Some Results on D*-Fuzzy Metric Spaces

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***Abstract*—In this paper $\mathcal{M}$-fuzzy metric space is redefined and establish a decomposition theorem of D*-fuzzy metric space into a family of D*-metric spaces. Also idea of generating spaces of D*-quasi metric family is introduced.**

***Index Terms*—$\mathcal{M}$-fuzzy metric space, D*-fuzzy metric space, generating spaces of D*-quasi metric family, continuous t-norm.**
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I. Introduction

The concept of fuzzy set is introduced by Zadeh [12] in 1965. After that, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application in different direction. Especially Zike-Deng [3], Kaleva and Seikkala [7], Kramosil and Michalek [6] have introduced the concept of fuzzy metric spaces in different ways.

On the other hand, there have been a number of generalizations of metric spaces. One of such generalization is generalized metric space (or D-metric space) introduced by Dhage [4] in 1992. He studied some basic properties viz. convergence of sequences, boundedness, completeness etc. and established some fixed point theorems. Many other authors viz. Sedghi et al [10] made a significant contribution in fixed point theory of D^*-metric spaces.Recently Sedghi et al. [9] introduced the concept of $\mathcal{M}$-fuzzy metric spaces which is a generalization of fuzzy metric spaces due to George and Veeramoni [5] and studied some related results. Other authors viz. J.H.Park et al. [8] established some fixed point theorems in $\mathcal{M}$-fuzzy metric spaces.

In this paper, definition of $\mathcal{M}$-fuzzy metric space introduced by Sedghi et al.[9] has been modified by omitting the condition of continuity and call it D^*-fuzzy metric space since its crisp form is similar to D^*-metric space. With the help of this modified definition it has been possible to achieve two decomposition theorems of the D^*-fuzzy metric space into a family of crisp D^*-metrices. The novelty of decomposition theorems are far fetching. In fact, it will be possible to establish many results of functional analysis in fuzzy setting. Also it has been possible to deduce a generating space of D^*-quasi metric family from D^*-fuzzy metric space.

The organization of the paper is as follows:

Section II is provided for preliminary results which are used in this paper. Definition of D^*-fuzzy metric space is given in Section III. In Section IV, it is shown that from D^*-fuzzy metric a generating space of D^*-quasi metric family can be deduced. In Section V, we choose "min" as t-norm and establish two decomposition theorems.

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II. Some preliminary results.

Definition 2.1 A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a,b,c,d \in [0,1]$.

Two typical examples of continuous t-norm are $a * b = ab$ and $a * b$=min (a,b).

Definition 2.2 [10] Let X be a nonempty set. A generalized metric (or D^* metric) on X is a function $D^* \; X^3 \to [0\ ,\ \infty)$ that satisfies the following conditions for each x,y,z,a $\in X$;

(1) $D^*(x,y,z) \geq 0$
(2) $D^*(x,y,z) = 0$ iff $x = y = z$
(3) $D^*(x,y,z) = D^*(p\{x,y,z,\})$ (symmetry) where p is a permutation function,
(4) $D^*(x,y,z) \ \leq D^*(x,y,a) + D^*(a,z,z)$

The pair (X, D^*) is called a generalized metric (D^*) space.

Definition 2.3 [9] A 3-tuple $(X, \mathcal{M}, *)$ is called an $\mathcal{M}$-fuzzy metric space if X is an arbitrary (nonempty) set and $\mathcal{M}$ is a fuzzy set on $X^3 \times [0\ ,\ \infty)$, satisfying the following conditions for each $x,y,z,a \in$ X and $t,s > 0$;

(1) $\mathcal{M}(x,y,z,t) \ > 0$
(2) $\mathcal{M}(x,y,z,t) = 1$ iff $x = y = z$
(3) $\mathcal{M}(x,y,z,t) = \mathcal{M}(p\{x,y,z,\},t)$, (symmetry) where p is a permutation function
(4) $\mathcal{M}(x,y,a,t) * \mathcal{M}(a,z,z,s) \leq \mathcal{M}(x,y,z,t+s)$
(5) $\mathcal{M}(x,y,z,.) : \ (0\ ,\ \infty) \to [0\ ,\ 1]$ is continuous

III. D*-fuzzy metric space.

In this Section $\mathcal{M}$-fuzzy metric space is redefined and call it D^*-fuzzy metric space.

Definition 3.1 A 3-tuple $(X, D^*, *)$ is called an D^*-fuzzy metric space if X is an arbitrary (non-empty) set and D^* is a fuzzy set on $X^3 \times [0\ ,\ \infty)$, satisfying the following conditions for each $x,y,z,a \in$ X and $t,s \in [0\ ,\ \infty)$;

(FD^*1) $D^*(x,y,z,0) = \ 0$.
(FD^*2) $(\forall t > 0, D^*(x,y,z,t) = 1)$ iff $x = y = z$.
(FD^*3) $D^*(x,y,z,t) = D^*(p\{x,y,z,\},t)$, (symmetry) where p is a permutation function.
(FD^*4) $D^*(x,y,a,t) * D^*(a,z,z,s) \leq D^*(x,y,z,t+s)$.
(FD^*5) $\lim_{t\to\infty} D^*(x,y,z,t) \ = 1$.

Example 2.1 Let X be a nonempty set and D' be a D^*-metric on X.
Choose $a * b = ab \ \forall a,b \in [0\ ,\ 1]$.
For each $t \in [0\ ,\ \infty)$ define

$$D^*(x,y,z,t) \ = \ \frac{t}{t + D'(x,y,z)} \ \forall x,y,z \in X.$$

It is easy to see that D^* is a fuzzy metric on X and hence $(X, D^*, *)$ is a D^*-fuzzy metric space.

Lemma 2.1 $D^*(x, y, z, .)$ is non-decreasing $\forall x, y, z \in$ X.

Proof. Proof follows from (FD^*1) and (FD^*4).

IV. Generating space of D^*-quasi metric family

Definition 4.1 Let X be a non-empty set and $\{D^*_\alpha : \alpha \in (0, 1]\}$ be a family of mappings of X^3 into R^+. $(X, D^*_\alpha : \alpha \in (0, 1])$ is called a generating space of D^*-quasi metric family if it satisfies the following conditions:

(QD^*1) $D^*_\alpha(x, y, z) = 0 \ \ \forall\alpha \in (0, 1]$ iff $x = y = z$;

(QD^*2) $D^*_\alpha(x, y, z) = D^*_\alpha(p\{x, y, z\})$(symmetry) where p is a permutation function;

(QD^*3) For any $\alpha \in (0, 1]$, $\exists$ a number $\mu \in (0, \alpha]$ such that
$D^*_\alpha(x, y, z) \le D^*_\mu(x, y, a) + D^*_\mu(a, z, z)$;

(QD^*4) For any $x, y, z \in X$, $D^*_\alpha(x, y, z)$ is non-increasing and left continuous for $\alpha \in (0, 1]$.

Example 4.1 Let (X, D^*) be a D^*-metric space. Define $D^*_\alpha(x, y, z) = \frac{1}{\alpha} D^*(x, y, z) \ \forall x, y, z \in X, \ \alpha \in (0, 1]$.
Then $\{D^*_\alpha : \alpha \in (0, 1]\}$ is a generating space of D^*-quasi metric family.

Theorem 4.1 *Let $(X, D^*, *)$ be a D^*-fuzzy metric space. We assume that $D^*(x, y, z, .)$ is continuous as well as strictly increasing for all $t > 0$.*
For, $\alpha \in (0, 1]$ define,
$D^*_\alpha(x, y, z) = \bigwedge\{t > 0 : D^*(x, y, z, t) \ge 1 - \alpha\}$ *and denote* $Q = \{D^*_\alpha : \alpha \in (0, 1]\}$.
Then (X, Q) is a generating space of D^*-quasi metric family.

Proof. **(QD^*1).** Let $x = y = z$.
Then $D^*(x, y, z, t) = 1 \ \ \forall t > 0$.
i.e. $D^*_\alpha(x, y, z) = 0 \ \ \forall\alpha \in (0, 1]$.
Conversely, if $D^*_\alpha(x, y, z) = 0 \ \ \forall\alpha \in (0, 1]$ then
$D^*(x, y, z, t) \ge 1 - \alpha \ \ \forall t > 0, \ \ \forall\alpha \in (0, 1]$
$\Rightarrow D^*(x, y, z, t) = 1 \ \ \forall t > 0$
$\Rightarrow x = y = z$.

(QD^*2). It is clear from definition,
$D^*_\alpha(x, y, z) = D^*_\alpha(p\{x, y, z\}) \ \ \forall\alpha \in (0, 1]$.

(QD^*3). Since $*$ is continuous, thus for any $\alpha \in (0, 1]$, $\exists\beta \in (0, \alpha]$ such that $(1 - \beta) * (1 - \beta) = 1 - \alpha$.
Now $D^*_\beta(x, y, a, t) + D^*_\beta(a, z, z, s) = \bigwedge\{t > 0 : D^*(x, y, a, t) \ge 1 - \beta\} + \bigwedge\{s > 0 : D^*(a, z, z, s) \ge 1 - \beta\}$.
i.e.$D^*_\beta(x, y, a, t) + D^*_\beta(a, z, z, s) = \bigwedge\{t + s > 0 : D^*(x, y, a, t) \ge 1 - \beta, \ D^*(a, z, z, s) \ge 1 - \beta\}$.
Now $D^*(x, y, a, t) \ge 1 - \beta, \ D^*(a, z, z, s) \ge 1 - \beta \Rightarrow D^*(x, y, a, t) * D^*(a, z, z, s) \ge (1 - \beta) * (1 - \beta)$
i.e.$D^*(x, y, a, t) \ge 1 - \beta, \ D^*(a, z, z, s) \ge 1 - \beta \Rightarrow D^*(x, y, z, t + s) \ge 1 - \alpha$.
Thus we have,
$D^*_\beta(x, y, a) + D^*_\beta(a, z, z) \ge D^*_\alpha(x, y, z)$.

(QD^*4). From definition of D^*_α it is clear that $D^*_\alpha(x, y, z)$ is non-increasing $\forall t > 0, \ \ \alpha \in (0, 1]$.
Let $\{\alpha_n\} \uparrow \alpha_0$ and $D^*_{\alpha_n}(x, y, z) = t_n, \ \ D^*_{\alpha_0}(x, y, z) = t_0$.
Then $D^*(x, y, z, t_n) = 1 - \alpha_n$ and $D^*(x, y, z, t_0) = 1 - \alpha_0$ (since $D^*(x, y, z, .)$ is continuous).
Now $D^*(x, y, z, t_n) - D^*(x, y, z, t_0) = \alpha_0 - \alpha_n$
$\Rightarrow \lim_{n\to\infty} D^*(x, y, z, t_n) = D^*(x, y, z, t_0)$.
$\Rightarrow D^*(x, y, z, \lim_{n\to\infty} t_n) = D^*(x, y, z, t_0)$.
$\Rightarrow \lim_{n\to\infty} t_n = t_0$ (since $D^*(x, y, z, .)$ is strictly increasing).
$\Rightarrow \lim_{n\to\infty} D^*_{\alpha_n}(x, y, z) = D^*_{\alpha_0}(x, y, z)$.
Thus for any $x, y, z \in X, \ D^*_\alpha(x, y, z)$ is non-increasing and left-continuous for each $\alpha \in (0, 1]$.

V. Decomposition Theorems of D^*-fuzzy metric

If we take t-norm $* = \min$, then the following theorem follows:

Theorem 5.1.(Decomposition Theorem 1). Let $(X, D^*, *)$ be a D^*-fuzzy metric space.
Assume that,
(M6) $D^*(x, y, z, t) > 0 \ \forall t > 0$ implies $x = y = z$.
Define $D^*_\alpha(x, y, z) = \bigwedge\{t > 0 : \ D^*(x, y, z, t) \ge \alpha\}, \ \ \alpha \in (0, 1)$.
Then $\{D^*_\alpha : \alpha \in (0, 1)\}$ is an ascending family of D^*-metrices on X.
We call these metrices as $\alpha - D^*$-metrices on X corresponding to the D^*-fuzzy metric on X.

Proof. (i) For $x, y, z \in X, \ D^*(x, y, z, t) = 0$ for $t < 0$
$\Rightarrow \ \bigwedge\{t > 0 : D^*(x, y, z, t) \ge \alpha\} \ge 0 \ \forall\alpha \in (0, 1)$
$\Rightarrow D^*_\alpha(x, y, z, t) \ge 0 \ \forall\alpha \in (0, 1)$.

(ii) If $D^*_\alpha(x, y, z, t) = 0 \ \forall t > 0$
$\Rightarrow \bigwedge\{t > 0 : D^*(x, y, z, t) \ge \alpha\} = 0 \ \forall t > 0$
$\Rightarrow \forall t \in R, \ t > 0, \ D^*(x, y, z, t) \ge \alpha > 0$
$\Rightarrow x = y = z$ by $(M6)$.
Conversely $x = y = z$
$\Rightarrow D^*(x, y, z, t) = 1 \ \forall t > 0$
$\Rightarrow \forall\alpha \in (0, 1), \ \wedge\{t : \ D^*(x, y, z, t) \ge \alpha\} = 0$
$\Rightarrow D^*_\alpha(x, y, z) = 0 \ \forall\alpha \in (0, 1)$.

(iii) From (FD^*3), it follows that $D^*_\alpha(x, y, z) = D^*_\alpha(p\{x, y, z\})$, (symmetry) where p is a permutation function.

(iv) $D^*_\alpha(x, y, a) + D^*_\alpha(a, z, z) = \bigwedge\{t > 0 : \ D^*(x, y, a, t) \ge \alpha\} + \bigwedge\{s > 0 : \ D^*(a, z, z, s) \ge \alpha\}$.
$= \bigwedge\{t + s > 0 : \ D^*(x, y, a, t) \ge \alpha, \ D^*(a, z, z, s) \ge \alpha\}$
$\ge \bigwedge\{r > 0 : \ D^*(x, y, z, r) \ge \alpha\}$ where $r = t + s$
(Since $D^*(x, y, a, t) \ge \alpha, \ D^*(a, z, z, s) \ge \alpha \Rightarrow D^*(x, y, z, s + t) \ge \alpha$).
Therefore, $D^*_\alpha(x, y, z) \le D^*_\alpha(x, y, a) + D^*_\alpha(a, z, z)$.

Now taking $0 < \alpha_1 < \alpha_2$.
We have $D^*_{\alpha_1}(x, y, z) = \bigwedge\{t > 0 : \ D^*(x, y, z, t) \ge \alpha_1\}$
$D^*_{\alpha_2}(x, y, z) = \bigwedge\{t > 0 : \ D^*(x, y, z, t) \ge \alpha_2\}$.
Since $\alpha_1 < \alpha_2$ we have
$\{t > 0 : \ D^*(x, y, z, t) \ge \alpha_2\} \subset \{t > 0 : \ D^*(x, y, z) \ge \alpha_1\}$
$\Rightarrow \ \bigwedge\{t > 0 : \ D^*(x, y, z, t) \ge \alpha_2\} \ge \bigwedge\{t > 0 : \ D^*(x, y, z, t) \ge \alpha_1\}$
$\Rightarrow D^*_{\alpha_2}(x, y, z) \ge D^*_{\alpha_1}(x, y, z)$.
Thus $\{D^*_\alpha : \alpha \in (0, 1)\}$ is an ascending family of D^*-metrices on X.

Theorem 5.2. *Let $\{D^*_\alpha : \alpha \in (0, 1)\}$ be an ascending family of D^*-metrices on a non-empty set X.*
Define a function $D' : X^3 \times [0, \infty) \to [0, 1]$ by

$$D'(x, y, z, t) = \begin{cases} 0 & \textit{when } x = y = z \textit{ and } t = 0 \\ \Omega & \textit{otherwise} \end{cases}$$

where

$$\Omega = \bigvee\{\alpha \in (0,1) : \; D^*_\alpha(x,y,z) \le t\}$$

Then D' is a D^*-fuzzy metric on X.

Proof. (FD^*1) For $t = 0$ and either $x \ne y$ or $x \ne z$ then $\{\alpha \in (0,1) : D^*_\alpha(x,y,z) \le t\} \; = \phi$
$\Rightarrow D'(x,y,z,t) = 0$
When $x = y = z$ and $t = 0$ then from definition $D'(x,y,z,t) = 0$.
Thus $D'(x,y,z,0) = 0 \; \forall x,y,z \in X$.

(FD^*2) Let $\forall t(>0) \; \in R, \; D'(x,y,z,t) = 1$
Choose any $\epsilon(>0) \in (0,1)$.
Then for any $t > 0, \; \exists\alpha_t \in (\epsilon, 1]$ such that $D^*_{\alpha_t}(x,y,z) \le t$ and hence $D^*_\epsilon(x,y,z) \le t$.
Since $t > 0$ is arbitrary, this implies that $D^*_\epsilon(x,y,z) = 0 \; \Rightarrow x = y = z$.
If $x = y = z$, then for $t > 0$,
$D'(x,y,z,t) = \bigvee\{\alpha \in (0,1) : \; D^*_\alpha(x,y,z) \le t\} \; = \; \bigvee\{\alpha \in (0,1) : \; \alpha \in (0,1)\} \; = 1$
Thus ($\forall t(>0) \in R, \; D'(x,y,z,t) = 1$) iff $x = y = z$.

(FD^*3) follows from definition of $D'(x,y,z,t)$.

(FD^*4) we have to show that $D'(x,y,a,t) * D'(a,z,z,s) \le D'(x,y,z,t+s)$.
If
(a) $s = t = 0$
(b) $s + t > 0; \; s > 0, t = 0; s = 0, t > 0$, then in these cases the relation is obvious.
If
(c) $s > 0, \; t > 0$, let $p = D'(x,y,a,s), \; q = D'(a,z,z,t)$ and $p \le q$.
If $p = 0, \; q = 0$ then obviously (FD^*4) holds.
Let $0 < r < p \le q$.
Then $\exists\alpha > r$ such that $D^*_\alpha(x,y,a) \le s$ and $\exists\beta > r$ such that $D^*_\alpha(a,z,z,s) \le t$
Let $\gamma = \alpha \wedge \beta > r$.
Therefore $D^*_\gamma(x,y,a) \le D^*_\alpha(x,y,a) \le s$ and $D^*_\gamma(a,z,z) \le D^*_\beta(a,z,z) \le t$.
Now $D^*_\gamma(x,y,z) \le D^*_\gamma(x,y,a) + D^*_\gamma(a,z,z) \le s + t$
Therefore $D^*(x,y,z,t+s) \ge \gamma > r$.
Since $0 < r < \gamma$ is arbitrary thus
$D'(x,y,z,t+s) \ge p \; = \min\{D'(x,y,a,s), \; D'(a,z,z,t)\}$.
Similarly if $p \ge q$, then the relation also holds .

(N5) Let $x \in X, \; \alpha \in (0,1)$.
Now $t > D^*_\alpha(x,y,z) \Rightarrow D'(x,y,z,t) = \bigvee\{\beta : D^*_\beta(x,y,z) \; \le t\} \; \ge \alpha$.
So $\lim_{t\to\infty} D'(x,y,z,t) = 1$.
Next we verify that $D'(x,y,z,.)$ is a non-decreasing function of R.
If $t_1 = t_2 = 0$, then $D'(x,y,z,t_1) = D'(x,y,z,t_2) = 0 \; \forall x,y,z \in X$.
If $t_2 > t_1 > 0$ then $\{\alpha \in (0,1) : D^*_\alpha(x,y,z) \; \le t_1\} \subset \{\alpha \in (0,1) : \; D^*_\alpha(x,y,z) \le t_2\}$
$\Rightarrow \vee\{\alpha : \; D^*_\alpha(x,y,z) \le t_1\} \; \le \vee\{\alpha : \; D^*_\alpha(x,y,z) \le t_2\}$
$\Rightarrow D'(x,y,z,t_1) \le D'(x,y,z,t_2)$.
Thus $D'(x,y,z,.)$ is a non-decreasing function on $[0 \; , \; \infty)$ and hence D' is a D^*-fuzzy metric on X.

Definition 5.1. Let $(X \; , \; D^*)$ be a D^*-fuzzy metric space. We define
$D^*(x,y,z,t+) = D^*_+(x,y,z,t) = \lim_{s\downarrow t} D^*(x,y,z,s)$ and
$D^*(x,y,z,t-) = D^*_-(x,y,z,t) = \lim_{s\uparrow t} D^*(x,y,z,s)$.

Theorem 5.3. *Let X be a non-empty set and D^*_1 and D^*_2 be two D^*-fuzzy metrices on X satisfying (D^*6).*
Then $\forall x,y,z \in X, \quad \forall t \in [0,\infty)$,
$D^*_1(x,y,z,t+) = D^*_2(x,y,z,t+)$ and $D^*_1(x,y,z,t-) = D^*_2(x,y,z,t-)$
iff $D^*_{1\alpha}(x,y,z) \; = D^*_{2\alpha}(x,y,z) \;\; \forall\alpha \in (0,1)$ where $D^*_{1\alpha}$ and $D^*_{2\alpha}$ denote the corresponding $\alpha - D^*$ metrices of D^*_1 and D^*_2 respectively.

Proof. First we suppose that $D^*_{1\alpha}(x,y,z) = D^*_{2\alpha}(x,y,z) \;\; \forall\alpha \in (0,1)$. If possible, suppose for some $t = t_0 \; \in [0,\infty), \; D^*_1(x,y,z,t_0+) \ne D^*_2(x,y,z,t_0+)$.
Without loss of generality we may suppose $D^*_1(x,y,z,t_0+) < D^*_2(x,y,z,t_0+)$.
Then for $t_0 < t < t_0 + \epsilon \; (\epsilon > 0), \; D^*_1(x,y,z,t) < D^*_2(x,y,z,t)$.

Choose β such that

$$D^*_1(x,y,z,t) < \beta < D^*_2(x,y,z,t) \;\; --\text{(i)}$$

Note that

$$D^*_{1\alpha}(x,y,z) = \wedge\{t > 0 : D^*_1(x,y,z,t) \ge \alpha\} \;\; --\text{(ii)}$$

where $\alpha \in (0,1)$. Also,

$$D^*_{2\alpha}(x,y,z) = \bigwedge\{t > 0 : D^*_2(x,y,z,t) \ge \alpha\} \;\; --\text{(iii)}$$

By using (i),(ii) and (iii) we have

$$D^*_{1\beta}(x,y,z) \le t_0, \; D^*_{2\beta}(x,y,z) \ge t_0 + \epsilon$$

which is a contradiction.

So

$$D^*_1(x,y,z,t+) = D^*_2(x,y,z,t+) \; \forall t \in [0,\infty).$$

Similarly

$$D^*_1(x,y,z,t-) = D^*_2(x,y,z,t-) \; \forall t \in [0,\infty).$$

Conversely suppose that

$$D^*_1(x,y,z,t+) = D^*_2(x,y,z,t+) \; \forall t \in [0,\infty).$$

We have to show that

$$D^*_{1,\alpha}(x,y,z) \; = D^*_{2,\alpha}(x,y,z) \;\; \forall\alpha \in (0,1).$$

If possible suppose that $\exists\alpha_0 \in (0,1)$ such that $D^*_{1,\alpha_0}(x,y,z) \; \ne D^*_{2,\alpha_0}(x,y,z)$

Without loss of generality we may suppose that

$$D^*_{1,\alpha_0}(x,y,z) \; > \; D^*_{2,\alpha_0}(x,y,z).$$

Choose k_1, k_2, k_3 such that

$$D^*_{1,\alpha_0}(x,y,z) > k_1 > k_2 > k_3 > D^*_{2,\alpha_0}(x,y,z) \;\; --\text{(iv)}$$

Then by using (ii) we have,

$$D^*_1(x,y,z,k_1) < \alpha_0, \; D^*_2(x,y,z,k_2) \ge \alpha_0 \;\; --\text{(v)}$$

Now from (iv) and (v) we get,

$$\alpha_0 > D^*_1(x,y,z,k_1) \ge D^*_1(x,y,z,k_2+),$$

$$D_2^*(x,y,z,k_2-) \geq D_2^*(x,y,z,k_3) \geq \alpha_0.$$

Combining the above two results we have,

$$D_1^*(x,y,z,k_2+) < \alpha_0 \leq D_2^*(x,y,z,k_2-) \leq D_2^*(x,y,z,k_2+)$$

$$\Rightarrow D_1^*(x,y,z,k_2+) \ < D_2^*(x,y,z,k_2+)$$

which is a contradiction to the assumption.

Thus $D_{1\alpha}^*(x,y,z) = D_{2\alpha}^*(x,y,z) \forall \alpha \in (0,1) \ \forall x,y,z \in X$. This completes the proof. □

Definition 5.2. Let X be a non-empty set and D_1^* and D_2^* be two D^*-fuzzy metrices on X. Then D_1^* and D_2^* are said to be equipotent if $D_1^*(x,y,z,t+) = D_2^*(x,y,z,t+)$ and $D_1^*(x,y,z,t-) = D_2^*(x,y,z,t-) \quad \forall x,y,z \in X$ and $\forall t \in [0,\infty)$.

Note 5.1. It can be easily verified that the above relation is an equivalence relation.

Theorem 5.4. *Let $(X\ ,\ D^*)$ be a D^*-fuzzy metric space satisfying (D^*6) and D_α^* denotes the $\alpha - D^*$-metric of D^* for each $\alpha \in (0,1)$.*
Define

$$D'(x,y,z,t) = \begin{cases} 0 & \text{when } x=y=z \text{ and } t=0 \\ \Omega & \text{otherwise} \end{cases}$$

where

$$\Omega = \bigvee \{\alpha \in (0,1): \ D_\alpha^*(x,y,z) \leq t\}$$

Then D' is a D^*-fuzzy metric on X and D^* & D' are equipotent.

Proof. By Theorem 5.2, it follows that D' is a D^*-fuzzy metric on X.
We have, $D_\alpha^*(x,y,z) = \wedge\{t>0 : D^*(x,y,z,t) \geq \alpha\}, \alpha \in (0,1) - - - - - - - - - - -(i)$.
Now we have to show that,
$D^*(x,y,z,t-) = D'(x,y,z,t-)$ and $D^*(x,y,z,t+) = D'(x,y,z,t+), \ \forall x,y,z \in X, \ \forall t \in R$.
If possible, suppose that for some $t = t_0 \in R$,
$D^*(x,y,z,t_0-) \neq D'(x,y,z,t_0-)$.
Without loss of generality we may suppose that $D^*(x,y,z,t_0-) < D'(x,y,z,t_0-)$.
Choose β such that $D^*(x,y,z,t_0-) < \beta < D'(x,y,z,t_0-)$.
For, $t_0 - \epsilon < t < t_0 \ (\epsilon > 0), \ D^*(x,y,z,t) < \beta < D'(x,y,z,t)$.
Now for $t_0 - \epsilon < t < t_0, \ D^*(x,y,z,t) < \beta \Rightarrow D_\beta^*(x,y,z,) \geq t_0$ by using (i).
$D'(x,y,z,t) > \beta \ \Rightarrow D_\beta^*(x,y,z) \leq t \ \forall t \in (t_0-\epsilon, t_0)$ (by using definition of D').
Thus we arrive at a contradiction.
Therefore $D^*(x,y,z,t_0-) = D'(x,y,z,t_0-)$.
Similarly we can verify that $D^*(x,y,z,t+) = D'(x,y,z,t+)$.
Hence D^* and D' are equipotent.

However if we assume that

(D^*7) for $(x,y,z) \neq (0,0,0), \ D^*(x,y,z,.)$ is a continuous function in $[0\ ,\ \infty)$ then
D^* & D' are identical and the following theorem follows:

Theorem 5.5(Decomposition Theorem 2). *Let $(X\ ,\ D^*)$ be a D^*-fuzzy metric space satisfying (D^*6) and(D^*7).*
Let $D_\alpha^*(x,y,z) = \bigwedge\{t>0 : \ D^*(x,y,z,t) \geq \alpha\}, \ \ \alpha \in (0,1)$.
$M' : X^3 \times [0\ ,\ \infty) \to [0,1]$ by

$$D'(x,y,z,t) = \begin{cases} 0 & \text{when } x=y=z \text{ and } t=0 \\ \Omega & \text{otherwise} \end{cases}$$

where

$$\Omega = \bigvee \{\alpha \in (0,1): \ D_\alpha^*(x,y,z) \leq t\}$$

Then
(i) $\{D_\alpha^* : \ \alpha \in (0,1)\}$ is an ascending family of D^*-metrices on X.
(ii) D' is a D^*-fuzzy metric on X.
(iii) $D^* = D'$.

To prove this theorem first we prove the following lemma:

Lemma 5.1. *Let $(X\ ,\ D^*)$ be a D^*-fuzzy metric space satisfying (D^*6), $(x_0,y_0,z_0) \neq (0,0,0) \in X$ and D_α^* be the corresponding $\alpha - D^*$-fuzzy metrices on X corresponding to the D^*-fuzzy metric on X and $\alpha \in (0\ ,\ 1)$.*
Then,
(1) if $D^*(x_0,y_0,z_0,.)$ is upper semicontinuous and if for $t_0 > 0$, $D^*(x_0,y_0,z_0,t_0) = \alpha_0 \in (0\ ,\ 1)$ then $D^*(x_0,y_0,z_0,D_{\alpha_0}^*(x_0,y_0,z_0)) = \alpha_0$.

(2) if $D^*(x_0,y_0,z_0,.)$ is continuous then for any $\alpha \in (0,1)$, $D^*(x_0,y_0,z_0,D_\alpha^*(x_0,y_0,z_0)) = \alpha$. (3) if $D^*(x_0,y_0,z_0,.)$ is continuous and strictly increasing for $t>0$ then $D^*(x_0,y_0,z_0,t) = \alpha \Leftrightarrow D_\alpha^*(x_0,y_0,z_0) = t$.

Proof. (1). From definition,
$D_\alpha^*(x_0,y_0,z_0) = \bigwedge\{s>0 : \ D^*(x_0,y_0,z_0,s) \geq \alpha_0\}$-$- - - - - - - - -(i)$.
Since $D^*(x_0,y_0,z_0,t_0) = \alpha_0$, we get from (i),
$D_{\alpha_0}^*(x_0,y_0,z_0) \leq t_0 - - - - - - - - - - -(ii)$.
Since $D^*(x_0,y_0,z_0,.)$ is non-decreasing, we have from (ii)
$\alpha_0 = D^*(x_0,y_0,z_0,t_0) \geq D^*(x_0,y_0,z_0,D_\alpha^*(x_0,y_0,z_0))$
i.e.$D^*(x_0,y_0,z_0, \ D_\alpha^*(x_0,y_0,z_0)) \leq \alpha_0 - - - - - - - - - -(iii)$.
If possible suppose that $D^*(x_0,y_0,z_0, \ D_{\alpha_0}^*(x_0,y_0,z_0)) < \alpha_0$.
Then by the upper semicontinuity of $D^*(x_0,y_0,z_0,.)$, $\exists t' > \ D_{\alpha_0}^*(x_0,y_0,z_0)$ such that
$D^*(x_0,y_0,z_0,t') < \alpha_0$.
Then
$D_{\alpha_0}^*(x_0,y_0,z_0) = \bigwedge\{s>0 : \ D^*(x_0,y_0,z_0,s) \geq \alpha_0\} \geq t' > D_{\alpha_0}^*(x_0,y_0,z_0)$- which is a contradiction.
So from (iii), $D^*(x_0,y_0,z_0,D_{\alpha_0}^*(x_0,y_0,z_0)) = \alpha_0$.

(2). Since $D^*(x_0,y_0,z_0,.)$ is continuous, by (D^*1) and (D^*5), for each $\alpha \in (0\ ,\ 1)$, $\exists t > 0$ such that $D^*(x_0,y_0,z_0,t) = \ \alpha$.
Then the proof follows.

(3). It follows from (1) and (2) and by using strict increasing property of $D^*(x_0,y_0,z_0,.)$.

Proof of the Theorem :

(i) and (ii) follows from Theorem 5.1 and Theorem 5.2 respectively.
For (iii), we consider the following cases:
Let $(x_0,y_0,z_0,t_0) \in X^3 \times [0\ ,\ \infty)$ and $D^*(x_0,y_0,z_0,t_0) = \alpha_0$.

Case I $x_0 = y_0 = z_0$ and $t_0 = 0$.
Then $D^*(x_0,y_0,z_0,t_0) = D'(x_0,y_0,z_0,t_0) = 0$.

Case II $x_0 = y_0 = z_0$ and $t_0 > 0$.
Then $D^*(x_0,y_0,z_0,t_0) = D'(x_0,y_0,z_0,t_0) = 1$.

Case III Other than $x_0 = y_0 = z_0$ and $t_0 = 0$.
Then $D^*(x_0, y_0, z_0, t_0) = D'(x_0, y_0, z_0, t_0) = 0$.

Case IV Other than $x_0 = y_0 = z_0$ and $t_0(> 0) \in [0, \infty)$ such that $D^*(x_0, y_0, z_0, t_0) = 0$.
For $\alpha \in (0,1)$, $D^*_\alpha(x_0, y_0, z_0) = \bigwedge\{t > 0 : D^*(x_0, y_0, z_0, t) \geq \alpha\}$.
By Lemma 5.1 (2), we have $D^*(x_0, y_0, z_0, D^*_\alpha(x_0, y_0, z_0)) = \alpha \; \forall \alpha \in (0,1)$.
Since $D^*(x_0, y_0, z_0, t_0) = 0 < \alpha$, it follows that $t_0 < D^*_\alpha(x_0, y_0, z_0) \; \forall \alpha \in (0, 1)$.
So $D'(x_0, y_0, z_0, t_0) = \bigvee\{\alpha \in (0,1) : D^*_\alpha(x_0, y_0, z_0) \leq t_0\} = \bigvee \phi = 0$.
Thus $D^*(x_0, y_0, z_0, t_0) = D'(x_0, y_0, z_0, t_0) = 0$.

Case V Other than $x_0 = y_0 = z_0$ and $t_0(> 0) \in [0, \infty)$ such that $0 < D^*(x_0, y_0, z_0, t_0) < 1$.
Let $D^*(x_0, y_0, z_0, t_0) = \alpha_0$. Then $0 < \alpha_0 < 1$.
Now

$$D'(x, y, z, t) = \bigvee\{\alpha \in (0,1) : D^*_\alpha(x, y, z) \leq t\} \; --((\text{i}))$$

$D^*_\alpha(x, y, z) = \bigwedge\{t > 0 : D^*(x, y, z, t) \geq \alpha\}$- $- - - - - - - - - - (ii)$.
Since $D^*(x_0, y_0, z_0, t_0) = \alpha_0$, we get from (ii), $D^*_{\alpha_0}(x_0, y_0, z_0) \leq t_0 - - - - - - (iii)$.
Using (iii), we get from (i), $D'(x_0, y_0, z_0, t_0) \geq \alpha_0$.
i.e. $D'(x_0, y_0, z_0, t_0) \geq D^*(x_0, y_0, z_0, t_0) - - - - - - - - - - (iv)$.
Again from Lemma 5.1(1), we get $D^*_{\alpha_0}(x_0, y_0, z_0) = t_0$.
For, $1 > \alpha > \alpha_0$, let $D^*_\alpha(x_0, y_0, z_0) = t'$. Then $t' \geq t_0$.
By Lemma 5.1(2), $D^*(x_0, y_0, z_0, t') = \alpha$.
So $D^*(x_0, y_0, z_0, t') = \alpha > \alpha_0 = D^*(x_0, y_0, z_0, t_0)$.
Since $D^*(x_0, y_0, z_0, .)$ is monotonically increasing, from above it follows that $t' > t_0$.
So for $1 > \alpha > \alpha_0$, $D^*_\alpha(x_0, y_0, z_0) = t' \nleq t_0$.
Hence $D'(x_0, y_0, z_0, t_0) \leq \alpha_0 = D^*(x_0, y_0, z_0, t_0)$- $- - - - - - - - - - (v)$.
From (iv) and (v),we have $D'(x_0, y_0, z_0, t_0) = D^*(x_0, y_0, z_0, t_0)$.

Case VI Other than $x_0 = y_0 = z_0$ and $t_0(> 0) \in [0, \infty)$ such that $D^*(x_0, y_0, z_0, t_0) = 1$.
Note that,

$$D'(x, y, z, t) = \begin{cases} 0 & \text{when } x = y = z \text{ and } t = 0 \\ \Omega & \text{otherwise} \end{cases} \quad - (\text{vi})$$

where

$$\Omega = \bigvee\{\alpha \in (0,1] : D^*_\alpha(x, y, z) \leq t\}$$

$D^*_\alpha(x, y, z) = \bigwedge\{t > 0 : D^*(x, y, z, t) \geq \alpha\}$ $-$ (vii)
From (vii) we get $D^*_\alpha(x_0, y_0, z_0) \leq t_0 \; \forall \alpha \in (0,1]$.
Thus from (vi) $D'(x_0, y_0, z_0, t_0) = D^*(x_0, y_0, z_0, t_0) = 1$.
Hence $D^* = D'$.

VI. Conclusion

In this paper, we redefine $\mathcal{M}$-fuzzy metric space introduced by Sedghi et al. by slightly changing its conditions with a view to formulate a decomposition theorem of D^*-fuzzy metric space into a family of D^*-metric spaces. The novelty of this decomposition theorem is far fetching.

It is hoped that the results of this paper, especially decomposition theorems, will be helpful for the researchers to develop many results in D^*-fuzzy metric spaces.

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