Asymptotic Behavior of Solutions of Second Order Delay Differential Equations with Impulses

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***Abstract*—The purpose of this paper is to investigate the asymptotic behavior of solutions of second order non linear delay differential equations with impulses of the form**

$$[r(t)(x'(t))]'-p(t)(x'(t))+\sum_{i=1}^{n} q_i(t)g(x(t-\sigma_i))+h(t)=0,\ \ t\neq t_k,$$

$$x(t_k^+)-x(t_k)=a_k x(t_k),\quad x'(t_k^+)-x'(t_k)=b_k x'(t_k),\quad k\in Z^+$$

and some sufficient conditions are obtained

***Index Terms*—Asymptotic behavior, second order, impulses.**

I. INTRODUCTION

Qualitative theory of delay differential equations has taken the shape of a developed theory presented in monographs [1], [2], [3]. In particular, asymptotic behavior of solutions of delay differential equations has been studied by many authors (see [4], [5], [6], [7], [8]). The theory of impulsive differential equations is emerging as an important area of investigation, since it is a lot richer than the corresponding theory of differential equations without impulse effects. Moreover, such equations may exhibit several real world phenomena, such as merging of solutions, rhythmical beating.

In this paper, we consider a second order non linear delay differential equation with impulses of the form

$$[r(t)(x'(t))]'-p(t)(x'(t)) + \sum_{i=1}^{n} q_i(t)g(x(t-\sigma_i))+h(t)=0,\quad t\neq t_k, \tag{1}$$

$$x(t_k^+)-x(t_k)=a_k x(t_k),\quad x'(t_k^+)-x'(t_k)=b_k x'(t_k), \quad k=1,2,3,\dots \tag{2}$$

where $r(t), p(t), q_i(t), h(t)\in C([0,\infty),R^+)$, $i=1,2,\dots,n$; $0\le\sigma_1<\sigma_2<\dots<\sigma_n$, $g\in C(R,R)$.

Let PC_{t_0} denotes the set of function $\phi:[t_0-\sigma_n,t_0]\to R$, which is continuous in the set $[t_0-\sigma_n,t_0]\backslash\{t_k:k=1,2,\dots\}$ and may have discontinuities of the first kind and is continuous from left at the points t_k situated in the interval $(t_0-\sigma_n,t_0]$. For any $t_0\ge 0$, $\phi\in PC_{t_0}$, a function x is called a solution of (1) and (2) satisfying the initial value condition

$$x(t)=\phi(t), x(t_0^+)=x_0, x'(t)=\phi'(t),$$
$$x'(t_0^+)=x'_0,\quad t\in[t_0-\sigma_n,t_0]$$

in the interval $[t_0-\sigma_n,\infty)$, if $x:[t_0-\sigma_n,\infty)\to R$ satisfies and

(i) for $t\in(t_0,\infty)$, $t\neq t_k$, $t\neq t_k+\sigma_i$, $i=1,2,\dots,n$, $k=1,2,\dots,x(t)$, $x'(t)$ is continuously differentiable and satisfies (1);

(ii) for $t_k\in[t_0,\infty)$, $x(t_k^+)$, $x'(t_k^+)$, $x(t_k^-)$ and $x'(t_k^-)$ exist, $x(t_k^-)=x(t_k)$, $x'(t_k^-)=x'(t_k)$ and satisfies (2).

As it is customary, a solution of (1), (2) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

II. MAIN RESULTS

Throughout this chapter, we assume that the following conditions hold:

(H0) there is a constant $M>0$ such that $|g(x)|\le M|x|, x\in R,\ \ xg(x)>0, x\neq 0.$

(H1) $r(t)\ge r$, $\int_0^\infty p(t)dt\le p$, $q_i(t)\le q_i$, $i=1,2,\dots,n,r,p,q_i\in R^+$.

(H2) for all $t\in[0,\infty)$, the integration $H(t)=\int_t^\infty h(s)ds$ converges, $\sum_{k=1}^{\infty} b_k^+<\infty$ where $b_k^+=\max\{b_k,0\}$.

(H3) $$\lim_{n\to\infty}\sum_{m=0}^{n-1}\prod_{k=m+1}^{n-1}\prod_{\ell=0}^{m}(a_k+1)(b_\ell+1) \int_{t_{j+m}}^{t_{j+m+1}}\left(\frac{1}{r(u)}\right)exp\left(\int_{t_j}^{u}\frac{p(s)}{r(s)}ds\right)du=+\infty.$$

(H4) $$\prod_{k=0}^{n-1}(b_{j+k}+1)\left(\frac{r(t_j)}{r(t_{j+n})}\right)exp\left[-\int_{t_j}^{t_{j+m}}\frac{p(s)}{r(s)}ds\right]>1.$$

Lemma 2.1: Suppose that $x(t)$ is a solution of equations (1)-(2), and there exists $T\ge t_0$ such that $x(t)>0$, $t\ge T$. If (H3) hold, then $x'(t_k)>0$, $x'(t)>0$ where $t\in(t_k,t_{k+1}]$, $k=1,2,\dots$.

Proof: First, we prove $x'(t_k)>0$ for all $t_k\ge T$. Otherwise, there exists some j such that $t_j\ge T$, $x'(t_j)<0$, then $x'(t_j^+)=(1+b_j)x'(t_j)$ from (1), we get

$$\left[r(t)(x'(t))exp\left(-\int_{t_j}^{t}\frac{p(s)}{r(s)}ds\right)\right]' = -\sum_{i=1}^{n}q_i(t)g(x(t-\sigma_i))exp\left(-\int_{t_j}^{t}\frac{p(s)}{r(s)}ds\right) - h(t)exp\left(-\int_{t_j}^{t}\frac{p(s)}{r(s)}ds\right)$$

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$$= \left[-\sum_{i=1}^{n} q_i(t) g(x(t-\sigma_i)) - h(t) \right] exp\left(-\int_{t_j}^{t} \frac{p(s)}{r(s)} ds \right) < 0$$

Hence, $r(t)(x'(t))exp\left(-\int_{t_j}^{t}\frac{p(s)}{r(s)}ds\right)$ is decreasing on $(t_j, t_{j+1}]$ and

$$r(t_{j+1})(x'(t_{j+1}))exp\left[-\int_{t_j}^{t_{j+1}} \frac{p(s)}{r(s)}ds\right] \leq r(t_j)(b_j+1)(x'(t_j))$$

$$x'(t_{j+1}) \leq (b_j+1)\left[\frac{r(t_j)}{r(t_{j+1})}\right] x'(t_j) exp\left[\int_{t_j}^{t_{j+1}} \frac{p(s)}{r(s)}ds\right]$$

$$x'(t_{j+2}) \leq (b_{j+1}+1)(b_j+1)\left[\frac{r(t_j)}{r(t_{j+2})}\right] x'(t_j) exp\left[\int_{t_j}^{t_{j+2}} \frac{p(s)}{r(s)}ds\right]$$

By induction, we have, for all $n \geq 2$.

$$x'(t_{j+n}) \leq \prod_{k=0}^{n-1}(b_{j+k}+1)\left(\frac{r(t_j)}{r(t_{j+n})}\right) x'(t_j) exp\left[\int_{t_j}^{t_{j+n}} \frac{p(s)}{r(s)}ds\right]$$

Since $r(t)(x'(t))exp\left[-\int_{t_j}^{t}\frac{p(s)}{r(s)}ds\right]$ is decreasing on $(t_j, t_{j+1}]$, so

$$x'(t) \leq (b_j+1)\left[\frac{r(t_j)}{r(t)}\right] x'(t_j) exp\left[\int_{t_j}^{u} \frac{p(s)}{r(s)}ds\right], \quad t \in (t_j, t_{j+1}]. \tag{3}$$

Integrating, the above inequality from s to t, we have,

$$x(t) \leq x(s) + (b_j+1)(r(t_j))x'(t_j) \int_{s}^{t} \left(\frac{1}{r(u)}\right) exp\left[\int_{t_j}^{t} \frac{p(s)}{r(s)}ds\right] du, \quad t_j < s < t \leq t_{j+1}.$$

Let $t \to t_{j+1}$, $s \to t_j^+$, we get

$$x(t_{j+1}) \leq x(t_j^+) + (b_j+1)(r(t_j))x'(t_j) \int_{t_j^+}^{t_{j+1}} \left(\frac{1}{r(u)}\right) exp\left[\int_{t_j}^{u} \frac{p(s)}{r(s)}ds\right] du$$
$$\leq (a_j+1)x(t_j) + (b_{j+1}+1)(r(t_j))x'(t_j) \int_{t_j}^{t_{j+1}} \left(\frac{1}{r(u)}\right) exp\left[\int_{t_j}^{u} \frac{p(s)}{r(s)}ds\right] du$$

$$x(t_{j+2}) \leq (a_{j+1}+1)(a_j+1)x(t_j) + (a_{j+1}+1)(b_j+1)(r(t_j))x'(t_j) \int_{t_j}^{t_{j+1}} \left(\frac{1}{r(u)}\right) exp\left[\int_{t_j}^{u} \frac{p(s)}{r(s)}ds\right] du + (b_{j+1}+1)(b_j+1)(r(t_j))x'(t_j) \int_{t_{j+1}}^{t_{j+2}} \left(\frac{1}{r(u)}\right) exp\left[\int_{t_j}^{u} \frac{p(s)}{r(s)}ds\right] du$$

By induction, we get, for all n,

$$x(t_{j+n}) \leq \prod_{k=0}^{n-1}(a_{j+k}+1)x(t_j) + (r(t_j))x'(t_j) \left[\sum_{m=0}^{n-1}\prod_{k=m+1}^{n-1}\prod_{\ell=0}^{m}(a_{j+k}+1)(b_{j+\ell}+1) \int_{t_{j+m}}^{t_{j+m+1}} \left(\frac{1}{r(u)}\right) exp\left[\int_{t_j}^{u} \frac{p(s)}{r(s)}ds\right] du\right]$$

because of $x(t) > 0$, $x'(t_j) < 0$ $(t_j \geq T)$. It is a contradiction to (H3). Hence, $x'(t_k) > 0$ for all $t_k \geq T$ and $r(t)(x'(t))exp\left(-\int_{t_j}^{t}\frac{p(s)}{r(s)}ds\right)$ is decreasing on $(t_j, t_{j+1}]$, thus

$$r(t)(x'(t))exp\left(-\int_{t_j}^{t}\frac{p(s)}{r(s)}ds\right) \geq r(t_{j+1})(x'(t_{j+1}))exp\left(-\int_{t_j}^{t_{j+1}}\frac{p(s)}{r(s)}ds\right) \geq 0.$$

It is clear that $x'(t) \geq \left[\frac{r(t_{j+1})}{r(t)}\right] x'(t_{j+1}) \geq 0$ for $t \in (t_j, t_{j+1}]$, $j \geq \ell$ and $x'(t) \geq \left[\frac{r(t_\ell)}{r(t)}\right] x'(t_\ell) \geq 0$ for any $t \in [T, t_\ell]$ therefore $x'(t) \geq 0$, $t \in (t_j, t_{j+1}]$. The proof is complete. ■

Theorem 2.1: Let (H0)-(H3) hold, assume that for any $\gamma > 0$ there exists $\beta > 0$ such that

$$|g(x)| \geq \beta \quad \text{for} \quad |x| \leq \gamma. \tag{4}$$

Suppose that

$$\sum_{i=1}^{n} q_i(t+\sigma_i) \geq 0, \qquad \int_0^\infty \sum_{i=1}^{n} q_i(s+\sigma_i)ds = \infty \tag{5}$$

and there exists constant $\lambda > 0$ such that for sufficiently large t

$$\sum_{i=1}^{n}\int_{t-\sigma_i}^{t-r} q_i(s+\sigma_i)ds \leq \lambda < \frac{r-p}{M} \tag{6}$$

where $r \in [0, \sigma_n]$, $q_i^+ = \max\{q_i(t), 0\}$, $q_i^- = \max\{-q_i(t), 0\}$. Then every non oscillatory solution of (1)-(2) tends to zero as $t \to \infty$.

Proof: Choose a positive integer N such that (6) holds for $t \geq t_N$ and $\sum_{k=N}^{\infty}(b_k^+) < r - p - \lambda M$. Let $x(t)$ be a non oscillatory solution of (1)-(2). We will assume that $x(t)$ is eventually

positive, the case where $x(t)$ is eventually negative is similar and omitted. Let $x(t) > 0$ for $t \geq t_N$. By Lemma 2.1, we know that $x'(t) > 0$ for $t \geq t_N$.
Define

$$y(t) = r(t)(x'(t)) - \int_{t_N}^{t} p(s)(x'(s))ds - \sum_{i=1}^{n} \int_{t-\sigma_i}^{t-r} q_i(s+\sigma_i)g(x(s))ds - H(t) - \sum_{t_N<t_k\leq t} [b_k^+(x'(t_k))] \quad (7)$$

Then for $t \neq t_k$, $t \neq t_k + \sigma_i$, $i = 1, 2, \ldots, n, k = 1, 2, \ldots$

$$y'(t) = -\sum_{i=1}^{n} q_i(t - r + \sigma_i)g(x(t - r)) \quad (8)$$

and

$$y(t_k^+) - y(t_k) \leq 0, \quad k = N, N+1, \ldots.$$

Thus, $y(t)$ is non increasing on $[t_N, \infty)$. Set $L = \lim_{t\to\infty} y(t)$ we claim that $L \in R$. Otherwise, $L = -\infty$, then $x'(t)$ must be unbounded by (H1) and (5). Hence, it is possible to choose $t^* > t_N + \sigma_N$ such that $y(t^*) + H(t^*) < 0$ and $x'(t^*) = \max\{x'(t); t_N \leq t \leq t^*\}$. Thus we have,

$$\begin{aligned} 0 &> y(t^*) + H(t^*) \\ &\geq r(t^*)(x'(t^*)) - \int_{t_N}^{t^*} p(s)(x'(s))ds \\ &\quad - M\sum_{i=1}^{n} \int_{t-\sigma_i}^{t-r} q_i(s+\sigma_i)x(s)ds - \sum_{t_N<t_k\leq t^*} [b_k^+(x'(t_k))] \\ &\geq (x'(t^*)) \left(r - p - \lambda M - \sum_{k=N}^{\infty} (b_k^+) \right) > 0 \end{aligned}$$

which is a contradiction and so $L \in R$.
By integrating equation (8) from t_N to t, we have,

$$\begin{aligned} &\int_{t_N}^{t} \sum_{i=1}^{n} q_i(s - r - \sigma_i)g(x(s - r))ds = -\int_{t_N}^{t} y'(s)ds \\ &= y(t_N^+) + \sum_{t_N<t_k\leq t} [y(t_k^+) - y(t_k)] - y(t) < y(t_N^+) - L \end{aligned}$$

which together with (5), implies that $g(x(t)) \in L^1([t_N, \infty), R)$ and so $\lim_{t\to\infty} \inf g(x(t)) = 0$. We claim that

$$\lim_{t\to\infty} \inf x(t) = 0. \quad (9)$$

Let $\{S_m\}$ be such that $S_m \to \infty$ as $m \to \infty$ and $\lim_{m\to\infty} g(x(S_m)) = 0$. We must have $\lim_{m\to\infty} \inf x(S_m) = a = 0$. In fact, if $a > 0$, then there is a subsequence $\{S_{m_k}\}$ such that $x(S_{m_k}) \geq a/2$ for sufficiently large k. By (4) we have $g(x(S_{m_k})) \geq \beta_a$ for some $\beta_a > 0$ and sufficiently large k, which yields a contradiction because of $\lim_{k\to\infty} g(x(S_{m_k})) = 0$. Therefore, $\lim_{t\to\infty} \inf x(t) = 0$ and so $\lim_{t\to\infty} x(t) = 0$. The proof is then complete. ∎

Lemma 2.2: Let $x(t)$ be a oscillatory solution of (1)-(2). Suppose that there exists some $T \geq t_0$, if (H4) hold, then $|x'(t_k)| \geq |x(t_k)|$, $|x'(t)| \geq |x(t)|$ where $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots$.

Proof: From the result of Lemma 2.1, it is known that, if $x(t) > 0$ then $x'(t_k) > 0$, $x'(t) > 0$, where $t \in (t_k, t_{k+1}]$. We will assume that when $x(t) > 0$ we have $x'(t_k) \geq x(t_k)$, $x'(t) \geq x(t)$, $t \in (t_k, t_{k+1}]$ the case $x(t)$ is negative is similar and omitted. From Lemma 2.1, we have $x'(t_k) > 0$, $x'(t) > 0$, $t \in (t_k, t_{k+1}]$, then the $x(t)$ is increased. We also obtained

$$\left\{ r(t)(x(t))exp\left(-\int_{t_j}^{t} \frac{p(s)}{r(s)}ds\right) \right\}' < \left[r(t)(x'(t))exp\left(-\int_{t_j}^{t} \frac{p(s)}{r(s)}ds\right) \right] < 0.$$

Hence, $r(t)(x(t))exp\left(-\int_{t_j}^{t} \frac{p(s)}{r(s)}ds\right)$ is decreasing on $(t_j, t_{j+1}]$ and

$$(x(t_{j+1})) \leq (b_j + 1)\frac{r(t_j)}{r(t_{j+1})}(x(t_j))exp\left[-\int_{t_j}^{t_{j+1}} \frac{p(s)}{r(s)}ds\right]$$

$$x(t_{j+1}) \leq (b_j + 1)\left[\frac{r(t_j)}{r(t_{j+1})}\right] x(t_j)exp\left[-\int_{t_j}^{t_{j+1}} \frac{p(s)}{r(s)}ds\right]$$

for all n, we obtain

$$x(t_{j+n}) \leq \prod_{k=0}^{n-1}(b_{j+k}+1)\left[\frac{r(t_j)}{r(t_{j+n})}\right] x(t_j)exp\left[-\int_{t_j}^{t_{j+1}} \frac{p(s)}{r(s)}ds\right]$$

by the condition (H4), we get $x(t_{j+n}) < x(t_j)$ which is a contradiction. The proof is complete. ∎

Theorem 2.2: Let (H0)-(H2) and (H4) holds. Suppose that

$$\sum_{k=1}^{\infty} |b_k| < \infty \quad (10)$$

and there exists positive constant λ and $r \in (0, \sigma_n]$ such that

$$\lim_{t\to\infty} \sup\ Q_1(t) + \lim_{t\to\infty} \sup\ Q_2(t) \leq \lambda < \frac{r - 2p}{M} \quad (11)$$

$$\sum_{i=1}^{n} q_i(t + \sigma_i) \neq 0 \quad \text{for large } t \quad (12)$$

where

$$Q_1(t) = \sum_{i=1}^{n} \int_{t-\sigma_i}^{t} |q_i(s + \sigma_i)|ds \quad (13)$$

$$Q_2(t) = \sum_{i=1}^{n} \int_{t-r}^{t-\sigma_i} sgn(r - \sigma_i)|q_i(s + \sigma_i)|ds \quad (14)$$

Then every oscillatory solution of (1)-(2) tends to zero as $t \to \infty$.

Proof: Let $x(t)$ be an oscillatory solutions of (1)-(2). We first show that $x'(t)$ and $x(t)$ are bounded. Otherwise, $x'(t)$ is unbounded which implies that there exists positive integer N such that $\lim_{t\to\infty} \sup_{t_N\leq s\leq t} |x'(s)| = \infty$ and

$$\sup_{t_N+\sigma_n\leq s\leq t} |x'(s)| = \sup_{t_N\leq s\leq t} |x'(s)|, \quad t \geq t_N + \sigma_n$$

and

$$\sum_{k=N}^{\infty}|b_k| < \frac{r-2|p|-\lambda M}{2} \tag{15}$$

Let

$$\begin{aligned} y(t) &= r(t)(x'(t)) - \int_{t_N}^{t} p(s)(x'(s))ds \\ &- \sum_{i=1}^{n}\int_{t-\sigma_i}^{t-r} q_i(s+\sigma_i)g(x(s))ds - H(t) \\ &- \sum_{t_N<t_k\le t}[b_k^{+}(x'(t_k))] \end{aligned}$$

where $b_k^{+} = \max\{b_k, 0\}$. Then equation (8) holds. For $t \ge t_N + \sigma_n$, using Lemma 2.2 we have

$$\begin{aligned} |y(t)| &\ge r|x'(t)| - p|x'(t)| \\ &- \sum_{i=1}^{n}\int_{t-\sigma_i}^{t-r} q_i(s+\sigma_i)|g(x(s))|ds - |H(t)| \\ &- \sum_{t_N<t_k\le t}|b_k(x'(t_k))| \\ &\ge (r-p)|x'(t)| - \left(MQ_2(t) + \sum_{k=N}^{\infty}|b_k|\right) \\ &\sup_{t_N\le s\le t}|x'(s)| - |H(t)| \end{aligned}$$

which implies

$$\begin{aligned} \sup_{t_N+\sigma_n\le s\le t}|y(s)| \ge \left(r-p-M\sup_{t_N\le s\le t}Q_2(s) - \sum_{k=N}^{\infty}|b_k|\right) \\ \sup_{t_N\le s\le t}|x'(s)| - \sup_{t_N+\sigma_n\le s\le t}|H(s)| \end{aligned} \tag{16}$$

Hence, $\lim_{t\to\infty}\sup|y(t)| = \infty$. From (8) we notice that $y'(t)$ is oscillatory, we see that there is a $\eta' \ge t_N + 2\sigma_n$ such that $|y(\eta')| = \sup_{t_N+\sigma_n\le s\le t}|y(s)|$ and $y'(\eta') = 0$. From (8) and (12), we get $x(\eta'-r) = 0$. By Lemma 2.2, it is known that $x'(t)$ is oscillatory, hence there is $\eta > \eta' + r$ such that $x'(\eta - r) = 0$. Integrating both sides of (8) from $\eta - r$ to η we obtain

$$\begin{aligned} y(\eta) &= y(\eta-r) - \int_{\eta-r}^{\eta}\sum_{i=1}^{n} q_i(s-r+\sigma_i)g(x(s-r))ds \\ &= -\int_{t_N}^{\eta-r} p(s)(x'(s))ds \\ &+ \sum_{i=1}^{n}\int_{\eta-2r}^{\eta-r-\sigma_i} q_i(s+\sigma_i)g(x(s))ds + H(\eta-r) \\ &- \sum_{t_N\le t_k\le \eta-r}[b_k(x'(t_k))] \\ &- \int_{\eta-r}^{\eta}\sum_{i=1}^{n} q_i(s-r+\sigma_i)g(x(s-r))ds \\ &= -\int_{t_N}^{\eta-r} p(s)(x'(s))ds + H(\eta-r) \\ &- \sum_{i=1}^{n}\int_{\eta-r-\sigma_i}^{\eta-r} q_i(s+\sigma_i)g(x(s))ds \\ &- \sum_{t_N\le t_k\le \eta-r}[b_k(x'(t_k))] \end{aligned}$$

which implies that,

$$\begin{aligned} |y(\eta)| \le \left(p + MQ_1(\eta-r) + \sum_{k=N}^{\infty}|b_k|\right) \\ \sup_{t_N\le s\le \eta}|x'(s)| + |H(\eta-r)| \end{aligned} \tag{17}$$

From (16) and (17), we have,

$$\begin{aligned} &-r + 2p + M\left(Q_1(\eta-r) + \sup_{t_N+\sigma_n\le s\le\eta}Q_2(s)\right) + 2\sum_{k=N}^{\infty}|b_k| \\ &+ \left(\sup_{t_N+\sigma_n\le s\le\eta}|H(s)| + |H(\eta-r)|\right)\left(\sup_{t_N\le s\le\eta}|x'(s)|\right)^{-1} \ge 0 \end{aligned}$$

Let $\eta \to \infty$ and noting that $\lim_{\eta\to\infty}\sup_{t_N\le s\le\eta}|x'(s)| = \infty$ we have

$$-r + 2p + \lambda M + 2\sum_{k=N}^{\infty}|b_k| \ge 0$$

by (11), which contradicts (15) and so $x'(t)$ is bounded. By Lemma 2.2, it is known that $x(t)$ is bounded. Next, we prove that $\mu = \lim_{t\to\infty}\sup|x'(t)| = 0$.
To this end, we define

$$\begin{aligned} z(t) &= r(t)(x'(t)) - \int_{t_N}^{t} p(s)(x'(s))ds \\ &+ \sum_{i=1}^{n}\int_{t-r}^{t-\sigma_i} q_i(s+\sigma_i)g(x(s))ds + H(t) \\ &+ \sum_{t_k\ge t}[b_k(x'(t_k))] \end{aligned} \tag{18}$$

then $z(t)$ is bounded and for sufficiently large t,

$$\begin{aligned} |z(t)| \ge r|x'(t)| - p|x'(t)| - MQ_2(t)\sup_{t-\sigma_n\le s\le t}|x'(s)| \\ -|H(t)| - \sum_{t_k\ge t}|b_k(x'(t_k))| \end{aligned}$$

Thus by (H2) and (11),

$$\begin{aligned} \delta = \lim_{t\to\infty}\sup\ |z(t)| &\ge (r-p)\mu - \mu M\lim_{t\to\infty}\sup\ Q_2(t) \\ &= \mu\left[r - p - M\lim\sup_{t\to\infty}Q_2(t)\right] \end{aligned} \tag{19}$$

On the other hand, we have by (18) for $t \ne t_k$, $t \ne t_k + \sigma_i$, $k = 1, 2, \ldots$, $i = 1, 2, \ldots$

$$z'(t) = -\sum_{i=1}^{n} q_i(t-r+\sigma_i)g(x(t-r)) \tag{20}$$

From this we see that $z'(t)$ is oscillatory. Hence there exists a sequence $\{\eta'_m\}$ such that $\lim_{m\to\infty}\eta'_m = \infty$, $\lim_{m\to\infty}|z(\eta'_m)| = \delta$, $z'(\eta'_m) = 0$, and $x(\eta'_m - r) = 0$, $m = 1, 2, \ldots$. Similar to

(17) we can obtain by (18) and (20), there is a $\eta_m > \eta'_m$, such that

$$|z(\eta_m)| \leq [p + MQ_1(\eta_m - r)] \sup_{\eta_m - 2\sigma_n \leq s \leq \eta_m} |x'(s)| + |H(\eta_m - r)| + \sum_{t_k \geq \eta_m - r} |b_k(x'(t_k))|,$$

which implies by (10) and (H2) that

$$\delta \leq \mu \left[p + M \lim_{t\to\infty} \sup Q_1(t) \right]$$

This, together with (19), yields

$$\mu \left[-r + 2p + M \left(\lim_{t\to\infty} \sup \ Q_1(t) + \lim_{t\to\infty} \sup \ Q_2(t) \right) \right] \geq 0$$

Therefore, by (11) we have,

$$\mu[-r + 2p + \lambda M] \geq 0,$$

which implies $\mu = 0$. This completes the proof. ■

III. An Example

Consider the second order nonlinear delay differential equation with impulses given by

$$[t(x'(t))]' - e^{-4t}(x'(t)) + \frac{2}{t+2} x(t-1)[1 + sin^2 x(t-1)] - \frac{1}{t+3} x(t-2)[1 + sin^2 x(t-2)] + e^{-2t} = 0, \quad t \geq 2, t \neq k \tag{21}$$

$$x(k^+) - x(k) = 2x(k), \quad x'(k^+) - x'(k) = \frac{1}{k^2+1} x'(k), \quad k = 1, 2, \ldots. \tag{22}$$

Since all the conditions of Theorem 2.1 are satisfied, it is clear that every non oscillatory solution of (21) and (22) tends to zero as $t \to \infty$.

References

[1] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, Oscillation theory of differential equations with Deviating Arguments, New York, Marcel Dekkar, 1987.

[2] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon, Oxford, 1991.

[3] J.K. Hale, Theory of functional differential equations, New York, Springer-Verlag, 1977.

[4] G. Ladas, Y.G. Sficas and I.P. Stavroulakis, Asymptotic behavior of solutions of retarded differential equations, *Proc. Amer. Math. Soc.*, 88 (1983), 247–253.

[5] M.R.S. Kulenovic, G. Ladas, A. Meimaridou, Stability solutions of linear delay differential equations, Proc. Amer. Math. Soc., 100 (1987), 433–441.

[6] J.H. Shen and Z. Wang, Oscillatory and asymptotic behavior of solutions of delay differential equations with impulses, *Ann. Diff. Eqn.*, 10 (1994), 61–67.

[7] X. Liu and J.H. Shen, Asymptotic behavior of solutions of impulsive neutral differential equations, *J. Appl. Math. Lett.*, 12 (1999), 51–58.

[8] A. Zhao and J. Yan, Asymptotic behavior of solutions of impulsive delay differential equations, *J. Math. Anal. Appl.*, 201 (1996), 943–954.