x, y\}\} \in \gamma$ then $x = y$.
(3) If $x \neq y$ then $\{X - \{x\}, X - \{y\}\} \in \mu$. ■

Definition 10: For a set X, a subset $\mathcal{B}^X \subset PX$ is called *prebornology* or shortly **B**-structure or **B**-set, on X, and its elements are called *bounded* sets, if the following axioms are satisfies:

(B1) $\emptyset \in \mathcal{B}^X$;
(B2) $x \in X$ implies $\{x\} \in \mathcal{B}^X$.
(B3) $B' \subseteq B \in \mathcal{B}^X$ implies $B' \in \mathcal{B}^X$;

Given a pair of **B**-structures $\mathcal{B}^X$, $\mathcal{B}^Y$ on sets X and Y, respectively, a map $f : X \longrightarrow Y$ is called *bounded* iff

(b) $\left\{f[B] | B \in \mathcal{B}^X\right\} \subseteq \mathcal{B}^Y$.

We denote by **BOUND** the corresponding category.

Also, a map $f : X \longrightarrow Y$ is called *rebounded* iff

(rb) $\left\{f^{-1}[B] : B \in \mathcal{B}^Y\right\} =: f^{-1}\mathcal{B}^Y \subseteq \mathcal{B}^X$. ■

Definition 11: A **B**-set $\mathcal{B}^X$ on X is called *saturated* iff

$$X \in \mathcal{B}^X ■$$

Definition 12: A *supertopology* on a set X is a pair $(\mathcal{B}^X, \Theta)$, where $\mathcal{B}^X$ is a B-set on X and Θ is a map from $\mathcal{B}^X$ to **FIL**(X) where **FIL**(X) is the set of all filters on X, such that:

(ST1) $\Theta(\emptyset) = PX$;
(ST2) $B \in \mathcal{B}^X$ and $U \in \Theta(B)$ imply $B \subseteq U$;
(ST3) $B \in \mathcal{B}^X$ and $U \in \Theta(B)$ imply there exists a set $V \in \Theta(B)$ such that $U \in \Theta(C)$ for each $C \in \mathcal{B}^X$ with $C \subseteq V$.

Then the triple $(X, \mathcal{B}^X, \Theta)$ is called a *supertopological space*.

For each $B \in \mathcal{B}^X$, a set $U \in \Theta(B)$ is called a *neighbourhood* of B, and $\Theta(B)$ is called the *neighbourhood system* of B with respect to Θ.

For supertopological spaces $(X, \mathcal{B}^X, \Theta_X)$ and $(Y, \mathcal{B}^Y, \Theta_Y)$, a bounded map $f : X \rightarrow Y$ is called *continuous* iff satisfies (st), e.g.

(st) $B \in \mathcal{B}^X$ and $V \in \Theta_Y(f[B])$ imply $f^{-1}[V] \in \Theta_X(B)$, where f^{-1} denotes the inverse image under f.

We denote by **STOP** the corresponding category.

A supertopology $(\mathcal{B}^X, \Theta)$ on X is said to be *symmetric* iff it additionally satisfies the following condition:

(sym) If $A, B \in \mathcal{B}^X$ and $U \cap B \neq \emptyset$ for every $U \in \Theta(A)$, then $V \cap A \neq \emptyset$ for every $V \in \Theta(B)$. ■

Remark 3: Let $(\mathcal{B}^X, \Theta)$ be an arbitrary supertopology on X, then following holds:

$B_2 \subseteq B_1 \in \mathcal{B}^X$ implies $\Theta(B_1) \subseteq \Theta(B_2)$. ■

Definition 13: For a set X, and a B-set $\mathcal{B}^X$ the relation $\delta \subset \mathcal{B}^X \times PX$ is called *superproximity* , and triple $(X, \mathcal{B}^X, \delta)$ is celled *superproximal space* iff it satisfies the following conditions:

(SP1) $B \in \mathcal{B}^X$ and $A \subseteq X$ imply $B\overline{\delta}\emptyset$ and $\emptyset\overline{\delta}A$, which means that the empty set is not in relation to B, nor is it in relation to A;
(SP2) $B\delta(C \cup D)$ iff $B\delta C$ or $B\delta D$, for $B \in \mathcal{B}^X$ and subsets $C, D \subseteq X$;
(SP3) $x \in X$, implies $\{x\}\delta\{x\}$;
(SP4) If $B\delta A$ and $B \subseteq B' \in \mathcal{B}^X$ then $B'\delta A$;
(SP5) $B \in \mathcal{B}^X$ and $B\overline{\delta}A$ imply there is a set $V \subseteq X$ such that $B\overline{\delta}X\backslash V$ and $C\overline{\delta}A$ for each $C \in \mathcal{B}^X$ with $C \subseteq V$.

For superproximal spaces $(X, \mathcal{B}^X, \delta_X)$ and $(Y, \mathcal{B}^Y, \delta_Y)$, a bounded map $f : X \rightarrow Y$ is called *continuous* iff

$$B\delta_X A \implies f[B]\delta_Y f[A]$$

We denote by **SUPPROX** the corresponding category. ■

Remark 4: If we replace (SP5) with (HP5),e.g.

(HP5) For $B \in \mathcal{B}^X$, $B\delta A$ and $A \subseteq cl_\delta(C)$ imply $B\delta C$,

where

$$cl_\delta(C) = \{x \in X |\, \{x\}\, \delta C\}$$

then we call δ a *hyperproximity* and $(X, \mathcal{B}^X, \delta)$ is *hyperproximal space*.

We denote by **HYPROX** the corresponding category.

A Superproximity (hyperproximity) is called *symmetric* if it satisfies the following condition:

(sym) If $B_1, B_2 \in \mathcal{B}^X$ and $B_1\delta B_2$ then $B_2\delta B_1$. ■

Remark 5: Let $(X, \mathcal{B}^X, \delta)$ be a superproximal space (or hyperproximal space) then δ satisfies the following condition:

$B \in \mathcal{B}^X$, $A \in PX$ and $A \cap B \neq \emptyset$ imply $B\delta A$. ■

Remark 6: Every supertopological space $(X, \mathcal{B}^X, \Theta)$ induces a superproximical space $(X, \mathcal{B}^X, \delta_\Theta)$ by setting

$$B\delta_\Theta A \text{ iff } A \in sec\Theta(B), ■$$

Remark 7: Every superproximal space $(X, \mathcal{B}^X, \delta)$ induces a supertopological space $(X, \mathcal{B}^X, \Theta_\delta)$ by setting
$A \in \Theta_\delta(B)$ iff $A \in sec\delta(B)$, where $\delta(B) = \{C \subset X | B\delta C\}$. ■

Corollary 1: Each superproximimal space is hyperproximal. ■

In general, the converse of above corollary does not hold.

Example 1 [8]: Let be given a T_1 topology that is not T_2, then cl on given set defines a *hyperproximity* on the set that it is not *superproximity* for $\mathcal{B}^X = PX$ by setting

$$B\delta A \text{ iff } cl(B) \cap cl(A) \neq \emptyset$$

Easily we can see it is *hyperproximity* but since the given topology is not T_2, there exist different points x and y in space such that there is not any disjoint neighborhood for them. Therefore by T_1 we have $cl(\{x\}) \cap cl(\{y\}) = \emptyset$ i.e. $\{x\}\,\overline{\delta}\,\{y\}$.

If (SP5) holds then we have $\exists V \subset X$ s.t. $\{x\}\,\overline{\delta}X\backslash V$ and $V\overline{\delta}\,\{y\}$. So either $y \in cl(V)$ or $y \notin cl(V)$.

If $y \in cl(V)$ then $cl(\{y\}) \cap cl(V) \neq \emptyset$ i.e. $V\delta\,\{y\}$ which is contradiction therefore $y \notin cl(V)$ i.e. there exists an open set U containing y s.t. $U \cap V = \emptyset$ therefore $U \cap int(V) = \emptyset$.

If $x \notin int(V)$ so $x \in cl(X\backslash V)$ i.e. $cl(\{x\}) \cap cl(X\backslash V) \neq \emptyset$ so $\{x\}\,\delta X\backslash V$ which is contradiction.

Therefore $x \in int(V)$. So we have disjoint open sets $int(V)$ and U which are respectively containing x and y. It is a contradiction to the given topology is not T_2, therefore (PX, δ) does not satisfy (SP5) so it is not *superproximity*. ■

Definition 14: For a **B**-set $\mathcal{B}^X$, a function

$$N : \mathcal{B}^X \longrightarrow P(P(PX))$$

is called a *supernear operator* or a *supernearness* on $\mathcal{B}^X$, and the pair $(\mathcal{B}^X, N)$ is called a *supernear space* (*supernearness space*), iff

(SN1) $B \in \mathcal{B}^X$ and $\mathcal{H}_2 << \mathcal{H}_1 \in N(B)$ imply $\mathcal{H}_2 \in N(B)$;
(SN2) $B \in \mathcal{B}^X$ implies $\mathcal{B}^X \notin N(B) \neq \emptyset$;
(SN3) $\mathcal{H} \in N(\emptyset)$ implies $\mathcal{H} = \emptyset$;
(SN4) $x \in X$ implies $\{\{x\}\} \in N(\{x\})$;
(SN5) $B' \subseteq B \in \mathcal{B}^X$ implies $N(B') \subseteq N(B)$;
(SN6) $B \in \mathcal{B}^X$ and $\mathcal{H}_1 \bigvee \mathcal{H}_2 \in N(B)$ imply $\mathcal{H}_1 \in N(B)$ or $\mathcal{H}_2 \in N(B)$;
(SN7) $B \in \mathcal{B}^X$ and $\{cl_N(F)|F \in \mathcal{H}\} \in N(B)$ for some $\mathcal{H} \subseteq P(PX)$ imply $\mathcal{H} \in N(B)$, where

$$cl_N(F) := \{x \in X| \, \{F\} \in N(\{x\})\}$$

Elements of $N(B)$ are called **B**-near collections.

Given a pair of supernear spaces $(\mathcal{B}^X, N_X)$, $(\mathcal{B}^Y, N_Y)$, a bounded map $f : \mathcal{B}^X \longrightarrow \mathcal{B}^Y$ is called a *supernear* map or shortly *sn-map*, iff satisfies (sn), e.g.

(sn) $B \in \mathcal{B}^X$ and $\mathcal{H} \in N_X(B)$ imply $\{f[F]|F \in \mathcal{H}\} \in N_Y(f[B])$.

A map will also be referred to as a *supernear map* by saying it preserves **B**-near collections in the above sense.

We denote by **SN** the corresponding category.

Also, we denote by $\mathbf{SN^S}$ full subcategory of **SN** where its corresponding B-set is saturated. ■

Definition 15: A supernearness is called *paranear* iff satisfies the following condition:
(sym) $(\emptyset \neq)B \in \mathcal{B}^X$ and $\mathcal{H} \in N(B)$ imply $\{B\} \cup \mathcal{H} \in \bigcap \{N(F)|F \in (\mathcal{H} \cap \mathcal{B}^X) \cup \{B\}\}$.
We denote by **PN** the corresponding category.
And we denote by $\mathbf{PN^S}$ full subcategory of **PN** where its corresponding B-set is saturated. ■

Definition 16: Any paranear space $(\mathcal{B}^X, N)$ is automatically *dense*, by satisfying (d), e.g.

(d) $B \subset X$ and $cl_N(B) \in \mathcal{B}^X$ imply $N(cl_N(B)) = N(B)$. ■

Definition 17 [9]: Let $(\mathcal{B}^X, N)$ be a supernear space for $B \in \mathcal{B}^X \setminus \{\emptyset\}$, $\mathcal{G} \subset PX$ is called a B- *clan in* N iff it satisfies in the following conditions:

(Cla_0) $\emptyset \notin \mathcal{G}$;
(Cla_1) $G_1 \in \mathcal{G}$ and $G_1 \subset G_2$ imply $G_2 \in \mathcal{G}$;
(Cla_2) G_1, $G_2 \in PX$ and $G_1 \cup G_2 \in \mathcal{G}$ imply $G_1 \in \mathcal{G}$ or $G_2 \in \mathcal{G}$;
(Cla_3) $B \in \mathcal{G} \in N(B)$;
(Cla_4) $A \subset X$ and $cl_N(A) \in \mathcal{G}$ imply $A \in \mathcal{G}$. ■

Remark 8: For each $B \in \mathcal{B}^X$ with $x \in B$

$$x_N := \{A \subset X | x \in cl_N(A)\}$$

is a B-clan in N.

Definition 18: A supernearness space $(\mathcal{B}^X, N)$ is called *Conic* iff it satisfies (cnc), e.g.
(cnc) $B \in \mathcal{B}^X$ implies $\bigcup \{\mathcal{H}|\mathcal{H} \in N(B)\} \in N(B)$, where $\bigcup \{\mathcal{H}|\mathcal{H} \in N(B)\} := \{F \subseteq X | \exists \mathcal{H} \in N(B), F \in \mathcal{H}\}$. ■

Definition 19: A supernearness space $(\mathcal{B}^X, N)$ is called *superclan space* iff it satisfies (Cla), e.g.
(Cla) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{H} \in N(B)$ imply there exists a B-clan $\mathcal{C} \in N$ such that $\mathcal{H} \subseteq \mathcal{C}$.
We denote by **CLA-SN** the corresponding category. ■

Remark 9: Analogously, we call a paranear space paraclan space and denote by **CLA-PN** the full subcategory of **PN** , whose objects are the paraclans. ■

Theorem 1: The category of **HYPROX** whose objects are the hyperproximity spaces, is isomorphic to a full subcategory of **SN**.

Proof: Let be given a hyperproximity space $(X, \mathcal{B}^X, \delta)$ then $(X, \mathcal{B}^X, N_\delta)$ is a *conic superclan* space, where $N_\delta(B) = \{\mathcal{H}|\mathcal{H} \subseteq \delta(B)\}$, and $\delta(B) = \{F \subseteq X | B\delta F\}$.

Conversely, we consider the hyperproximity δ_M by setting $B\delta_M A$ iff $\{A\} \in N(B)$. We get a bijection between the set of all hyperproximities and the set of all so defined supernear operators on $\mathcal{B}^X$. (Let us call them proximal supernear operators, and the corresponding spaces proximal supernear spaces, respectively) ■

Corollary 2: **STOP** is up to isomorphism a subcategory of **SN**. ■

Theorem 2: The category **NEAR** is isomorphic to the full subcategory $\mathbf{PN^S}$ of $\mathbf{SN^S}$, whose objects are saturated paranear spaces.

Proof: Let be given a nearness space (X, ξ), we obtain a paranear operator on $\mathcal{B}^X := PX$ by setting

$$N_\xi(B) = \begin{cases} \{\emptyset\} & if B = \emptyset \\ \{\mathcal{H}| \, \{B\} \cup \mathcal{H} \in \xi\} & otherwise. \end{cases}$$

Now by setting $\mathcal{A} \in \xi_N$ iff $\mathcal{A} \in \bigcap \{N(A)|A \in \mathcal{A}\}$ for a given paranear operator N on PX, we get a bijection between the set of all nearness structures and all so-defined paranear operators on PX. With respect to the corresponding morphisms this yield an isomorphism between the above-mentioned categories.

Lemma 1: For a paranear space $(\mathcal{B}^X, N)$ with $x \in X \supseteq A$, the following statements are equivalent:
(i) $x \in cl_N(A)$;
(ii) $\{A, \{x\}\} \in N(\{x\})$. ■

Corollary 3: For a nearness space (X, ξ) the following statements are equivalent:
(i) $x \in cl_\xi(A)$;
(ii) $x \in cl_{N_\xi}(A)$. ■

Definition 20: Let $(\mathcal{B}^X, N)$ be a supernear space, then a subset $\mathcal{C}$ of PX is called $N-cluster$ iff $\mathcal{C}$ is *maximal element* of $N(B) \setminus \{\emptyset\}$ for some $B \in \mathcal{B}^X$, ordered by inclusion.

We call $(\mathcal{B}^X, N)$ complete iff every $N-cluster$ contains an element $\{x\}$ for some $x \in X$. ■

Corollary 4: Symmetric supertopologies can be essentially subsumed by symmetric hyperproximities. ■

Corollary 5: For a symmetric supertopology Θ, if $\mathcal{B}^X$ is saturated then δ_Θ is EF-proximity. ■

Remark 10: Let $(\mathcal{B}^X, N)$ be a supernear space then any $N-cluster$, $\mathcal{H}$ is a *grill* in N which is satisfying the following condition:

$$A \subset X \text{ and } cl_N(A) \in \mathcal{H} \implies A \in \mathcal{H} \blacksquare$$

III. SUPERFARNESS, SUPERSMALLNESS AND SUPERCOVERING

Definition 21: For a **B**-set $\mathcal{B}^X$, a function $F : \mathcal{B}^X \longrightarrow P(P(PX))$ is called a *superfar operator* or a *superfarness* on $\mathcal{B}^X$, and the pair $(\mathcal{B}^X, F)$ is called a *superfar space* (*superfarness space*), iff

(SF1) $B \in \mathcal{B}^X$, $\mathcal{H}_2 << \mathcal{H}_1$ and $\mathcal{H}_2 \in F(B)$ imply $\mathcal{H}_1 \in F(B)$;
(SF2) $B \in \mathcal{B}^X$ implies $\mathcal{B}^X \in F(B) \neq P^2X$;
(SF3) $\mathcal{H} \neq \emptyset$ implies $\mathcal{H} \in F(\emptyset)$;
(SF4) $x \in X$ implies $\{\{x\}\} \notin F(\{x\})$;
(SF5) $B' \subseteq B \in \mathcal{B}^X$ implies $F(B) \subseteq F(B')$;
(SF6) $B \in \mathcal{B}^X, \mathcal{H}_1 \in F(B)$ and $\mathcal{H}_2 \in F(B)$ imply $\mathcal{H}_1 \bigvee \mathcal{H}_2 \in F(B)$; (See definition 1 (N4).)
(SF7) $B \in \mathcal{B}^X$ and $\mathcal{H} \in F(B)$ imply $\{cl_F(H) | H \in \mathcal{H}\} \in F(B)$, where

$$cl_F(H) := \{x \in X |\, \{H\} \notin F(\{x\})\} \blacksquare$$

Definition 22: For a **B**-set $\mathcal{B}^X$, a function $S : \mathcal{B}^X \longrightarrow P(P(PX))$ is called a *supersmall operator* or a *supersmallness* on $\mathcal{B}^X$, and the pair $(\mathcal{B}^X, S)$ is called a *supersmall space* (*supersmallness space*), iff

(SS1) $B \in \mathcal{B}^X, \mathcal{H}_2 << \mathcal{H}_1$ and $\mathcal{H}_2 \in S(B)$ imply $\mathcal{H}_1 \in S(B)$;
(SS2) $B \in \mathcal{B}^X$ implies $\emptyset \notin S(B) \neq \emptyset$;
(SS3) $sec\mathcal{H} \in S(\emptyset)$ implies $\mathcal{H} = \emptyset$; (equivalently $\{PX\} = S(\emptyset)$.)
(SS4) $x \in X$ implies $\{\{x\}\} \in S(\{x\})$;
(SS5) $B' \subseteq B \in \mathcal{B}^X$ implies $S(B') \subseteq S(B)$;
(SS6) $B \in \mathcal{B}^X$ and $\mathcal{H}_1 \bigcup \mathcal{H}_2 \in S(B)$ imply $\mathcal{H}_1 \in S(B)$ or $\mathcal{H}_2 \in S(B)$;
(SS7) $B \in \mathcal{B}^X$ and $sec\{cl_S(F) | F \in \mathcal{H}\} \in S(B)$ for some $\mathcal{H} \subseteq P(PX)$ imply $sec\mathcal{H} \in S(B)$, where

$$cl_S(F) := \{x \in X | sec\{F\} \in S(\{x\})\} \blacksquare$$

Definition 23: For a **B**-set $\mathcal{B}^X$, a function $C : \mathcal{B}^X \longrightarrow P(P(PX))$ is called a *supercovering operator* or a *supercovering* on $\mathcal{B}^X$, and the pair $(\mathcal{B}^X, C)$ is called a *supercovering space*, iff

(SC1) $B \in \mathcal{B}^X$, $\mathcal{H}_2 \prec \mathcal{H}_1$ and $\mathcal{H}_2 \in C(B)$ imply $\mathcal{H}_1 \in C(B)$;
(SC2) $B \in \mathcal{B}^X$ implies $\emptyset \notin C(B)$ and $\{X\} \in C(B)$;
(SC3) $\mathcal{H} \neq \emptyset$ implies $\mathcal{H} \in C(\emptyset)$;
(SC4) $x \in X$ implies $\{X - \{x\}\} \notin C(\{x\})$;
(SC5) $B' \subseteq B \in \mathcal{B}^X$ implies $C(B) \subseteq C(B')$;
(SC6) $B \in \mathcal{B}^X, \mathcal{H}_1 \in C(B)$ and $\mathcal{H}_2 \in C(B)$ imply $\mathcal{H}_1 \bigwedge \mathcal{H}_2 \in C(B)$;
(SC7) $B \in \mathcal{B}^X$ and $\mathcal{H} \in C(B)$ imply $\{int_C(H) | H \in \mathcal{H}\} \in C(B)$, where

$$int_C(H) := \{x \in X |\, \{H\} \in C(\{x\})\}$$

Given a pair of supercovering spaces $(\mathcal{B}^X, C_X)$ and $(\mathcal{B}^Y, C_Y)$, a rebounded map

$$f : X \longrightarrow Y$$

is called *supercover* map or shortly *sc-map*, iff satisfies (sc):

(sc) $B \in \mathcal{B}^Y$ and $\mathcal{H} \in C_Y(B)$ imply $f^{-1}\mathcal{H} \in C_X(f^{-1}[B])$.

We denote by **SC** the corresponding category. $\blacksquare$

Proposition 2: Let $(\mathcal{B}^X, C)$ be a supercovering space then following holds:
If $B \in \mathcal{B}^X$ and $\mathcal{H} \in C(B)$ then $B \subset \cup\mathcal{H}$.

Proof: For every $b \in B$ by (SC4), we have $\{X - \{b\}\} \notin C(\{b\})$ and since $\{b\} \subset B$, by (SC5) for every $b \in B$ we have $\{X - \{b\}\} \notin C(B)$.

Now let $\mathcal{H} \in C(B)$ if B is not subset of $\cup\mathcal{H}$ then for some $b \in B$, for every $H \in \mathcal{H}$, $b \notin H$ i.e. $H \subset X - \{b\}$ therefore $\mathcal{H} \prec \{X - \{b\}\}$ then by (SC1) we have $\mathcal{H} \notin C(B)$ which is contradiction so $B \subset \cup\mathcal{H}$. $\blacksquare$

Definition 24: A supercovering is called *paracovering* iff C in addition satisfies (sym), e.g.

(sym) $B \in \mathcal{B}^X \setminus \{\emptyset\}$, $\mathcal{H} \subset PX$ and $\{X - A : A \in \{B\} \cup \mathcal{H}\} \in \bigcup\{C(F) : F \in (\mathcal{H} \cap \mathcal{B}^X) \cup \{B\}\}$ implies $\{X - H : H \in \mathcal{H}\} \in C(B)$.

We denote by **PC** the corresponding full subcategory of **SC**. $\blacksquare$

Proposition 3: Let $(\mathcal{B}^X, N)$ be a supernear space then

(i) $F(B) = P^2X - N(B)$ is the superfarness induced by N on $\mathcal{B}^X$;
(ii) $C(B) = \{\mathcal{A} \subset PX |\, \{X - A | A \in \mathcal{A}\} \in F(B)\}$ is the supercovering induced by N on $\mathcal{B}^X$.
(iii) $S(B) = \{\mathcal{A} \subset PX | \forall \mathcal{H} \in C(B)\ \ \mathcal{H} \cap stack\mathcal{A} \neq \emptyset\}$ is the supersmallness induced by N on $\mathcal{B}^X$.

Proof: (i) We show that F is a superfarness on $\mathcal{B}^X$.

To (SF1): Let $B \in \mathcal{B}^X$, $\mathcal{H}_2 << \mathcal{H}_1$ and $\mathcal{H}_2 \in F(B)$ by definition
$\mathcal{H}_2 \notin N(B)$ hence $\mathcal{H}_1 \notin N(B)$ by (SN1). So $\mathcal{H}_1 \in F(B)$.

To (SF2): By (SN2) obviously.

To (SF3): By (SN3) obviously.

To (SF4): By (SN4) obviously.

To (SF5): Let $B' \subset B \in \mathcal{B}^X$ and $\mathcal{H} \in F(B)$ so $\mathcal{H} \notin N(B)$ and by (SN5) $\mathcal{H} \notin N(B')$ so $\mathcal{H} \in F(B')$.

To (SF6): Let $B \in \mathcal{B}^X$, $\mathcal{H}_1 \in F(B)$ and $\mathcal{H}_2 \in F(B)$ so $\mathcal{H}_1 \notin N(B)$ and $\mathcal{H}_2 \notin N(B)$ by (SN6) $\mathcal{H}_1 \bigvee \mathcal{H}_2 \notin N(B)$ therefore $\mathcal{H}_1 \bigvee \mathcal{H}_2 \in F(B)$.

To (SF7): Let $B \in \mathcal{B}^X$ and $\mathcal{H} \in F(B)$ so $\mathcal{H} \notin N(B)$ by (SN7) $\{cl_N(H) | H \in \mathcal{H}\} \notin N(B)$.
But $cl_F(H) = \{x \in X |\, \{H\} \notin F(\{x\})\}$
$= \{x \in X |\, \{H\} \in N(\{x\})\} = cl_N(H)$.
Therefore $\{cl_F(H) | H \in \mathcal{H}\} \notin N(B)$ i.e.
$\{cl_F(H) | H \in \mathcal{H}\} \in F(B)$.
Therefore F is a superfarness on $\mathcal{B}^X$.

(ii) Now we show that C is a supercovering on $\mathcal{B}^X$.

To (SC1): Let $B \in \mathcal{B}^X$, $\mathcal{H}_2 \prec \mathcal{H}_1$ and $\mathcal{H}_2 \in C(B)$.

Since $\mathcal{H}_2 \prec \mathcal{H}_1$ means
$\{X - H'' | H'' \in \mathcal{H}_2\} << \{X - H' | H' \in \mathcal{H}_1\}$.

Also, $\mathcal{H}_2 \in C(B)$ means $\{X - H'' | H'' \in \mathcal{H}_2\} \in F(B)$.

By (SF1), $\{X - H' | H' \in \mathcal{H}_1\} \in F(B)$ i.e. $\mathcal{H}_1 \in C(B)$.

To (SC2): By (SF2), $F(B) \neq P^2X$ so by (SF1), $\emptyset \notin F(B)$ therefore $\emptyset \notin C(B)$.

Now by (SF2) $\mathcal{B}^X \in F(B)$ and by (SF1), $\{\emptyset\} \in F(B)$ it implies $\{X\} \in C(B)$.

To (SC3): By (SF3) obviously.

To (SC4): By (SF4) obviously.

To (SC5): Let $B' \subset B \in \mathcal{B}^X$ and $\mathcal{H} \in C(B)$ so $\{X-H|H\in\mathcal{H}\} \in F(B)$ by (SF5)
$\{X-H|H\in\mathcal{H}\} \in F(B')$ therefore $\mathcal{H} \in C(B')$

To (SC6): Let $B \in \mathcal{B}^X$, $\mathcal{H}_1 \in C(B)$ and $\mathcal{H}_2 \in C(B)$ so $\{X-H'|H'\in H_1\} \in F(B)$ and $\{X-H''|H''\in H_2\} \in F(B)$, by (SF6)
$\{X-H'|H'\in H_1\} \bigvee \{X-H''|H''\in H_2\} \in F(B)$
i.e. $\{(X-H')\cup(X-H'')|H'\in\mathcal{H}_1, H''\in\mathcal{H}_2\} \in F(B)$
i.e. $\{X-(H'\cap H'')|H'\in\mathcal{H}_1, H''\in\mathcal{H}_2\} \in F(B)$ i.e.
$\{X-H|H\in\mathcal{H}_1\bigwedge\mathcal{H}_2\} \in F(B)$ so $\mathcal{H}_1\bigwedge\mathcal{H}_2 \in C(B)$.
To (SC7): Let $B \in \mathcal{B}^X$ and $\mathcal{H} \in C(B)$ so
$\{X-H|H\in\mathcal{H}\} \in F(B)$ and by (SF7)
$\{cl_F(X-H)|H\in\mathcal{H}\} \in F(B)$.
But $cl_F(X-H) = \{x\in X|\,\{X-H\}\notin F(\{x\})\}$
$= \{x\in X|\,\{H\}\notin C(\{x\})\}$
$= X - int_C(H)$
So $\{X-int_C(H)|H\in\mathcal{H}\} \in F(B)$ therefore
$\{int_C(H)|H\in\mathcal{H}\} \in C(B)$.
Therefore C is a supercovering on $\mathcal{B}^X$.
(iii) Now we show that S is a supersmallness on $\mathcal{B}^X$.
To (SS1): Let $\mathcal{H}_2 << \mathcal{H}_1$ and $\mathcal{H}_2 \in S(B)$.
Let $\mathcal{A}$ be arbitrary member of $C(B)$ since $\mathcal{H}_2 \in S(B)$ it implies
$\mathcal{A}\cap stack\mathcal{H}_2 \neq \emptyset$ i.e. there exists $A\in\mathcal{A}$ s.t. $A \in stack\mathcal{H}_2$ so there exists $H_2\in\mathcal{H}_2$ s.t. $H_2\subset A$. On the other hand there exists $H_1\in\mathcal{H}_1$ s.t. $H_1\subset H_2$ therefore $H_1\subset A$ and it implies $A\in stack\mathcal{H}_1$ i.e. $\mathcal{A}\cap stack\mathcal{H}_1\neq\emptyset$ so $\mathcal{H}_1\in S(B)$.
To (SS2): Since $stack\emptyset=\emptyset$ so $\emptyset\notin S(B)$.
Since $stack\,\{\emptyset\} = PX$ so $\forall\mathcal{A}\in C(B)$, $\mathcal{A}\cap stack\,\{\emptyset\}\neq\emptyset$ so $S(B)\neq\emptyset$.
To (SS3): Let be $sec\mathcal{H} \in S(\emptyset)$ i.e. $\forall\mathcal{A}\in C(\emptyset)$, $\mathcal{A}\cap stack(sec\mathcal{H})\neq\emptyset$ i.e. $\mathcal{A}\cap sec\mathcal{H}\neq\emptyset$. By (SC3), $\{\emptyset\}\in C(\emptyset)$ so it implies $\mathcal{H}=\emptyset$.
To (SS4): Let $x\in X$ then by proposition 2, we have for every $\mathcal{A}\in C(\{x\})$ there is $A\in\mathcal{A}$ s.t. $x\in A$ so $\mathcal{A}\cap stack\,\{\{x\}\}\neq\emptyset$ i.e. $\{\{x\}\}\in S(\{x\})$.
To (SS5): Let $B'\subset B\in\mathcal{B}^X$ and $\mathcal{H}\in S(B')$ so $\forall\mathcal{A}\in C(B')$, $\mathcal{A}\cap stack\mathcal{H}\neq\emptyset$ and by (SC5) $C(B)\subset C(B')$ so $\forall\mathcal{A}\in C(B)$, $\mathcal{A}\cap stack\mathcal{H}\neq\emptyset$ i.e. $\mathcal{H}\in S(B)$.
To (SS6): Let $\mathcal{H}_1\bigcup\mathcal{H}_2 \in S(B)$ so $\forall\mathcal{A}\in C(B)$, $\mathcal{A}\cap stack(\mathcal{H}_1\bigcup\mathcal{H}_2)\neq\emptyset$ i.e. there exists $A\in\mathcal{A}$ s.t. $A\in stack(\mathcal{H}_1\bigcup\mathcal{H}_2)$ so $A\in stack\mathcal{H}_1$ or $A\in stack\mathcal{H}_2$.
Suppose for $\mathcal{A}_1\in C(B)$, $\mathcal{A}_1\cap stack\mathcal{H}_2=\emptyset$ and for $\mathcal{A}_2\in C(B)$,
$\mathcal{A}_2\cap stack\mathcal{H}_1=\emptyset$. By (SC6) $\mathcal{A}_1\bigwedge\mathcal{A}_2\in C(B)$ so
$(\mathcal{A}_1\bigwedge\mathcal{A}_2)\cap stack(\mathcal{H}_1\bigcup\mathcal{H}_2)\neq\emptyset$.
If $(\mathcal{A}_1\bigwedge\mathcal{A}_2)\cap stack\mathcal{H}_1\neq\emptyset$ then there is $A\in\mathcal{A}_1\bigwedge\mathcal{A}_2$ s.t. $A\in stack\mathcal{H}_1$ where $A=A_1\cap A_2$ for some $A_1\in\mathcal{A}_1$ and $A_2\in\mathcal{A}_2$ i.e. for some $H_1\in\mathcal{H}_1$, $H_1\subset A\subset A_2$ and it implies $\mathcal{A}_2\cap stack\mathcal{H}_1\neq\emptyset$ and which is a contradiction to our supposition.
Similarly if $(\mathcal{A}_1\bigwedge\mathcal{A}_2)\cap stack\mathcal{H}_2\neq\emptyset$ then there is $A\in\mathcal{A}_1\bigwedge\mathcal{A}_2$ s.t.
$A\in stack\mathcal{H}_2$ where $A=A_1\cap A_2$ for some $A_1\in\mathcal{A}_1$ and $A_2\in\mathcal{A}_2$ i.e. for some $H_2\in\mathcal{H}_2$, $H_2\subset A\subset A_1$ and it implies $\mathcal{A}_1\cap stack\mathcal{H}_2\neq\emptyset$ and again we have a contradiction to our supposition.
So either $\forall\mathcal{A}\in C(B)$, $\mathcal{A}\cap stack\mathcal{H}_1\neq\emptyset$ or $\forall\mathcal{A}\in C(B)$, $\mathcal{A}\cap stack\mathcal{H}_2\neq\emptyset$ i.e. either $\mathcal{H}_1\in S(B)$ or $\mathcal{H}_2\in S(B)$.
To (SS7): Let $sec\{cl_S(F)|F\in\mathcal{H}\}\in S(B)$ i.e. for every $\mathcal{A}\in C(B)$,
$\mathcal{A}\cap stack(sec\{cl_S(F)|F\in\mathcal{H}\})\neq\emptyset$ and it means $\mathcal{A}\cap sec\{cl_S(F)|F\in\mathcal{H}\}\neq\emptyset$ and by (SC7) $\{int_C(A)|A\in\mathcal{A}\}\in C(B)$
so for some $A\in\mathcal{A}$, $int_C(A)\cap cl_S(F)\neq\emptyset$ for all $F\in\mathcal{H}$ i.e. for every $F\in\mathcal{H}$, $\exists x\in X$ s.t. $\{A\}\in C(\{x\})$ and $sec\{F\}\in S(\{x\})$ and it implies $\{A\}\cap sec\{F\}\neq\emptyset$ so $A\cap F\neq\emptyset$ i.e. $A\in sec\mathcal{H}$ i.e. $\mathcal{A}\cap stack(sec\mathcal{H})\neq\emptyset$ i.e. $sec\mathcal{H}\in S(B)$.
Therefore S is supersmallness on $\mathcal{B}^X$.

Corollary 6: For a supernear space $(\mathcal{B}^X, N)$ the following statements are equivalent:
(i) $(\mathcal{B}^X, N)$ is paranear space;
(ii) $(\mathcal{B}^X, C_N)$ is paracovering space.

Proof: (i $\Rightarrow$ ii) Let $B\in\mathcal{B}^X\setminus\{\emptyset\}$, $\mathcal{H}\subset PX$ and
$\{X-A|A\in\{B\}\cup\mathcal{H}\}\in\bigcup\{C_N(F)|F\in(\mathcal{H}\cap\mathcal{B}^X)\cup\{B\}\}$
by Proposition 3 i.e.
$\mathcal{H}\cup\{B\}\notin N(F)$ for some $F\in(\mathcal{H}\cap\mathcal{B}^X)\cup\{B\}$. (1)
In other hand if we assume $\{X-H|H\in\mathcal{H}\}\notin C_N(B)$ by Proposition 3 we have
$\mathcal{H}\in N(B)$ and since N is symmetric we have
$\mathcal{H}\cup\{B\}\in\cap\{N(F)|F\in(\mathcal{H}\cap\mathcal{B}^X)\cup\{B\}\}$ i.e.
$\mathcal{H}\cup\{B\}\in N(F)$ for every $F\in(\mathcal{H}\cap\mathcal{B}^X)\cup\{B\}$ that is contradiction to (1).
(ii $\Rightarrow$ i) Let $B\in\mathcal{B}^X\setminus\{\emptyset\}$ and $\mathcal{H}\in N(B)$ then
by Proposition 3, $\{X-H|H\in\mathcal{H}\}\notin C_N(B)$ and since C_N is symmetric we have
$\{X-A|A\in\{B\}\cup\mathcal{H}\}\notin\bigcup\{C_N(F)|F\in(\mathcal{H}\cap\mathcal{B}^X)\cup\{B\}\}$
then by Proposition 3
$\{B\}\cup\mathcal{H}\in\bigcap\{N(F)|F\in(\mathcal{H}\cap\mathcal{B}^X)\cup\{B\}\}$. ■

Proposition 4: Let N, F, S and C be respectively supernearness, superfarness, supersmallness and supercovering structures induced by each other on $\mathcal{B}^X$. Then following relations for $B\in\mathcal{B}^X$ are hold:
(1) $\mathcal{H}\in N(B)$ iff $\mathcal{H}\notin F(B)$;
(2) $\mathcal{H}\in F(B)$ iff $\mathcal{H}\notin N(B)$;
(3) $\mathcal{H}\in F(B)$ iff $\{X-H|H\in\mathcal{H}\}\in C(B)$;
(4) $\mathcal{H}\in C(B)$ iff $\{X-H|H\in\mathcal{H}\}\in F(B)$;
(5) $\mathcal{H}\in C(B)$ iff $\forall\mathcal{A}\in N(B)$ $\mathcal{H}\cap sec\mathcal{A}\neq\emptyset$;
(6) $\mathcal{H}\in N(B)$ iff $\forall\mathcal{A}\in C(B)$ $\mathcal{A}\cap sec\mathcal{H}\neq\emptyset$.
(7) $\mathcal{H}\in S(B)$ iff $\forall\mathcal{A}\in C(B)$, $\mathcal{A}\cap stack\mathcal{H}\neq\emptyset$;
(8) $\mathcal{H}\in C(B)$ iff $\forall\mathcal{A}\in S(B)$, $\mathcal{H}\cap stack\mathcal{A}\neq\emptyset$;
(9) $\mathcal{H}\in N(B)$ iff $sec\mathcal{H}\in S(B)$;
(10) $\mathcal{H}\in S(B)$ iff $sec\mathcal{H}\in N(B)$;
(11) $\mathcal{H}\in S(B)$ iff $\forall\ \mathcal{A}\in F(B)$ $\exists H\in\mathcal{H}$ $\exists A\in\mathcal{A}$, $H\cap A=\emptyset$;
(12) $\mathcal{H}\in F(B)$ iff $\forall\ \mathcal{A}\in S(B)$ $\exists H\in\mathcal{H}$ $\exists A\in\mathcal{A}$, $H\cap A=\emptyset$.

Proof:
By proposition 3, (1) -(4) is clear.
(5) Let be $\mathcal{H}\in C(B)$ so by (4) and (2), $\{X-H|H\in\mathcal{H}\}\notin N(B)$.
If $\exists\mathcal{A}\in N(B)$ s.t. $\mathcal{H}\cap sec\mathcal{A}=\emptyset$ so $\forall H\in\mathcal{H}, H\notin sec\mathcal{A}$ i.e.

$\forall H \in \mathcal{H}, \exists A \in \mathcal{A}$ s.t. $A \cap H = \emptyset$ therefore $\forall H \in \mathcal{H}, \exists A \in \mathcal{A}$ s.t. $A \subset X - H$ i.e. $\{X - H | H \in \mathcal{H}\} << \mathcal{A}$. By assumption and (SN1), hence $\{X - H | H \in \mathcal{H}\} \in N(B)$, which is a contradiction.
Conversely, let be $\forall \mathcal{A} \in N(B)$ $\mathcal{H} \cap sec\mathcal{A} \neq \emptyset$. If $\mathcal{H} \notin C(B)$ so by (4) and (1), $\{X - H | H \in \mathcal{H}\} \in N(B)$.
But $\mathcal{H} \cap sec\{X - H | H \in \mathcal{H}\} = \emptyset$ which is a contradiction to assumption.
(6) Let be $\mathcal{H} \in N(B)$ so by (1) and (3),
$\{X - H | H \in \mathcal{H}\} \notin C(B)$.
If $\exists \mathcal{A} \in C(B)$ s.t. $\mathcal{A} \cap sec\mathcal{H} = \emptyset$ so $\forall A \in \mathcal{A}, A \notin sec\mathcal{H}$ i.e. $\forall A \in \mathcal{A}, \exists H \in \mathcal{H}$ s.t. $A \cap H = \emptyset$ therefore $\forall A \in \mathcal{A}, \exists H \in \mathcal{H}$ s.t. $A \subset X - H$ i.e. $\mathcal{A} \prec \{X - H | H \in \mathcal{H}\}$. By assumption and (SC1), hence
$\{X - H : H \in \mathcal{H}\} \in C(B)$, which is a contradiction.
Conversely, let be $\forall \mathcal{A} \in C(B)$ $\mathcal{A} \cap sec\mathcal{H} \neq \emptyset$. If $\mathcal{H} \notin N(B)$ so by(2) and (3),
$\{X - H | H \in \mathcal{H}\} \in C(B)$.
But $\{X - H | H \in \mathcal{H}\} \cap sec\mathcal{H} = \emptyset$ which is a contradiction to assumption.
(7) By Proposition 3, it is clear.
(9) Let $\mathcal{H} \notin N(B)$ so by (1) and (3),
$\{X - H | H \in \mathcal{H}\} \in C(B)$ also we know,
$\{X - H | H \in \mathcal{H}\} \cap sec\mathcal{H} = \emptyset$ i.e.
$\{X - H | H \in \mathcal{H}\} \cap stack(sec\mathcal{H}) = \emptyset$ by (7) it implies, $sec\mathcal{H} \notin S(B)$.
Conversely, Let $sec\mathcal{H} \notin S(B)$ so by (7), there exists $\mathcal{A} \in C(B)$ s.t. $\mathcal{A} \cap stack(sec\mathcal{H}) = \emptyset$ i.e.
$\mathcal{A} \cap \sec\mathcal{H} = \emptyset$ i.e. $\forall A \in \mathcal{A}$, $\exists H \in \mathcal{H}$ s.t. $H \cap A = \emptyset$ so $A \subset X - H$ therefore $\mathcal{A} \prec \{X - H | H \in \mathcal{H}\}$ and by (SC1) it implies $\{X - H | H \in \mathcal{H}\} \in C(B)$ so $\mathcal{H} \notin N(B)$.
(10) Let $\mathcal{H} \in S(B)$ so by (7), $\forall \mathcal{A} \in C(B)$, $\mathcal{A} \cap stack\mathcal{H} \neq \emptyset$.
We consider $\mathcal{A} = \{X - F | \forall\ H \in \mathcal{H},\ F \cap H \neq \emptyset\}$.
For every $X - F \in \mathcal{A}$, there is not any $H \in \mathcal{H}$ s.t. $H \subseteq X - F$ i.e. for every $X - F \in \mathcal{A}$, $X - F \notin stack\mathcal{H}$ so
$\{X - F | \forall\ H \in \mathcal{H},\ F \cap H \neq \emptyset\} \notin C(B)$ therefore
$\{F | \forall\ H \in \mathcal{H},\ F \cap H \neq \emptyset\} \in N(B)$ i.e. $sec\mathcal{H} \in N(B)$.
Conversely, $sec\mathcal{H} \in N(B)$ by (9) $sec^2\mathcal{H} \in S(B)$ i.e. $stack\mathcal{H} \in S(B)$ and since $stack\mathcal{H} << \mathcal{H}$ by (SS1) it implies $\mathcal{H} \in S(B)$.
(8) Let $\mathcal{H} \in C(B)$ and $\mathcal{A} \in S(B)$ then by (7) we have $\mathcal{H} \cap stack\mathcal{A} \neq \emptyset$.
Conversely, Let $\forall \mathcal{A} \in S(B)$, $\mathcal{H} \cap stack\mathcal{A} \neq \emptyset$, if $\mathcal{H} \notin C(B)$ then $\{X - H | H \in \mathcal{H}\} \in N(B)$ and by (9), $sec\{X - H | H \in \mathcal{H}\} \in S(B)$ and by assumption $\mathcal{H} \cap stack(sec\{X - H | H \in \mathcal{H}\}) \neq \emptyset$ i.e.
$\mathcal{H} \cap sec\{X - H | H \in \mathcal{H}\} \neq \emptyset$ which is a contradiction so $\mathcal{H} \in C(B)$.
(11) Let $\mathcal{H} \in S(B)$ and let $\mathcal{A} \in F(B)$ so by (3) $\{X - A | A \in \mathcal{A}\} \in C(B)$, and by (7), $\{X - A | A \in \mathcal{A}\} \cap stack\mathcal{H} \neq \emptyset$ i.e. $\exists H \in \mathcal{H}$ $\exists A \in \mathcal{A}$ s.t. $H \subset (X - A)$ i.e. $H \cap A = \emptyset$.
Conversely, let $\forall$ $\mathcal{A} \in F(B)$ $\exists H \in \mathcal{H}$ $\exists A \in \mathcal{A}$, $H \cap A = \emptyset$ holds. Now suppose $\mathcal{A} \in C(B)$ so by (4), $\{X - A | A \in \mathcal{A}\} \in F(B)$ and by our supposition, $\exists H \in \mathcal{H}$ $\exists A \in \mathcal{A}$, $H \cap (X - A) = \emptyset$ i.e. $H \subset A$ therefore, $A \in stack\mathcal{H}$ so $\mathcal{A} \cap stack\mathcal{H} \neq \emptyset$. By (7) it implies $\mathcal{H} \in S(B)$.
(12) Let $\mathcal{H} \in F(B)$ and let $\mathcal{A} \in S(B)$ so by (3) $\{X - H | H \in \mathcal{H}\} \in C(B)$, and by (8), $\{X - H | H \in \mathcal{H}\} \cap stack\mathcal{A} \neq \emptyset$ i.e. $\exists H \in \mathcal{H}$ $\exists A \in \mathcal{A}$ s.t. $A \subset X - H$ i.e. $H \cap A = \emptyset$.
Conversely, let $\forall$ $\mathcal{A} \in S(B)$ $\exists H \in \mathcal{H}$ $\exists A \in \mathcal{A}$, $H \cap A = \emptyset$ holds. Now suppose $\mathcal{A} \in S(B)$, so $\exists H \in \mathcal{H}$ $\exists A \in \mathcal{A}$, $A \subset (X - H)$. i.e. $(X - H) \in stack\mathcal{A}$ which implies $\{X - H | H \in \mathcal{H}\} \cap stack\mathcal{A} \neq \emptyset$.
By (8) we have $\{X - H | H \in \mathcal{H}\} \in C(B)$ and by (3), $\mathcal{H} \in F(B)$. ■

Theorem 3 : $\mathbf{SC^S}$ denotes the full subcategory of **SC**, whose objects are the saturated supercovering spaces, then $\mathbf{SC^S}$ and $\mathbf{SN^S}$ are isomorphic.

Corollary 7 : The category $\mathbf{PN^S}$ is isomorphic to the full subcategory $\mathbf{PC^S}$ of **PC**, whose objects are the saturated paracovering spaces. ■

IV. Compact and Precompact

Definition 25: We call a supernear space $(\mathcal{B}^X, N)$ *neartopological* iff satisfies in the following equivalence relations,

(nt) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{H} \in N(B)$ imply $\bigcap\{cl_N(H) | H \in \mathcal{H} \cup \{B\}\} \neq \emptyset$;
(nt$'$)$B \in \mathcal{B}^X \setminus \{\emptyset\}$ and
$X = \bigcup\{int_C(H) | H \in \mathcal{H} \cup \{X - B\}\}$ imply $\mathcal{H} \in C(B)$ where C is supercovering induced by N.

Remark 11: For a nearness space (X, ξ) we have the equivalence:
(i) (X, ξ) is topological ;
(ii) (PX, N_ξ) is neartopological.

Definition 26: A supernear space $(\mathcal{B}^X, N)$ is called *contiguous* iff N satisfies
(ct) $B \in \mathcal{B}^X$ and $\mathcal{H} \notin N(B)$ imply there exists $\mathcal{H}' \subset PX$ finite ($\mathcal{H}' \notin N(B)$ and $\mathcal{H}' \subset \{B\} \cup \mathcal{H}$).

Lemma 2 : For a nearness space (X, ξ) the following statements are equivalent:
(i) (X, ξ) is contigual;
(ii) (PX, N_ξ) is contiguous.

Proof: (i $\Rightarrow$ ii) Let $B \in \mathcal{B}^X$ and $\mathcal{H} \notin N_\xi(B)$ by Theorem 2, $\mathcal{H} \cup \{B\} \notin \xi$ since ξ is contigual there exists finite $\mathcal{H}' \subset \mathcal{H} \cup \{B\}$ s.t. $\mathcal{H}' \notin \xi$ by (N1) $\mathcal{H}' \cup \{B\} \notin \xi$ by Theorem 2 $\mathcal{H}' \notin N_\xi(B)$ i.e. N_ξ is contiguous.
(ii $\Rightarrow$ i) $\mathcal{H} \notin \xi \Rightarrow \mathcal{H} \neq \emptyset$. Choose $F \in \mathcal{H}$ and without restriction $F \neq \emptyset$, elsewise finished. Hence , $\{F\} \cup \mathcal{H} \notin \xi \Rightarrow$ $\mathcal{H} \notin N_\xi(F)$ by suppose $\exists \mathcal{H}' \subset PX$ finite ($\mathcal{H}' \notin N_\xi(F)$ and $\mathcal{H}' \subset \{F\} \cup \mathcal{H} \subset \mathcal{H}$). Therefore $\mathcal{H}'' := \{F\} \cup \mathcal{H}' \notin \xi$ and $\mathcal{H}'' \subset \mathcal{H}$ with $\mathcal{H}'' \subset PX$ finite. ■

Lemma 3 : For each hyperproximity space $(\mathcal{B}^X, \delta)$ the accordant supernear space $(\mathcal{B}^X, N_\delta)$ is contiguous.

Proof : Let $B \in \mathcal{B}^X$ and $\mathcal{H} \notin N_\delta(B)$ by Theorem 1 $\exists H \in \mathcal{H}$ s.t. $B\overline{\delta}H$ therefore $\{H\} \notin N_\delta(B)$ and obviously $\{H\} \subset \{B\} \cup \mathcal{H}$ i.e. $(\mathcal{B}^X, N_\delta)$ is contiguous. ■

Lemma 4 : For a supernear space $(\mathcal{B}^X, N)$ following are equivalent.

(i) $B \in \mathcal{B}^X$ and $\mathcal{H} \notin N(B)$ imply there exists $\mathcal{H}' \subset PX$ finite ($\mathcal{H}' \notin N(B)$ and $\mathcal{H}' \subset \{B\} \cup \mathcal{H}$).
(ii) $B \in \mathcal{B}^X$ and $\mathcal{H} \in F_N(B)$ imply there exists $\mathcal{H}' \subset PX$ finite ($\mathcal{H}' \in F_N(B)$ and $\mathcal{H}' \subset \{B\} \cup \mathcal{H}$); (Where F_N is superfarness induced by N)
(iii) $B \in \mathcal{B}^X$ and $\mathcal{H} \in C_N(B)$ there exists $\mathcal{H}' \subset PX$ finite ($\mathcal{H}' \in C_N(B)$ and
$\mathcal{H}' \subset \{X - B\} \cup \mathcal{H}$). (Where C_N is supercovering induced by N)

Proof :
(i) and (ii) obviously are equivalent.
(i) $\Rightarrow$ (iii):
Let $B \in \mathcal{B}^X$ and $\mathcal{H} \in C_N(B)$ i.e. $\{X - H | H \in \mathcal{H}\} \notin N(B)$ by (i), there exists finite $\mathcal{A} \subset PX$ s.t. $\mathcal{A} \notin N(B)$ and $\mathcal{A} \subset \{B\} \cup \{X - H | H \in \mathcal{H}\}$ i.e. $\{X - A | A \in \mathcal{A}\} \in C_N(B)$ and $\{X - A | A \in \mathcal{A}\} \subset \{X - B\} \cup \mathcal{H}$.
If we call $\{X - A | A \in \mathcal{A}\} = H'$ then (iii) holds.
(iii) $\Rightarrow$ (i):
Let $B \in \mathcal{B}^X$ and $\mathcal{H} \notin N(B)$ i.e. $\{X - H | H \in \mathcal{H}\} \in C_N(B)$ so by (iii), there exists finite $\mathcal{A} \subset PX$ s.t. $\mathcal{A} \in C_N(B)$ and $\mathcal{A} \subset \{X - B\} \cup \{X - H | H \in \mathcal{H}\}$ therefore $\{X - A | A \in \mathcal{A}\} \notin N(B)$ and $\{X - A | A \in \mathcal{A}\} \subset \{B\} \cup \mathcal{H}$.
If we call $\{X - A | A \in \mathcal{A}\} = H'$ then (i) holds.

Definition 27: A supernear space $(\mathcal{B}^X, N)$ is called *uniform* iff N satisfies (u), e.g.
(u) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{A} \notin N(B)$ therefore $\exists \mathcal{H} \notin N(B)$, $\forall H \in \mathcal{H}\ \exists D \in \{B\} \cup \mathcal{A}$, $D \subset \bigcap\{C \in \mathcal{H} | H \cup C \neq X\}$. ■

Lemma 5 : For each hyperproximity space $(X, \mathcal{B}^X, \delta)$ as equivalent $(\mathcal{B}^X, \delta)$ the accordant supernear space $(\mathcal{B}^X, N_\delta)$ is uniform.

Proof : Let $\mathcal{A} \notin N_\delta(B)$ for $B \in \mathcal{B}^X \setminus \{\emptyset\}$; by definition of N_δ there exists an $A \in \mathcal{A}$ s.t. $B\bar{\delta}A$ i.e. $\{A\} \notin N_\delta(B)$. By setting $\mathcal{H} := \{A\}$, we obtain the desired property for being uniform if choosing $A \in \{B\} \cup \mathcal{A}$. ■

Lemma 6 : For a nearness space (X, ξ) we have the equivalence:
(i) (X, ξ) is uniform nearness space;
(ii) (PX, N_ξ) is uniform paranear space.

Proof:
$(i \Rightarrow ii)$ Let $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{A} \notin N_\xi(B)$ then $\mathcal{A} \cup \{B\} \notin \xi$. Since (X, ξ) is uniform $\exists \mathcal{H} \notin \xi$ s.t. $\forall H \in \mathcal{H}\ \exists D \in \mathcal{A} \cup \{B\}$ with $D \subset \bigcap\{C \in \mathcal{H} | H \cup C \neq X\}$. But by (N1) $\mathcal{H} \notin \xi$ implies $\mathcal{H} \cup \{B\} \notin \xi$ i.e. $\mathcal{H} \notin N_\xi(B)$ so (PX, N_ξ) is uniform paranear space.
$(ii \Rightarrow i)$ $\mathcal{A} \notin \xi \Rightarrow \mathcal{A} \neq \emptyset$, choose $A \in \mathcal{A}$, for the first case $A = \emptyset$ set $\mathcal{B} := \{\emptyset\}$, hence $\mathcal{B} \notin \xi$ with the desired property. Second case $\mathcal{A} \neq \emptyset$, hence $\mathcal{A} \notin N_\xi(A)$ by supposition there exists $\mathcal{H} \notin N_\xi(A)\ \forall H \in \mathcal{H}\ \exists F \in \{A\} \cup \mathcal{A} = \mathcal{A}$, $F \subset \bigcap\{C \in \mathcal{H} | C \cup H \neq X\}$. Set $\mathcal{B} := \{A\} \cup \mathcal{H}$, hence $\mathcal{B} \notin \xi$; for $B \in \mathcal{B}$ the desired property follows in both of cases. ■

Lemma 7 : For a supernear space $(\mathcal{B}^X, N)$ following are equivalent:
(i) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{A} \notin N(B)$ therefore
$\exists \mathcal{H} \notin N(B)$, $\forall H \in \mathcal{H}\ \exists D \in \{B\} \cup \mathcal{A}$, $D \subset \bigcap\{C \in \mathcal{H} | H \cup C \neq X\}$.
(ii) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{A} \in F_N(B)$ therefore
$\exists \mathcal{H} \in F_N(B)$, $\forall H \in \mathcal{H}\ \exists D \in \{B\} \cup \mathcal{A}$, $D \subset \bigcap\{C \in \mathcal{H} | H \cup C \neq X\}$; (Where F_N is superfarness induced by N)
(iii) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{A} \in C_N(B)$ therefore
$\exists \mathcal{H} \in C_N(B)$, $\forall H \in \mathcal{H}\ \exists D \in \{X - B\} \cup \mathcal{A}$, $\bigcup\{C \in \mathcal{H} | H \cap C \neq \emptyset\} \subset D$; (Where C_N is supercovering induced by N)
(iv) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{A} \notin S_N(B)$ therefore $\exists \mathcal{H} \subset PX$ s.t. $sec\mathcal{H} \notin S_N(B)$, $\forall H \in \mathcal{H}\ \ \exists D \in \{B\} \cup sec\mathcal{A}$, $D \subset \bigcap\{C \in \mathcal{H} | H \cup C \neq X\}$.
(Where S_N is supersmallness induced by N)

Proof:
(i) and (ii) Obviously are equivalent.
(i) $\Leftrightarrow$ (iii)
Let (i) holds and $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{A} \in C_N(B)$ therefore $\mathcal{A}' = \{X - A | A \in \mathcal{A}\} \notin N(B)$ since $(\mathcal{B}^X, N)$ is uniform so $\exists \mathcal{H}' \notin N(B)$, $\forall H' \in \mathcal{H}\ \exists D' \in \{B\} \cup \mathcal{A}'$, $D' \subset \bigcap\{C' \in \mathcal{H}' | H' \cup C' \neq X\}$. We have $\mathcal{H}' \notin N(B)$ implies $\mathcal{H} = \{X - H' | H' \in \mathcal{H}'\} \in C_N(B)$,
and by De Morgan's laws $D' \subset \bigcap\{C' \in \mathcal{H}' | H' \cup C' \neq X\}$ implies
$\bigcup\{X - C' \in \mathcal{H} | (X - H') \cap (X - C') \neq \emptyset\} \subset X - D'$.
Since $D' \in \{B\} \cup \mathcal{A}'$, $D = X - D' \in \{X - B\} \cup \mathcal{A}$. So we have
$\exists \mathcal{H} \in C_N(B)$, $\forall H \in \mathcal{H}\ \exists D \in \{X - B\} \cup \mathcal{A}$, $\bigcup\{C \in \mathcal{H} | H \cap C \neq \emptyset\} \subset D$.
Similarly its converse holds.
(i) $\Rightarrow$ (iv)
Let $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{A} \notin S_N(B)$ therefore by proposition 4 (10)
$sec\mathcal{A} \notin N(B)$ since $(\mathcal{B}^X, N)$ is uniform so
$\exists \mathcal{H} \notin N(B)$, $\forall H \in \mathcal{H}\ \exists D \in \{B\} \cup sec\mathcal{A}$, $D \subset \bigcap\{C \in \mathcal{H} | H \cup C \neq X\}$. And by proposition 4 (9), $sec\mathcal{H} \notin S_N(B)$. So we have $\exists \mathcal{H} \subset PX$ s.t. $sec\mathcal{H} \notin S_N(B)$, $\forall H \in \mathcal{H}$ $\exists D \in \{B\} \cup sec\mathcal{A}$, $D \subset \bigcap\{C \in \mathcal{H} | H \cup C \neq X\}$.
(iv) $\Rightarrow$ (i)
Let $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{A} \notin N(B)$ so by proposition 4 (9), $sec\mathcal{A} \notin S_N(B)$, therefore
$\exists \mathcal{H} \subset PX$ s.t. $sec\mathcal{H} \notin S_N(B)$, $\forall H \in \mathcal{H}\ \ \exists D \in \{B\} \cup sec^2\mathcal{A}$, $D \subset \bigcap\{C \in \mathcal{H} | H \cup C \neq X\}$.
But by proposition 4 (9), $sec\mathcal{H} \notin S_N(B)$ implies that $\mathcal{H} \notin N(B)$ and we know that if $D \in stack\mathcal{A} (= sec^2\mathcal{A})$ then there exists $A \in \mathcal{A}$ s.t. $A \subset D$ so we have $\exists \mathcal{H} \subset PX$ s.t.
$\mathcal{H} \notin N(B)$, $\forall H \in \mathcal{H}\ \ \exists D_1 \in \{B\} \cup \mathcal{A}$, $D_1 \subset \bigcap\{C \in \mathcal{H} | H \cup C \neq X\}$. ■

Above lemma leads us to following remark:

Remark 12 : A nearness space (X, ξ) is uniform iff the following condition holds:
If $\mathcal{A} \notin \gamma$ therefore $\exists \mathcal{B} \subset PX$ s.t. $sec\mathcal{B} \notin \gamma$, $\forall B \in \mathcal{B}\ \exists D \in sec\mathcal{A}$ with $D \subset \bigcap\{C \in \mathcal{B} | B \cup C \neq X\}$.
(Where γ is smallness induced by ξ) ■

Remark 13 : **NT** denotes the full subcategory of **SN**, whose objects are neartopological.
SNT denotes the full subcategory of **PN**, whose objects are symmetric neartopological.■

Corollary 8: **R$_0$-TOP** is isomorphic to the full subcategory **SNT$^{\mathbf{S}}$** of **SNT**, whose objects are the saturated symmetric neartopological supernear spaces. ■

Remark 14 : **CTSN** denotes the full subcategory of **SN**, whose objects are contiguous supernear spaces.
SCTSN denotes the full subcategory of **CTSN**, whose objects are symmetric contiguous supernear spaces. ■

Corollary 9 : **C-NEAR** is isomorphic to the full subcategory
SCTSN^S of **SCTSN**, whose objects are the saturated symmetric contiguous supernear spaces. ■

Remark 15 : **USN** denotes the full subcategory of **SN**, whose objects are uniform supernear spaces. ■

SUSN denotes the full subcategory of **USN**, whose objects are symmetric uniform supernear spaces.

Theorem 4: The category **UNIF** of uniform spaces and uniformly continuous maps is isomorphic to the full subcategory **SUSN^S** of **SUSN**, whose objects are the saturated symmetric uniform supernear spaces. ■

Definition 28: A supernear space $(\mathcal{B}^X, N)$ is called *separated* iff N satisfies (sep), e.g.
(sep) $x, z \in X$ and $\{\{z\}\} \in N(\{x\})$ imply $x = z$.

Proposition 5: For a nearness space (X, ξ) we have the equivalence:
(i) (X, ξ) is a $N1-space$;
(ii) (PX, N_ξ) is separated paranear space.

Definition 29: A separated supernear space $(\mathcal{B}^X, N)$ is called *paracompact* iff it is neartopological and uniform.

Proposition 6: For a nearness space (X, ξ) we have the equivalence:
(i) (X, ξ) is a paracompact nearness space;
(ii) (PX, N_ξ) is paracompact supernear space.

Proposition 7: For a neartopological supernear space $(\mathcal{B}^X, N)$ following are equivalent:
(i) $(\mathcal{B}^X, N)$ is separated and uniform;
(ii) $(\mathcal{B}^X, N)$ is paracompact.

Proof: By Definition 29, it is obviously.

Definition 30: A supernear space $(\mathcal{B}^X, N)$ is called *compact* iff it is neartopological and contiguous. ■

Remark 16: Every finite neartopological supernear space $(\mathcal{B}^X, N)$ is compact. ■

Proposition 8: For a nearness space (X, ξ) we have the equivalence:
(i) (X, ξ) is a compact nearness space;
(ii) (PX, N_ξ) is compact paranear space.

Definition 31: A supernear space $(\mathcal{B}^X, N)$ is called *precompact* (= totally bounded) iff $\mathcal{F} \in FIL(X)\backslash\{PX\} \Rightarrow \mathcal{F} \in \bigcap\{N(F) | F \in \mathcal{F} \cap \mathcal{B}^X\}$. ■

Proposition 9: For a nearness space (X, ξ) following are equivalent:
(i) (X, ξ) is totally bounded;
(ii) (PX, N_ξ) is precompact.

Lemma 8: Each compact supernear space is precompact.

Proof: Let $\mathcal{F} \in FIL(X)\backslash\{PX\}$ and for some $F \in \mathcal{F} \cap \mathcal{B}^X$, $\mathcal{F} \notin N(F)$ then since N is contiguous, there exists $\mathcal{H} \subset PX$ finite ($\mathcal{H} \notin N(F)$ and $\mathcal{H} \subset \mathcal{F}$) since $\mathcal{H}$ is finite so $\mathcal{H} \cup \{F\}$ is also finite subset of $\mathcal{F}$ and since $\mathcal{F}$ is a proper filter, $\bigcap(\mathcal{H} \cup \{F\}) \neq \emptyset$ so $\exists f \in F$ s.t. $f \in \bigcap(\mathcal{H} \cup \{F\})$ therefore $\mathcal{H} << \{\{f\}\} \in N(\{f\}) \subset N(F)$ and by (SN1) we have $\mathcal{H} \in N(F)$ that is contradiction so $\mathcal{F} \in N(F)$ for every $F \in \mathcal{F} \cap \mathcal{B}^X$ i.e. N is precompact. ■

Proposition 10: For a uniform paranear space $(\mathcal{B}^X, N)$ following are equivalent:
(i) $(\mathcal{B}^X, N)$ is complete and precompact;
(ii) $(\mathcal{B}^X, N)$ is compact.

Proof: $(i \Rightarrow ii)$
Let $\mathcal{H} \in N(B)$ for some $B \in \mathcal{B}^X \backslash \{\emptyset\}$ since N is symmetric $\mathcal{H} \cup \{B\} \in N(B)$ so there exists a $N-cluster$, $\mathcal{A} \in N(B)$ s.t. $\mathcal{H} \cup \{B\} \subset \mathcal{A}$ and since N is complete for some $x \in X$, $\{x\} \in \mathcal{A}$, since N is symmetric $\mathcal{A} \in N(\{x\})$. By (SN1), $\forall A \in \mathcal{A}$, $\{A\} \in N(\{x\})$ i.e. $x \in \bigcap\{cl_N(A) | A \in \mathcal{A}\} \subset \bigcap\{cl_N(H) | H \in \mathcal{H} \cup \{B\}\}$ so $\bigcap\{cl_N(H) | H \in \mathcal{H} \cup \{B\}\} \neq \emptyset$. Therefore N is neartopological.
Now let $\mathcal{A} \notin N(B)$ and every finite subset of $\mathcal{A} \cup \{B\}$ belongs to $N(B)$. Since N is neartopological $\bigcap\{cl_N(H) | H \in \mathcal{H} \cup \{B\}\} \neq \emptyset$ where $\mathcal{H}$ is arbitrary finite subset of $\mathcal{A} \cup \{B\}$, so there exists a proper filter $\mathcal{F}$ s.t. $\{cl_N(A) | A \in \mathcal{A} \cup \{B\}\} \subset \mathcal{F}$. Since N is precompact, for every $F \in \mathcal{F} \cap \mathcal{B}^X$, $\mathcal{F} \in N(F)$ and by (SN1), $\{cl_N(A) | A \in \mathcal{A} \cup \{B\}\} \in N(F)$ so by (SN7), $\mathcal{A} \cup \{B\} \in N(F)$. Since N is symmetric and by (SN1) $\mathcal{A} \in N(B)$ which is a contradiction. So N is contiguous.
$(ii \Rightarrow i)$
We have N is compact by lemma 8 it is precompact. Now let $\mathcal{A}$ be a $N-cluster$ hence $\mathcal{A}$ is maximal in $N(B)$ for some $B \in \mathcal{B}^X \backslash \{\emptyset\}$. Since N is symmetric
$\mathcal{A} \subset \mathcal{A} \cup \{B\} \in N(B)$ so $B \in \mathcal{A}$ i.e. $\mathcal{A} \cup \{B\} = \mathcal{A}$. We have N is neartopological so $\bigcap\{cl_N(A) | A \in \mathcal{A} \cup \{B\}\} \neq \emptyset$ therefore there exists $x \in X$ s.t. $x \in \bigcap\{cl_N(A) | A \in \mathcal{A}\}$ i.e. $\{cl_N(A) | A \in \mathcal{A}\} << \{\{x\}\} \in N(\{x\})$ by (SN1) and (SN7) $\mathcal{A} \in N(\{x\})$, since N is symmetric and $B \in \mathcal{A}$ so $\mathcal{A} \cup \{\{x\}\} \in N(B)$. But $\mathcal{A}$ is maximal therefore $\{x\} \in \mathcal{A}$ i.e. N is complete.

V. SIMPLE COMPLETION OF PARACOVERING SPACE

Proposition 11 : Let $(\mathcal{B}^X, N)$ be a supernear space, then each of the conditions below implies all the following ones:

(1) $(\mathcal{B}^X, N)$ is uniform;
(2) If $\mathcal{A} \in F(B)$, then for each $A \in \mathcal{A} \cup \{B\}$ there exists $H_A \subset X$ such that $\{H_A | A \in \mathcal{A} \cup \{B\}\} \in F(B)$ and
$\{X - \bigcap\{H_A | A \in \mathcal{A}'\}\} \cup \mathcal{A}' \in F(B)$ for any $\mathcal{A}' \subset \mathcal{A} \cup \{B\}$;
(3) If $\mathcal{A} \in F(B)$, then for each $A \in \mathcal{A} \cup \{B\}$ there exists $H_A \subset X$ such that
$\{A, (X - H_A)\} \in F(B)$ and $\{H_A | A \in \mathcal{A} \cup \{B\}\} \in F(B)$;

(4) If $\mathcal{A} \in F(B)$, then
$\{H \subset X | \exists A \in \mathcal{A} \cup \{B\} \ \ with \ \ \{A, (X-H)\} \in F(B)\} \in F(B)$;
(5) If $\mathcal{A} \in F(B)$ and $\mathcal{H} \in S(B)$, then there exists $A \in \mathcal{A} \cup \{B\}$ and $H \in \mathcal{H}$ with $\{A, H\} \in F(B)$;
(6) If $\mathcal{A} \in S(B)$ and $\mathcal{A} \cup \{B\} \in N(B)$, then
$\{H \subset X | \{H\} \cup \mathcal{A} \in N(B)\}$ is $N-cluster$ in $N(B)$ containing $\mathcal{A}$. In addition if $(\mathcal{B}^X, N)$ is symmetric then it is unique $N-cluster$ containing $\mathcal{A} \cup \{B\}$.

Proof:
$(1 \Rightarrow 2)$ Let N be uniform and $\mathcal{A} \in F(B)$ so by lemma 7,
$\exists\, \mathcal{H} \in F(B)$ s.t.$\forall H \in \mathcal{H} \quad \exists A \in \mathcal{A} \cup \{B\}$ with
$A \subset \bigcap\{C \in \mathcal{H} | C \cup H \neq X\}$ (*)
Therefore $\mathcal{A} \cup \{B\} = \mathcal{A}_1 \cup \mathcal{A}_2$ where $\mathcal{A}_1$ is set of members of $\mathcal{A} \cup \{B\}$ such that for some $H \in \mathcal{H}$ it holds (*) and $\mathcal{A}_2 = \mathcal{A} \backslash \mathcal{A}_1$. Let for $A \in \mathcal{A}_1$ we say H_A is $\cap H_i$ where $H_i \in \mathcal{H}$ and $A \subset \bigcap\{C \in \mathcal{H} | C \cup H_i \neq X\}$ and for $A \in \mathcal{A}_2$ we say $H_A = X$ then we call $\mathcal{H}' = \{H_A | A \in \mathcal{A} \cup \{B\}\}$. Easily we can see $\mathcal{H} << \mathcal{H}'$ and by (SF1) $\mathcal{H}' \in F(B)$.
Let $\mathcal{A}' \subset \mathcal{A} \cup \{B\}$ we have three cases.
First case: $\mathcal{A}' \subset \mathcal{A}_2$ so
$\mathcal{D} = \{X - \bigcap\{H_A | A \in \mathcal{A}'\}\} \cup \mathcal{A}' = \{\emptyset\} \cup \mathcal{A}'$ and obviously, $\mathcal{H} << \mathcal{D}$.
Second case: $\mathcal{A}' \subset \mathcal{A}_1$, we consider $A \in \mathcal{A}'$ so there exists $H_i \in \mathcal{H}$ such that $A \subset \bigcap\{C \in \mathcal{H} | C \cup H_i \neq X\}$.
Now let $C \in \mathcal{H}$ so if $C \cup H_i \neq X$ for some H_i then $A \subset C$ and if not so $C \cup \mathcal{H}_i = X$ for every H_i therefore $X - H_i \subset C$ for every H_i therefore $\cup(X - H_i) \subset C$ i.e. $X - \cap H_i \subset C$ so $X - H_A \subset C$. Therefore similarly for every $A \in \mathcal{A}'$ we have either $A \subset C$ or $X - H_A \subset C$. If for some $A \in \mathcal{A}'$, $A \subset C$ so there exists $A \in \mathcal{D}$ s.t. $A \subset C$, and if for every $A \in \mathcal{A}'$, $X - H_A \subset C$ so $\bigcup\{X - H_A | A \in \mathcal{A}'\} \subset C$ i.e. $X - \bigcap\{H_A | A \in \mathcal{A}'\} \subset C$ therefore $\mathcal{H} << \mathcal{D}$.
Third case: $\mathcal{A}' \cap \mathcal{A}_1 \neq \emptyset$ and $\mathcal{A}' \cap \mathcal{A}_2 \neq \emptyset$.
By second case
$\mathcal{H} << \{X - \bigcap\{H_A | A \in \mathcal{A}' \cap \mathcal{A}_1\}\} \cup (\mathcal{A}' \cap \mathcal{A}_1)$,
by definition of H_A we have
$X - \bigcap\{H_A | A \in \mathcal{A}'\} = X - \bigcap\{H_A | A \in \mathcal{A}' \cap \mathcal{A}_1\}$ so $\mathcal{H} << \mathcal{D}$.
And since $\mathcal{H} \in F(B)$ by (SF1) $\mathcal{D} \in F(B)$.
$(2 \Rightarrow 3)$ For $\mathcal{A}' = \{A\}$ 2 implies 3.
$(3 \Rightarrow 4)$ By (3)
$\{H_A | A \in \mathcal{A} \cup \{B\}\} <<$
$\{H \subset X | \exists A \in \mathcal{A} \cup \{B\}\, with\, \{A, (X-H)\} \in F(B)\}$
and since $\{H_A | A \in \mathcal{A} \cup \{B\}\} \in F(B)$ by (SF1)
$\{H \subset X | \exists A \in \mathcal{A} \cup \{B\} \ \ with \ \ \{A, (X-H)\} \in F(B)\}$
$\in F(B)$
$(4 \Rightarrow 5)$ Let $\mathcal{A} \in F(B)$ and $sec\mathcal{H} \in N(B)$ then by (4),
$\mathcal{D} = \{D \subset X | \exists A \in \mathcal{A} \cup \{B\} \ \ with \ \ \{A, (X-D)\} \in F(B)\}$
$\in F(B)$
by Proposition 3, we have $\exists D \in \mathcal{D}\ \exists H \in \mathcal{H}$, $D \cap H = \emptyset$ therefore $H \subset X - D$ so $\{A, (X-D)\} << \{A, H\}$ but for some $A \in \mathcal{A} \cup \{B\}$ we have $\{A, (X-D)\} \in F(B)$ so by (SF1) $\{A, H\} \in F(B)$.
$(5 \Rightarrow 6)$ If $\mathcal{A} \in S(B)$ and
$\mathcal{C} := \{H \subset X | \{H\} \cup \mathcal{A} \in N(B)\} \notin N(B)$ so $\mathcal{C} \in F(B)$, by (5), $\exists C \in \mathcal{C} \cup \{B\}$ and $A \in \mathcal{A}$ with $\{C, A\} \in F(B)$.
But since $\mathcal{A} \cup \{B\} \in N(B)$, $B \in \mathcal{C}$ so $\mathcal{C} \cup \{B\} = \mathcal{C}$.
Therefore for every $A \in \mathcal{A}$ and for every $C \in \mathcal{C}$ we have $\{C, A\} << \{C\} \cup \mathcal{A} \in N(B)$. By (SN1), $\{C, A\} \in N(B)$ that is contradiction. So $\mathcal{C} \in N(B)$ and obviously it contains $\mathcal{A}$.
Let $\mathcal{C} \subset \mathcal{D} \in N(B)$ so for all $D \in \mathcal{D}$, by (SN1), $\mathcal{C} \cup \{D\} \in N(B)$ then we have $\mathcal{A} \cup \{D\} << \mathcal{C} \cup \{D\}$ and by (SN1) we have $\mathcal{A} \cup \{D\} \in N(B)$ therefore $D \in \mathcal{C}$ which implies $\mathcal{C}$ is $N-cluster$.
Now let $(\mathcal{B}^X, N)$ be symmetric and $\mathcal{D}$ be $N-cluster$ containing $\mathcal{A} \cup \{B\}$. Note that $\mathcal{D} \in N(B')$ for some $B' \in \mathcal{B}^X$ and maximal in $N(B') \backslash \{\emptyset\}$. By symmetric property $\mathcal{D} \in N(B)$.
For every $D \in \mathcal{D}$, we have $\mathcal{A} \cup \{D\} << \mathcal{D} \in N(B)$ and by (SN1),
$\mathcal{A} \cup \{D\} \in N(B)$ so $\mathcal{D} \subset \mathcal{C}$.
On the other hand again by symmetric property and maximality of $\mathcal{D}$ in $N(B')$ we have $B' \in \mathcal{D}$ so $B' \in \mathcal{C}$ and it implies that $\mathcal{C} \in N(B')$. But by maximality of $\mathcal{D}$ in $N(B')$ and $\mathcal{D} \subset \mathcal{C}$ we have $\mathcal{C} = \mathcal{D}$. Therefore $\mathcal{C}$ is unique $N-cluster$ containing $\mathcal{A} \cup \{B\}$.

Definition 32: Let $(\mathcal{B}^X, C)$ be a supercovering space and $(\mathcal{B}^X, N_C)$ its corresponding supernear space. $\mathcal{H}$ is called near Cauchy-system in C iff for some $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mathcal{H} \in N_C(B)$ and $sec\mathcal{H} \in N_C(B)$. Or equivalently, $\mathcal{H} \in N_C(B) \cap S_C(B)$. Where S_C is supersmallness generated by C.

Definition 33: A paracovering space $(\mathcal{B}^X, C)$ is called
(i) *C_1-space* iff N_C is separated. (Where N_C is supernearness generated by C)
(ii) *graded* iff for every near Cauchy-system $\mathcal{M}$ in C,
$\mathcal{D} = \{D \subset X : \mathcal{M} \cup \{D\} \ \ is \ \ N_C - system\}$
is $N_C - system$. ■

Remark 17: Each uniform paracovering space is graded.

Proof: Let $(\mathcal{B}^X, C)$ be a uniform paracovering space,N_C its corresponding supernear operator and $\mathcal{M}$ be a near Cauchy-system in C, hence $\mathcal{M} \in N_C(B)$ and
$sec\mathcal{M} \in N_C(B)$ for some $B \in \mathcal{B}^X \backslash \{\emptyset\}$. Therefore $\{B\} \cup \mathcal{M} \in N_C(B)$, since $B \neq \emptyset$ and C is symmetric. Now since N_C is uniform by Proposition 11 we have
$\{D \subset X | \mathcal{M} \cup \{D\} \in N_C(B)\} \in N_C(B)$ i.e. $(\mathcal{B}^X, C)$ is graded. ■

Recall 2: A separated N_1 space in the sense of Herrlich is a N_1 covering space (X, ξ) which satisfies (sep), e.g.
(sep) For every near Cauchy-system $\mathcal{A}$ in that covering, $\{D \subset X | \mathcal{A} \cup \{D\} \in \xi\} \in \xi$.

Remark 18: A nearness space is separated N_1 space iff the corresponding paracovering space is graded C_1 space.

Proof: Let (X, ξ) be a nearness space then (PX, N) and (PX, C) are respectively its corresponding paranear and paracovering spaces. Obviously (X, ξ) is N_1 iff (PX, C) is C_1 so it is enough to show that (X, ξ) is separated iff (PX, C) is graded.
Suppose (X, ξ) is separated. Now let $\mathcal{H}$ be a near Cauchy-system in C i.e. for some $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mathcal{H} \in N(B)$ and

$sec\mathcal{H} \in N(B)$ so $\mathcal{H} \cup \{B\} \in \xi$ and $(sec\mathcal{H}) \cup \{B\} \in \xi$ by (N1) property $\mathcal{H} \in \xi$ and $sec\mathcal{H} \in \xi$ i.e $\mathcal{H}$ is a Cauchy-system in ξ.
Let for $D \subset X$ we have $\mathcal{H} \cup \{D\} \in N(B)$ so $\mathcal{H} \cup \{D, B\} \in \xi$ and by (N1) $\mathcal{H} \cup \{D\} \in \xi$ so
$\{D \subset X | \mathcal{H} \cup \{D\} \in N(B)\} \subset \{D \subset X | \mathcal{H} \cup \{D\} \in \xi\}$ (1)
Since $\mathcal{H} \cup \{B\} \in \xi$ it implies $B \in \{D \subset X | \mathcal{H} \cup \{D\} \in \xi\}$ and by assumption we have $\{D \subset X | \mathcal{H} \cup \{D\} \in \xi\} \in \xi$ so by definition of ξ, $\{D \subset X | \mathcal{H} \cup \{D\} \in \xi\} \in N(B)$.
By (SN1) and (1) we have
$\{D \subset X | \mathcal{H} \cup \{D\} \in N(B)\} \in N(B)$ i.e. (PX, C) is graded.
Conversely, assume (PX, C) be graded. Let $\mathcal{H}$ be a near Cauchy-system in ξ i.e. $\mathcal{H} \in \xi$ and $sec\mathcal{H} \in \xi$ now by definition $\mathcal{H} \in N(B)$ for every $B \in \mathcal{H}$ so by (SN5) $\mathcal{H} \in N(X)$ similarly $sec\mathcal{H} \in N(X)$ therefore $\mathcal{H}$ is a near Cauchy-system in C.
Let for $D \subset X$ we have $\mathcal{H} \cup \{D\} \in \xi$ so $\mathcal{H} \cup \{D\} \in N(B)$ for every $B \in \mathcal{H}$ and by (SN5) $\mathcal{H} \cup \{D\} \in N(X)$ so
$\{D \subset X | \mathcal{H} \cup \{D\} \in \xi\} \subset \{D \subset X | \mathcal{H} \cup \{D\} \in N(X)\}$ (2)
Since $(\mathcal{B}^X, C)$ is graded
$\{D \subset X | \mathcal{H} \cup \{D\} \in N(X)\} \in N(X)$ by symmetric property of N and definition of ξ, $\{D \subset X | \mathcal{H} \cup \{D\} \in N(X)\} \in \xi$.
Now by (N1) and (2) we have $\{D \subset X | \mathcal{H} \cup \{D\} \in \xi\} \in \xi$ i.e. (X, ξ) is separated.

Definition 34: A paracovering space $(\mathcal{B}^X, C)$ is called closed iff N_C satisfies (cl), e.g. (Where N_C is paranearness generated by C)
(cl) $B \in \mathcal{B}^X$ implies $cl_{N_C}(B) \in \mathcal{B}^X$.

Theorem 5: Let $(\mathcal{B}^X, C)$ be a closed graded C_1 paracovering space. And X^* be set of all clusters in $(\mathcal{B}^X, N_C)$, where N_C is supernearness generated by C.
Let rebounded $e : X \to X^*$ be defined by $e(x) := \{A \subset X : x \in cl_{N_C}(A)\}$ for each $x \in X$.
$\mathcal{B}^{X^*}$ be the **B**-set defined by
$\mathcal{B}^{X^*} := \{B^* \subset X^* : \exists B \in \mathcal{B}^X \ \ B^* \subset N_C(B)\}$;
and $\Omega \subset PX^*$ belongs to $C^*(B^*)$ iff $\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$ such that for each cluster $\mathcal{D} \in X^*$ there exists $A^* \in \Omega \cup \{X^* - B^*\}$ with $\mathcal{D} \in A^*$ and then $e^{-1}[A^*] \in sec\mathcal{D}$.
Then $(\mathcal{B}X^*, C^*)$ is complete C_1 paracovering space and $e : X \to X^*$ dense embedding.

Proof:
First we show that $\mathcal{B}^{X^*}$ is a **B**-set.
To (B1): $\emptyset \in \mathcal{B}^{X^*}$ obviously.
To (B2): Let $\mathcal{H}$ be an $N - cluster$ then $\mathcal{H} \in N(B) \setminus \{\emptyset\}$ for some $B \in \mathcal{B}^X$ so $\{\mathcal{H}\} \in \mathcal{B}^{X^*}$.
To (B3): Let $B_1^* \subseteq B_2^* \in \mathcal{B}^{X^*}$ then there exists $B \in \mathcal{B}^X$ such that for every $\mathcal{C} \in B_2^*, \mathcal{C} \in N(B)$ therefore for every $\mathcal{C} \in B_1^*$, $\mathcal{C} \in N(B)$ i.e. $B_1^* \in \mathcal{B}^{X^*}$.
Therefore $\mathcal{B}^{X^*}$ is a **B**-set.
Now let $B^* \in \mathcal{B}^{X^*}$ so $B^* \subset N_C(B)$ for some $B \in \mathcal{B}^X$, if $e^{-1}[B^*] = \emptyset$ so by (B1), $e^{-1}[B^*] \in \mathcal{B}^X$. Otherwise there exists $x \in e^{-1}[B^*]$ therefore there exists $\mathcal{D} \in B^*$ s.t. $e^{-1}(\mathcal{D}) = x$ and since C is symmetric and $\mathcal{D}$ is maximal, $B \in \mathcal{D}$ so $x \in cl_{N_C}(B)$. Therefore $e^{-1}[B^*] \subset cl_{N_C}(B)$ so since C is closed and by (B3), $e^{-1}[B^*] \in \mathcal{B}^X$.
Therefore e is rebounded map.
Now we show that $(\mathcal{B}^{X^*}, C^*)$ is supercovering space;
To (SC1): Let $\Omega_2 \prec \Omega_1$ and $\Omega_2 \in C^*(B^*)$, so
$\{e^{-1}[A_2^*] | A_2^* \in \Omega_2 \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$.
Let $A_2 \in \{e^{-1}[A_2^*] | A_2^* \in \Omega_2 \cup \{X^* - B^*\}\}$
so $\exists A_2{}^* \in \Omega_2 \cup \{X^* - B^*\}$ s.t. $A_2 = e^{-1}[A_2^*]$.
Since $\Omega_2 \prec \Omega_1$, so $\Omega_2 \cup \{X^* - B^*\} \prec \Omega_1 \cup \{X^* - B^*\}$ therefore $\exists A_1^* \in \Omega_1 \cup \{X^* - B^*\}$ s.t. $A_2^* \subseteq A_1^*$.
So $e^{-1}[A_2^*] \subset e^{-1}[A_1^*]$ therefore
$\{e^{-1}[A_2^*] | A_2^* \in \Omega_2 \cup \{X^* - B^*\}\} \prec$
$\{e^{-1}[A_1^*] | A_1^* \in \Omega_1 \cup \{X^* - B^*\}\}$
we have $\{e^{-1}[A_2^*] | A_2^* \in \Omega_2 \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$ so by (SC1) in C,
$\{e^{-1}[A_1^*] | A_1^* \in \Omega_1 \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$.
On the other hand for each cluster $\mathcal{D} \in X^*$ there exists $A_2^* \in \Omega_2 \cup \{X^* - B^*\}$ with $\mathcal{D} \in A_2^*$ and $e^{-1}[A_2^*] \in sec\mathcal{D}$.
Since $\Omega_2 \cup \{X^* - B^*\} \prec \Omega_1 \cup \{X^* - B^*\}$ there exists $A_1^* \in \Omega_1 \cup \{X^* - B^*\}$ s.t. $A_2^* \subseteq A_1^*$ therefore $\mathcal{D} \in A_1^*$ and $e^{-1}[A_2^*] \subseteq e^{-1}[A_1^*]$ also since $e^{-1}[A_2^*] \in sec\mathcal{D}$ i.e. $\forall D \in \mathcal{D}, e^{-1}[A_2^*] \cap D \neq \emptyset$ so $\forall D \in \mathcal{D}, e^{-1}[A_1^*] \cap D \neq \emptyset$ i.e. $e^{-1}[A_1^*] \in sec\mathcal{D}$.
To (SC2): Let $B^* \in \mathcal{B}^{X^*}$, if $\emptyset \in C^*(B^*)$ then $\{e^{-1}[X^* - B^*]\} \in C(e^{-1}[B^*])$ so
$\{X - e^{-1}[B^*]\} \in C(e^{-1}[B^*])$ that is a contradiction to proposition 2, in C.
And $\{X^*\} \in C^*(B^*)$ since $\{e^{-1}[X^*]\} = \{X\}$ and by (SC2) in C we have $\{X\} \in C(e^{-1}[B^*])$ since $\{e^{-1}[X^*]\} \prec \{e^{-1}[X^*], e^{-1}[X^* - B^*]\}$ by (SC1), $\{e^{-1}[X^*], e^{-1}[X^* - B^*]\} \in C(e^{-1}[B^*])$. Also for every $\mathcal{D} \in X^*$, $\mathcal{D} \in X^*$ and $e^{-1}[X^*] = X \in sec\mathcal{D}$.
To (SC3): Let $\Omega \neq \emptyset$ then obviously $\Omega \in C^*(\emptyset)$. Since
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - \emptyset\}\} \in C(e^{-1}[\emptyset])$ holds and for every $\mathcal{D} \in X^*$,
$\mathcal{D} \in X^*$ and $e^{-1}[X^*] = X \in sec\mathcal{D}$.
To (SC4): Let $\mathcal{D} \in X^*$ then $\{X^* - \{\mathcal{D}\}\} \notin C^*(\{\mathcal{D}\})$ since $\mathcal{D} \in X^*$ but $\mathcal{D} \notin \{X^* - \{\mathcal{D}\}\}$.
To (SC5): Let $B_1^* \subset B_2^* \in \mathcal{B}^{X^*}$ and $\Omega \in C^*(B_2^*)$ so
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B_2^*\}\} \in C(e^{-1}[B_2^*])$.
Since $B_1^* \subset B_2^*$ then $e^{-1}[B_1^*] \subset e^{-1}[B_2^*]$ so by (SC5) in C, we have $\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B_2^*]\} \in C(e^{-1}[B_1^*])$.
Also $B_1^* \subset B_2^*$ implies $X^* - B_2^* \subset X^* - B_1^*$ so
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B_2^*]\} \prec$
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B_1^*]\}$
and by (SC1) in C we have
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B_1^*\}\} \in C(e^{-1}[B_1^*])$.
And for each cluster $\mathcal{D} \in X^*$ there exists $A^* \in \Omega \cup \{X^* - B_2^*\}$ with $\mathcal{D} \in A^*$ and $e^{-1}[A^*] \in sec\mathcal{D}$. since $B_1^* \subset B_2^*$ so $A^* \in \Omega \cup \{X^* - B_1^*\}$.
To (SC6): Let $B^* \in \mathcal{B}^{X^*}$, $\Omega_1 \in C^*(B^*)$ and $\Omega_2 \in C^*(B^*)$ so $\{e^{-1}[A_1^*] | A_1^* \in \Omega_1 \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$ and
$\{e^{-1}[A_2^*] | A_2^* \in \Omega_2 \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$,
by (SC6) on C,
$\{e^{-1}[A_1^*] | A_1^* \in \Omega_1 \cup \{X^* - B^*\}\} \bigwedge$
$\{e^{-1}[A_2^*] | A_2^* \in \Omega_2 \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$.
Also we have
$\{e^{-1}[A_1^*] | A_1^* \in \Omega_1 \cup \{X^* - B^*\}\} \bigwedge$
$\{e^{-1}[A_2^*] | A_2^* \in \Omega_2 \cup \{X^* - B^*\}\} \prec$

$\{e^{-1}[A^*]|A^* \in \Omega_1 \wedge \Omega_2 \cup \{X^* - B^*\}\}$
Therefore by (SC1),
$\{e^{-1}[A^*]|A^* \in \Omega_1 \wedge \Omega_2 \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$.
And for each $\mathcal{D} \in X^*$ there exist $A_1^* \in \Omega_1 \cup \{X^* - B^*\}$ and $A_2^* \in \Omega_2 \cup \{X^* - B^*\}$ s.t. $\mathcal{D} \in A_1^*$ and $e^{-1}[A_1^*] \in sec\mathcal{D}$ as well as $\mathcal{D} \in A_2^*$ and $e^{-1}[A_2^*] \in sec\mathcal{D}$.
So $\mathcal{D} \in A_1^* \cap A_2^* \in (\Omega_1 \wedge \Omega_2) \cup \{X^* - B^*\}$ and $e^{-1}[A_1^* \cap A_2^*] \in sec\mathcal{D}$, since by remark 10, $\mathcal{D}$ is grill, and consequently, $sec\mathcal{D} \in FIL(X)$.
Therefore we have $\Omega_1 \bigwedge \Omega_2 \in C^*(B^*)$.
To (SC7): Let be $\mathcal{B}^* \in \mathcal{B}^{X^*}$ and $\Omega \in C^*(B^*)$ we have to verify $\{int_{C^*}(A^*)|A^* \in \Omega\} \in C^*(B^*)$.
By supposition we get
$\{e^{-1}[A^*]|A^* \in \Omega \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$,
hence by (SC7) on C,
$\{int_C(e^{-1}[A^*]) : A^* \in \Omega \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$.
Now, let be $A^* \in \Omega \cup \{X^* - B^*\}$ we will show that
$int_C(e^{-1}[A^*]) \subset e^{-1}[int_{C^*}(A^*)]$ because according to (SC1) the assertion holds.
$x \in int_C(e^{-1}[A^*])$ implies $\{e^{-1}[A^*]\} \in C(\{x\})$. $(*)$
But $e(x) \in int_{C^*}(A^*)$ is equivalent the statement $\{A^*\} \in C^*(\{e(x)\})$ which again is equivalent to the statements that
firstly $\{e^{-1}[A^*], e^{-1}[X^* - \{e(x)\}]\} \in C(e^{-1}[\{e(x)\}])$ and secondly for each $\mathcal{D} \in X^*$, $\mathcal{D} \in \omega$ for some $\omega \in \{A^*, (X^* - \{e(x)\})\}$ and $e^{-1}[\omega] \in sec\mathcal{D}$ hold.
Since C is separated equivalent to first we have to show
$\{e^{-1}[A^*], e^{-1}[X^* - \{e(x)\}]\} \in C(\{x\})$.
Easily we can see
$\{e^{-1}[A^*]\} \prec \{e^{-1}[A^*], e^{-1}[X^* - \{e(x)\}]\}$ and by (*) and (SC1), the first one of these is valid.
By proposition 2, and (*), we have $\{x\} \subset e^{-1}[A^*]$, hence for $\mathcal{D} = e(x)$, $\mathcal{D} \in A^*$ now we show that, $e^{-1}[A^*] \in sec\mathcal{D}$.
Let $D \in \mathcal{D}$ so $\{D\} \in N_C(\{x\})$ i.e. $\{X - D\} \notin C(\{x\})$, since
$\{e^{-1}[A^*]\} \in C(\{x\})$ so by (SC1), $\{e^{-1}[A^*]\}$ does not refine $\{X - D\}$ i.e. $e^{-1}[A^*]$ is not subset of $X - D$
therefore $D \cap e^{-1}[A^*] \neq \emptyset$. So $e^{-1}[A^*] \in sec\mathcal{D}$.
And for $\mathcal{D} \neq e(x)$, $\mathcal{D} \in X^* - \{e(x)\}$. And we have $e^{-1}[X^* - \{e(x)\}] = X - \{x\}$ and since C is symmetric $\{x\} \notin \mathcal{D}$ so obviously $X - \{x\} \in sec\mathcal{D}$.
Therefore $int_C(e^{-1}[A^*]) \subset e^{-1}[int_{C^*}(A^*)]$.
So by (SC1) and property of interior,
$\{e^{-1}[int_{C^*}(A^*)]|A^* \in \Omega\} \cup \{e^{-1}[X^* - B^*]\} \in C(e^{-1}[B^*])$.
Now, let be $\mathcal{D} \in X^*$, choose $A^* \in \Omega \cup \{X^* - B^*\}$ such that $\mathcal{D} \in A^*$ and $e^{-1}[A^*] \in sec\mathcal{D}$.
If $\mathcal{D} \in X^* - B^*$ s.t. $e^{-1}[X^* - B^*] \in sec\mathcal{D}$ no thing remain to prove. But if not so $A^* \in \Omega$ and We have to verify
(i) $\mathcal{D} \in int_{C^*}(A^*)$ and
(ii) $e^{-1}[int_{C^*}(A^*)] \in sec\mathcal{D}$.
To (i): $\{A^*\} \in C^*(\{\mathcal{D}\})$ is equivalent to
(1) $\{e^{-1}[A^*], e^{-1}[X^* - \{\mathcal{D}\}]\} \in C(e^{-1}[\{\mathcal{D}\}]$ and
(2) For $\mathcal{A} \in X^*$, $\mathcal{A} \in \omega$ for some $\omega \in \{A^*, X^* - \{\mathcal{D}\}\}$ and $e^{-1}[\omega] \in sec\mathcal{A}$.
First case: $e^{-1}[\{\mathcal{D}\}] = \emptyset$.
Since $e^{-1}[A^*] \in sec\mathcal{D}$ so $e^{-1}[A^*] \neq \emptyset$, and also since $e^{-1}[X^* - \{\mathcal{D}\}] \neq \emptyset$ so according to (SC3), $\{e^{-1}[A^*], e^{-1}[X^* - \{\mathcal{D}\}]\} \in C(\emptyset)$ follows.

Second case: $e^{-1}[\{\mathcal{D}\}] \neq \emptyset$, consequently there exists $x \in X$ with $e(x) = \mathcal{D}$.
We will proving that $\{e^{-1}[A^*], e^{-1}[X^* - \{\mathcal{D}\}]\} \in C(\{x\})$ (note that C is separated.) is valid.
Suppose $\{e^{-1}[A^*], e^{-1}[X^* - \{\mathcal{D}\}]\} \notin C(\{x\})$
so $\{X\backslash\ e^{-1}[A^*], \{x\}\} \in N_C(\{x\})$ by (SN1), $\{X\backslash e^{-1}[A^*]\} \in N_C(\{x\})$, so $x \in cl_{N_C}(X\backslash e^{-1}[A^*])$ i.e. $X\backslash e^{-1}[A^*] \in \mathcal{D}$ but $e^{-1}[A^*] \cap (X\backslash e^{-1}[A^*]) = \emptyset$ so $e^{-1}[A^*] \notin sec\mathcal{D}$ which is contradiction.
To (2) : We have two cases, either $\mathcal{A} = \mathcal{D}$ or $\mathcal{A} \neq \mathcal{D}$.
If $\mathcal{A} = \mathcal{D}$ so by assumption $\mathcal{A} \in A^*$ and $e^{-1}[A^*] \in sec\mathcal{A}$.
And if $\mathcal{A} \neq \mathcal{D}$ so $\mathcal{A} \in X^* - \{\mathcal{D}\}$.
Now if $e^{-1}[\{\mathcal{D}\}] = \emptyset$ then $e^{-1}[X^* - \{\mathcal{D}\}] = X$ so $e^{-1}[X^* - \{\mathcal{D}\}] \in sec\mathcal{A}$ and if $e^{-1}[\{\mathcal{D}\}] = x$ for some $x \in X$, since $\mathcal{D} \neq \mathcal{A}$ and C is symmetric so $\{x\} \notin \mathcal{A}$ therefore $e^{-1}[X^* - \{\mathcal{D}\}] \in sec\mathcal{A}$.
To (ii): It remains to verify $e^{-1}[int_{C^*}(A^*)] \in sec\mathcal{D}$. We will show that $int_C(e^{-1}[A^*]) \in sec\mathcal{D}$
therefore get desired assertion,
because $int_C(e^{-1}[A^*]) \subset e^{-1}[int_{C^*}(A^*)]$ is valid.
Suppose $int_C(e^{-1}[A^*]) \notin sec\mathcal{D}$, so there exists $D \in \mathcal{D}$ s.t.
$D \cap int_C(e^{-1}[A^*]) = \emptyset$ therefore $D \subset X - int_C(e^{-1}[A^*])$ and since $\mathcal{D}$ is N- cluster
therefore by remark 10, $\mathcal{D}$ is grill so $X - int_C(e^{-1}[A^*]) \in \mathcal{D}$
but $X - int_C(e^{-1}[A^*]) = cl_{N_C}(X - e^{-1}[A^*])$
therefore $cl_{N_C}(X - e^{-1}[A^*]) \in \mathcal{D}$ and again
by remark 10, $X - e^{-1}[A^*] \in \mathcal{D}$ therefore $e^{-1}[A^*] \notin sec\mathcal{D}$ which is contradiction.
Therefore $(\mathcal{B}^{X^*}, C^*)$ is supercovering.

Now we show $(\mathcal{B}^{X^*}, C^*)$ is symmetric.
Let $B^* \in \mathcal{B}^{X^*} \setminus \{\emptyset\}$, $\Omega \subset PX^*$ and
$\{X^* - A^*|A^* \in \{B^*\} \cup \Omega\} \in$
$\cup \{C^*(F^*)|F^* \in (\Omega \cap \mathcal{B}^{X^*}) \cup \{B^*\}\}$
i.e. $\{X^* - A^*|A^* \in \{B^*\} \cup \Omega\} \in C^*(F^*)$ for some $F^* \in (\Omega \cap \mathcal{B}^{X^*}) \cup \{B^*\}$ therefore $\{e^{-1}[X^* - A^*]|A^* \in \{B^*\} \cup \Omega\} \in C(e^{-1}[F^*])$ and
for every $\mathcal{D} \in X^*$, there exists $A^* \in \{B^*\} \cup \Omega$ s.t. $\mathcal{D} \in X^* - A^*$ and $e^{-1}[X^* - A^*] \in sec\mathcal{D}$ i.e. $\mathcal{D} \notin A^*$ and $X - e^{-1}[A^*] \in sec\mathcal{D}$. (1)
Since we have
$\{X - e^{-1}[A^*]|A^* \in \{B^*\} \cup \Omega\} \in C(e^{-1}[F^*])$ and C is symmetric it implies $\{X - e^{-1}[\omega]|\omega \in \Omega\} \in C(e^{-1}[B^*])$ i.e. $\{e^{-1}[X^* - \omega]|\omega \in \Omega\} \in C(e^{-1}[B^*])$. (2)
(1) and (2) imply that, $(\mathcal{B}^{X^*}, C^*)$ is symmetric. Therefore $(\mathcal{B}^{X^*}, C^*)$ is paracovering space.

Now we have to show that it is complete.
Suppose Ω is $N_{C^*} - cluster$ so for some $B^* \in \mathcal{B}^{X^*}$, Ω is maximal in $N_{C^*}(B^*)$. By definition of N_{C^*} we have $\{X^* - A^*|A^* \in \Omega\} \notin C^*(B^*)$ therefore by definition of C^* there are three cases,
Case 1: $\exists\mathcal{D} \in X^*$ s.t. $\forall A^* \in \Omega \cup \{B^*\}$, $\mathcal{D} \notin X^* - A^*$ i.e. $\forall A^* \in \Omega \cup \{B^*\}$, $\mathcal{D} \in A^*$. Now we consider $\Omega \cup \{\{\mathcal{D}\}\}$ that easily we can see $\{X^* - A^*|A^* \in \Omega \cup \{\{\mathcal{D}\}\}\} \notin C^*(B^*)$
i.e. $\Omega \cup \{\{\mathcal{D}\}\} \in N_{C^*}(B^*)$.
Since Ω is maximal it implies $\{\mathcal{D}\} \in \Omega$.
Case 2: $\exists\mathcal{D} \in X^*$ such that for every $A^* \in \Omega \cup \{B^*\}$ where $\mathcal{D} \in X^* - A^*$ we have $e^{-1}[X^* - A^*] \notin sec\mathcal{D}$.

So for $\Omega \cup \{\{\mathcal{D}\}\}$ also we have for every $A^* \in \Omega \cup \{\{\mathcal{D}\}\}$ where $\mathcal{D} \in X^* - A^*$ we have $e^{-1}[X^* - A^*] \notin sec\mathcal{D}$ so $\{X^* - A^* | A^* \in \Omega \cup \{\{\mathcal{D}\}\}\} \notin C^*(B^*)$ i.e.
$\Omega \cup \{\{\mathcal{D}\}\} \in N_{C^*}(B^*)$.
Since Ω is maximal it implies $\{\mathcal{D}\} \in \Omega$.
Case 3: $\{e^{-1}[X^* - A^*] | A^* \in \Omega \cup \{B^*\}\} \notin C(e^{-1}[B^*])$ i.e. $\{X - e^{-1}[A^*] | A^* \in \Omega \cup \{B^*\}\} \notin C(e^{-1}[B^*])$ therefore $\{e^{-1}[A^*] | A^* \in \Omega \cup \{B^*\}\} \in N_C(e^{-1}[B^*])$ so there exists a N_C - cluster, $\mathcal{D}$, s.t.
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{B^*\}\} \subset \mathcal{D} \in N_C(e^{-1}[B^*])$.
So for every $A^* \in \Omega \cup \{B^*\}$ we have $e^{-1}[A^*] \in \mathcal{D}$ therefore $e^{-1}[X^* - A^*] \notin sec\mathcal{D}$ which lead us to case 1 or case 2.
Therefore $(\mathcal{B}^{X^*}, C^*)$ is complete.

Now we show that $(\mathcal{B}^{X^*}, C^*)$ is C_1.
Let $\{\{\mathcal{H}\}\} \in N_{C^*}(\{\mathcal{A}\})$ i.e. $\{X^* - \{\mathcal{H}\}\} \notin C^*(\{\mathcal{A}\})$ so there are two cases,
Case 1: There is $\mathcal{D} \in X^*$ s.t. if $\mathcal{D} \in X^* - \{\mathcal{H}\}$ then $e^{-1}[X^* - \{\mathcal{H}\}] \notin sec\mathcal{D}$ and if $\mathcal{D} \in X^* - \{\mathcal{A}\}$ then $e^{-1}[X^* - \{\mathcal{A}\}] \notin sec\mathcal{D}$. So
subcase 1.1: if $\mathcal{D} \notin X^* - \{\mathcal{H}\}$ and $\mathcal{D} \notin X^* - \{\mathcal{A}\}$ therefore $\mathcal{D} = \mathcal{A} = \mathcal{H}$.
subcase 1.2: if $\mathcal{D} \notin X^* - \{\mathcal{H}\}$ and $\mathcal{D} \in X^* - \{\mathcal{A}\}$ then $\mathcal{D} = \mathcal{H}$ and $e^{-1}[X^* - \{\mathcal{A}\}] \notin sec\mathcal{D}$ so $e^{-1}[\{\mathcal{A}\}] \neq \emptyset$ i.e. there exists $x \in X$ s.t. $e^{-1}(\mathcal{A}) = x$ therefore $X - \{x\} \notin sec\mathcal{D}$ so $\{x\} \in \mathcal{D}$ i.e. $e^{-1}[\{\mathcal{A}\}] \in \mathcal{D}$ and since C is symmetric and $\mathcal{D}$, $\mathcal{A}$ are maximal in $N(\{e^{-1}[\{\mathcal{A}\}]\})$, so $\mathcal{A} = \mathcal{D}$. Which implies $\mathcal{H} = \mathcal{D} = \mathcal{A}$.
subcase 1.3: if $\mathcal{D} \in X^* - \{\mathcal{H}\}$ and $\mathcal{D} \notin X^* - \{\mathcal{A}\}$ then $\mathcal{D} = \mathcal{A}$ and $e^{-1}[X^* - \{\mathcal{H}\}] \notin sec\mathcal{D}$ so $e^{-1}[\{\mathcal{H}\}] \neq \emptyset$ i.e. there exists $x \in X$ s.t. $e^{-1}(\mathcal{H}) = x$ therefore $X - \{x\} \notin sec\mathcal{D}$ so $\{x\} \in \mathcal{D}$ i.e. $e^{-1}[\{\mathcal{H}\}] \in \mathcal{D}$ and since C is symmetric and $\mathcal{D}$, $\mathcal{H}$ are maximal in $N(\{e^{-1}[\{\mathcal{H}\}]\})$, so $\mathcal{H} = \mathcal{D}$. Which implies $\mathcal{H} = \mathcal{D} = \mathcal{A}$.
subcase 1.4: if $\mathcal{D} \in X^* - \{\mathcal{H}\}$ and $\mathcal{D} \in X^* - \{\mathcal{A}\}$ then $e^{-1}[X^* - \{\mathcal{H}\}] \notin sec\mathcal{D}$ and $e^{-1}[X^* - \{\mathcal{A}\}] \notin sec\mathcal{D}$ so $e^{-1}[\{\mathcal{H}\}] \neq \emptyset$ and $e^{-1}[\{\mathcal{A}\}] \neq \emptyset$ i.e. there exists $x, y \in X$ s.t. $e^{-1}(\mathcal{A}) = x$ and $e^{-1}(\mathcal{H}) = y$ therefore $X - \{x\} \notin sec\mathcal{D}$ and $X - \{y\} \notin sec\mathcal{D}$ so $\{x\} \in \mathcal{D}$ and $\{y\} \in \mathcal{D}$ i.e. $e^{-1}[\{\mathcal{A}\}] \in \mathcal{D}$ and $e^{-1}[\{\mathcal{H}\}] \in \mathcal{D}$. Since C is symmetric and $\mathcal{D}$, $\mathcal{H}$, are maximal in $N(\{e^{-1}[\{\mathcal{H}\}]\})$, so $\mathcal{H} = \mathcal{D}$ and similarly $\mathcal{D}$, $\mathcal{A}$, are maximal in $N(\{e^{-1}[\{\mathcal{A}\}]\})$, so $\mathcal{A} = \mathcal{D}$ Which implies $\mathcal{D} = \mathcal{H} = \mathcal{A}$.
Case 2: $\{e^{-1}[X^* - \{\mathcal{H}\}], e^{-1}[X^* - \{\mathcal{A}\}]\} \notin C(\{e^{-1}(\mathcal{A})\})$ i.e. $\{X - \{e^{-1}(\mathcal{H})\}, X - \{e^{-1}(\mathcal{A})\}\} \notin C(\{e^{-1}(\mathcal{A})\})$ so $\{\{e^{-1}(\mathcal{H})\}, \{e^{-1}(\mathcal{A})\}\} \in N_C(\{e^{-1}(\mathcal{A})\})$ so $\{\{e^{-1}(\mathcal{H})\}\} \in N_C(\{e^{-1}(\mathcal{A})\})$
since N_C is separated it implies $e^{-1}(\mathcal{H}) = e^{-1}(\mathcal{A})$ and since by proposition 4, e is one to one, $\mathcal{H} = \mathcal{A}$.
Therefore $(\mathcal{B}^{X^*}, C^*)$ is C_1.

Now we show that e is dense embedding.
Let $\mathcal{H} \in X^*$ we prove that $\mathcal{H} \in cl_{N_{C^*}}(e[X])$ i.e. $\{e[X]\} \in N_{C^*}(\{\mathcal{H}\})$ i.e. $\{X^* - e[X]\} \notin C^*(\{\mathcal{H}\})$
We know $e^{-1}[X^* - e[X]] = \emptyset$ then we can say
$\{e^{-1}[X^* - e[X]]\} \notin C(\{e^{-1}(\mathcal{H})\})$. And it implies $\{X^* - e[X]\} \notin C^*(\{\mathcal{H}\})$. So e is dense embedding.

Definition 35: Let $(\mathcal{B}^X, C)$ be a closed graded C_1 paracovering space, then $(\mathcal{B}^{X^*}, C^*)$ is called its simple completion.

VI. Conclusion

At the end we can recall following statement:

Recall 3 [10]: Let (X, μ) be N_1 separated covering space (i.e. (X, ξ_μ) is N_1 separated nearness space) and
(i) X^* be set of all $\xi_\mu - clusters$;
(ii) $e : X \to X^*$ be defined by $e(x) := \{A \subset X | x \in cl_{\xi_\mu}(A)\}$ for each $x \in X$.
A cover Ω of X^* belongs to μ^* iff the following are satisfied:
(1) $\{e^{-1}[\omega] | \omega \in \Omega\} \in \mu$;
(2) For each $\mathcal{A} \in X^*$, there exists some $\omega \in \Omega$ such that $\mathcal{A} \in \omega$ and $e^{-1}[\omega] \in sec\mathcal{A}$.
Then (X^*, μ^*) is a complete N_1 separated covering space and e is a dense embedding hence an epimorphism in the category of complete N_1 separated covering spaces.
Note that the underlying topological space is T_2 space.
Then (X^*, μ^*) is called simple completion of (X, μ).

And by following theorem we can see the simple completion of closed graded C_1 -paracovering spaces is a special case of the simple completion of a separated N_1- space in the sense of Herrlich. ■

Theorem 6: Let (X, μ) be a N_1 separated covering space, and (PX, C) be the saturated supercovering generated by its corresponding nearness. Then (PX, C) is closed graded C_1 paracovering and its simple completion is coincide with supercovering generated by simple completion of (X, μ).

Proof: Let (X, μ) be a N_1 separated covering space then (X, ξ) is its corresponding near space and (PX, N) is saturated paranear spaces induced by (X, ξ) and easily by proposition 4, we can get its corresponding supercovering i.e. (PX, C).

By corollary 6, we have (PX, C) symmetric so it is paracovering and by remark 17, (PX, C) is graded C_1, also since it is saturated obviously it is closed. Also here we mention that, ξ-clusters and N-clusters are same.

Now for $B \subset X$, we have:
$\mathcal{A} \in C(B)$ iff $\{X - A | A \in \mathcal{A}\} \notin N(B)$ iff $\{X - A | A \in \mathcal{A}\} \cup \{B\} \notin \xi$ iff $\{X - A | A \in \mathcal{A} \cup \{X - B\}\} \notin \xi$ iff $\mathcal{A} \cup \{X - B\} \in \mu$. (*)
Since (PX, C) is saturated, by definition of $\mathcal{B}^{X^*}$, we have $\mathcal{B}^{X^*} = PX^*$.

Now let for some $B^* \in \mathcal{B}^{X^*}$, $\Omega \in C^*(B^*)$, so
(1)$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$ and
(2) For each $\mathcal{D} \in X^*$ there exists $A^* \in \Omega \cup \{X^* - B^*\}$ s.t. $\mathcal{D} \in A^*$ and $e^{-1}[A^*] \in sec\mathcal{D}$.
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$ by (*) implies,
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B^*\}\} \cup \{X - e^{-1}[B^*]\} \in \mu$ i.e.
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B^*\}\} \in \mu$ which by (2) together according to definition of simple completion in covering space in recall 3, implies $\Omega \cup \{X^* - B^*\} \in \mu^*$ which by(*) implies $\Omega \in C_{\mu^*}(B^*)$.

Conversely, let for some $B^* \in \mathcal{B}^{X^*}$, $\Omega \in C_{\mu^*}(B^*)$ therefore by (*),
$\Omega \cup \{X^* - B^*\} \in \mu^*$. According to definition of simple completion in covering space in recall 3, so
(1)$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B^*\}\} \in \mu$ and
(2) For each $\mathcal{D} \in X^*$ there exists $A^* \in \Omega \cup \{X^* - B^*\}$ s.t.

$\mathcal{D} \in A^*$ and $e^{-1}[A^*] \in sec\mathcal{D}$.
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B^*\}\} \in \mu$ i.e.
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B^*\}\} \cup \{X - e^{-1}[B^*]\} \in \mu$ therefore by (*),
$\{e^{-1}[A^*] | A^* \in \Omega \cup \{X^* - B^*\}\} \in C(e^{-1}[B^*])$ which by (2) together according to definition of simple completion in supercovering space , implies $\Omega \in C^*(B^*)$. ■

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