Common Fixed Points for Integral Type Contractive Condition in A Menger Space

P.P. Murthy, M.S. Khan, S. Kumar, K. Tas, and Rashmi

***Abstract*—The intent of this note is to focus on various issues of Fixed Point Theorems among them is *various kinds of compatible maps* (for example Compatible Maps of Type(A), Type(B), Type(C), Type(P), weakly compatible maps, etc.) and these pairs satisfying a contractive condition of integral type in Menger Spaces, which improve in particular the results of Branciari (2002), Rhoades (2003), Kumar et al. (2007) and results cited in the literature of Fixed Point Theory. Also, we have introduced the notion of *Universal weakly compatible maps* and prove a fixed point theorem for weakly compatible maps along with the notion of any kind of weakly compatible. At the end of this note, we prove a fixed point theorem for variants of R-Weakly commuting mappings .**

***Index Terms*—Compatible map, weakly compatible maps, Universal weakly compatible maps, variants of R-Weakly commuting mappings.**

***MSC 2010 Codes* – 47H10, 54H25.**

I. INTRODUCTION

IN metric spaces $d(x, y)$ stands to be distance between the point x and y. If the exact distance is not known, then the concept of probable distances comes. The concept of a Probabilistic Metric Space (in short PM-space) deals with such situations. The notion of Probabilistic Metric Space (or Statistical Metric Space) was initially introduced by Menger [10] in 1944, which is a generalization of Metric spaces. The idea in Probabilistic Metric space is associated with distribution functions with a pair of points, say (p, q), denoted by $F(p, q; t)$ where $t > 0$ and interpreted this function as the probability that distance between p and q is less than t, whereas in the metric space the distance function is a single positive number. The study of PM-spaces was expanded rapidly with the pioneering works of Schweizer-Sklar [20].

First , we recall that a real valued function defined on the set of real numbers is known as a distribution function if it is non-decreasing, left-continuous and

$$\inf f(x) = 0, \quad \sup f(x) = 1$$

Dr. P.P. Murthy is a Professor in Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavdyalaya (A Central University), Bilaspur(C.G.) ,495 009, INDIA E-mail: ppmurthy@gmail.com

Dr. M.S. Khan is an Associate Professor in the Department of Mathematics and Statistics, Sultan Qaboos University Post Box 36, Postal Code 123, Al-Khod, Muscat, Oman

Dr. S. Kumar is a Lecturer in the Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Haryana, INDIA E-mail: sanjaymudgal2004@yahoo.com

Dr. K. Tas is a Professor in the Cankaya University, Department of Mathematics and Computer Science, Ankara, TURKEY

Rashmi is a Research Scholar in the Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavdyalaya (A Central University), Bilaspur(C.G.), 495 009, INDIA E-mail: rashmikwt@gmail.com

An example of a distribution function is Heavyside function $\mathsf{H}z(x)$, defined by

$$H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Definition 1.1 A probabilistic metric space (PM-space) is a pair (X, F) where X is a set and F is a function defined on X x X into L, the set of distribution functions, such that if x, y and z are points of X, then

(i) $F(x, y; 0) = 0$,
(ii) $F(x, y; t) = H(t)$ if and only if $x = y$,
(iii) $F(x, y; t) = F(y, x; t)$,
$(iiii)$ $F(x, y; s) = 1$ and $F(y, z; t) = 1$ then

$$F(x, z; s + t) = 1 \;\forall\; x, y, z \in X \text{ and } s, t > 0$$

For each x and y in X and for each real number $t \to 0$, $F(x, y; t)$ is to be thought of as the probability that the distance between x and y is less than t.

Of course, any metric space (X, d) may be regarded as a PM-space. Indeed, if (X, d) is a metric space, then the distribution function $F(x, y; t)$ defined by

$$F(x, y; t) = H(\tilde{t}d(x, y))$$

induces a PM-space , where d is a usual metric.

In this section, (X, F) is considered as probabilistic metric spaces with the condition that

$$\lim_{t\to\infty} F(x, y, t) = 1$$

for all x, y in X.

Definition 1.2 A t-norm is a 2-place function

$$\triangle : [0, 1] \times [0, 1] \to [0, 1]$$

satisfying the following;

(i) $\triangle(0, 0) = 0$,
(ii) $\triangle(0, 1) = 1$,
(iii) $\triangle(a, b) = \triangle(b, a)$,
(iv) If $a \leq c, b \leq d$ then $\triangle(a, b) \leq \triangle(c, d)$,
(v) $\triangle(\triangle(a, b), c) = \triangle(a, \triangle(b, c))$ for all a, b, c in $[0, 1]$.

Definition 1.3 A Menger PM-space is a triplet $(X, F, \triangle)$ where (X, F) is a PM-space and $\triangle$ is a t-norm with the following condition:

$$F(x, z; s + t) \geq \triangle(F(x, y; s), F(y, z; t))$$

for all x, y, z in X and $s, t > 0$.

This inequality is known as *Menger's triangle inequality.*

In 1966, the notion of contraction mappings on PM-space was first introduced by Seghal [22]. Moreover, every contraction mapping on a complete Menger space has a unique fixed point (For more detail see, [59],[10]-[15],[20]-[22].)

Definition 1.4 Let (X, F) be a PM-space and $f : X \to X$ be an arbitrary mapping on X. Then f is called a contraction if there exist $k \in (0, 1)$ such that for x, y in X and $t > 0$, we have $F(fx, fy; kt) \geq F(x, y; t)$.

In 1991, Mishra [2] introduced the concept of compatible mappings in PM-space.

Definition 1.5 Let f and g be self mappings on a Menger space $(X, F, \triangle)$. The mappings f and g are called compatible if $\lim_{n\to\infty} F(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.

In 1992, Cho,Murthy and Stojakovic [2] introduced the notion of Compatible mappings of type (A) in Menger Spaces.

Definition 1.6 Let f and g be self mappings on a Menger space $(X, F, \triangle)$. The mappings f and g are called *compatible maps of type (A)* if

$$\lim_{n\to\infty} F(fgx_n, ggx_n, t) = 1$$

and

$$\lim_{n\to\infty} F(fgx_n, ggx_n, t) = 1$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$$

for some $u \in X$ and for all $t > 0$.

Definition 1.7 [15] Let f and g be self mappings on a Menger space $(X, F, \triangle)$. The mappings f and g are called weak compatible maps of type (A) if

$$\lim_{n\to\infty} F(fgx_n, ggx_n, t) = 1 \quad \text{or} \quad \lim_{n\to\infty} F(fgx_n, ggx_n, t) = 1$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$$

for some $u \in X$ and for all $t > 0$.

Remark 1.1 Now we state some results, which highlight relation among compatible mappings, compatible mappings of type (A),compatible mappings of type (P) and weak compatible mappings of type (A).

(i) Let f and g be compatible (resp. compatible mappings of type (P), compatible mappings of type (A) and weak compatible mappings of type (A)) self-mappings of a Menger space (X, F, Δ) . If $f(t) = g(t)$ for some $t \in X$, then $fg(t) = gf(t)$.

(ii) Let $f, g : (X, F, \Delta) \to (X, F, \Delta)$ be continuous mappings .Then f and g are compatible if and only if f and g are compatible mappings of type (A) (or compatible mappings of type (P), weak compatible mappings of type (A)).

(iii) Let (X, F, Δ) be a Menger space. Let $f, g : X \to X$ be compatible mappings of type (A) (compatible mappings of type (P) ,weak compatible mappings of type (A)). If one of f and g is continuous, then f and g are compatible.

(iv) Let $f, g : (X, F, \Delta) \to (X, F, \Delta)$ be compatible mappings of type (A) (compatible mappings of type (P) ,weak compatible mappings of type (A)) and $fx = gx$ for some $x \in X$, then $fgx = ggx = gfx = ggx$.

II. Variants of Compatible Maps

In 2002, Branciari [1] proved a unique fixed point theorem of integral type which further strengthens the Banach Fixed Point Theorem.

Rhoades [19] generalized Branciari results for pair of mappings. Kumar ,Chugh and Vats [7] improved Rhoades results for a pair of compatible maps as follows:

Theorem 2.1 Let f and g be compatible self maps of a complete metric space (X, d) satisfying the following conditions:

(2.1) $f(X) \subset g(X)$.
(2.2) one of the mapping f or g is continuous .
(2.3) $\int_0^{d(fx,fy)} \phi(t)dt < c.\int_0^{d(gx,gy)} \phi(t)dt$, for each $x, y \in X, c \in [0, 1)$ where $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ is a Lebesgue-integrable mapping which is a summable, non-negative, and such that
(2.4) $\int_0^{\epsilon} \phi(p)dp > 0$ for each $\epsilon > 0$.

Then f and g have a unique common fixed point. □

Remark 2.1 Rhoades pointed out that inequality (2.3) should be as $d(fx, fy) < d(gx, gy)$.

Since $c < 1$,

$$\int_0^{d(fx,fy)} \phi(t)dt < c.\int_0^{d(gx,gy)} \phi(t)dt$$

implies that

$$d(fx, fy) < d(gx, gy), \quad \forall\ x, y \in X$$

For if there exists some $x, y \in X$ such that

$$d(fx, fy) \geq d(gx, gy),$$

then from using (2.3), we have

$$\begin{aligned} \int_0^{d(gx,gy)} \phi(t)dt &\leq \int_0^{d(fx,fy)} \phi(t)dt \\ &< \int_0^{d(gx,gy)} \phi(t)dt \end{aligned}$$

which is a contradiction.

Now in this section, we give analogue of the Theorem 2.1 in the probabilistic setting using compatible of type (A) (or compatible of type (P) or weak compatible of type (A)) maps.

Theorem 2.2. Let (X, F, Δ) be a complete Menger space .Let f and g be compatible of type (A) (or compatible of type (P) or weak compatible of type (A)) self maps of X satisfying the following conditions:

(2.5) $f(X) \subseteq g(X)$,
(2.6) one of the mappings f or g is continuous,
(2.7) $\int_0^{1-F(fx,fy,ct)} \phi(p)dp \leq \int_0^{1-F(gx,gy,ct)} \phi(p)dp$, for each $x, y \in X, t > 0, c \in [0, 1)$, where $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ is

a Lebesgue-integral mapping which is summable (with finite integral) on each compact subset of $\mathbf{R}^+$, non-negative, and such that for each $\epsilon > 0$,

(2.8) $\int_0^\epsilon \phi(p)dp > 0$.

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $f(X) \subseteq g(X)$, choose $x_1 \in X$ such that $gx_1 = fx_0$. In general, choose x_{n+1} such that $y_n = gx_{n+1} = fx_n$.

For each integer $n \geq 1$, we have from (2.7),
$\int_0^{1-F(y_n,y_{n+1},t)} \phi(p)dp = \int_0^{1-F(fx_n,fx_{n+1},t)} \phi(p)dp$
$\leq \int_0^{1-F(gx_n,gx_{n+1},\frac{t}{c})} \phi(p)dp$
$= \int_0^{1-F(fx_{n-1},fx_n,\frac{t}{c})} \phi(p)dp$
$\leq \int_0^{1-F(gx_{n-1},gx_n,\frac{t}{c^2})} \phi(p)dp$.

In this fashion one obtains that
$\int_0^{1-F(y_n,y_{n+1},t)} \phi(p)dp = \int_0^{1-F(fy_0,y_1,\frac{t}{c^n})} \phi(p)dp$.

Letting $n \to \infty$ and using Lebesgue dominated convergence theorem and $c \in [0,1)$ it follows in view of (2.8) that $\lim_{n\to\infty} F(y_n, y_{n+1}, t) = 1$. We now show that $\{y_n\}$ is a Cauchy sequence.

For each integer $n \geq 1$, from (2.7), we have
$\int_0^{1-F(y_n,y_{n+1},t)} \phi(t)dt = \int_0^{1-F(fx_n,fx_{n+1},t)} \phi(t)dt$
$\leq \int_0^{1-F(gx_n,gx_{n+1},\frac{t}{c})} \phi(t)dt = \int_0^{1-F(fx_{n-1},fx_n,\frac{t}{c})} \phi(t)dt$
$\leq \int_0^{1-F(gx_{n-1},gx_n,\frac{t}{c^2})} \phi(t)dt$

In this fashion one obtains after n-th iteration

$$\int_0^{1-F(y_n,y_{n+1},t)} \phi(t)dt \leq \int_0^{1-F(y_0,y_1,\frac{t}{c^n})} \phi(t)dt$$

Letting $\lim_{n\to\infty}$ and using Lebesgue dominated convergence theorem and $c \in [0,1)$ it follows in view of (2.8) that $\lim_{n\to\infty} F(y_n, y_{n+1}, t) = 1$.
Similarly, $\lim_{n\to\infty} F(y_{n+1}, y_{n+2}, t) = 1$.
For any positive integer p, we get
$F(y_n, y_{n+p}, t) \geq F(y_n, y_{n+1}, \frac{t}{p}) * \ldots p - times \ldots * F(y_{n+p-1}, y_{n+p}, \frac{t}{p})$
$\geq F(y_n, y_{n+1}, \frac{t}{p}) * \ldots p - times \ldots * F(y_n, y_{n+1}, \frac{t}{p})$ for any positive integer p.

Since

$$\lim_{n\to\infty} F(y_n, y_{n+1}, t) = 1,$$

for $t > 0$, it follows that

$$\lim_{n\to\infty} F(y_n, y_{n+p}, t) \geq 1 * \ldots * 1 \geq 1$$

Thus $\{y_n\}$ is a Cauchy sequence in X.

But f and g are compatible mappings of type (A) (or compatible of type (P)or weak compatible of type (A)) ,and g is continuous ,therefore, by Remark 1.1., we have $\lim_{n\to\infty} fgx_n = gz$.

Now from (2.7) , we have

$$\int_0^{1-F(fgx_n,fx_n,t)} \phi(t)dt \leq \int_0^{1-F(ggx_n,gx_n,\frac{t}{c})} \phi(t)dt$$

Letting $\lim_{n\to\infty}$ and using Lebesgue dominated convergence theorem it follows in view of (2.8) that $gz = z$.

Again from (2.7)

$$\int_0^{1-F(fx_n,fz,t)} \phi(t)dt \leq \int_0^{1-F(gx_n,gz,\frac{t}{c})} \phi(t)dt$$

Taking $\lim_{n\to\infty}$ and using Lebesgue dominated convergence theorem it follows in view of (2.8) that $fz = z$. Hence z is a common fixed point of f and g.

Uniqueness: Suppose that $w(\neq z)$ is also another fixed point of f and g. From (iii), we have
$\int_0^{1-F(z,w,t)} \phi(t)dt = \int_0^{1-F(fz,fw,t)} \phi(t)dt$
$\leq \int_0^{1-F(gx_n,gw,\frac{t}{c})} \phi(t)dt = \int_0^{1-F(z,w,\frac{t}{c})} \phi(t)dt$
since $c \in [0,1)$, therefore $z = w$ and so uniqueness follows.

Example 2.1 Let $X = [0,1]$ be equipped with the usual metric space.

Define $F(x,y,t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and for each $t > 0$. Define mappings $f, g : X \to$ by $fx = \frac{x}{3}$ and $gx = \frac{x}{2}$ for all $x \in X$.

Clearly $f(X) = [0, \frac{1}{3}] \subset gX = [0, \frac{1}{2}]$.

Moreover, ϕ is defined by $\phi(t) = t$ for $t > 0$ is a Lebesgue integral mapping which is summable (with finite integral) on each compact subset of $\mathbf{R}^+$, non-negative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$.

Now

$$\int_0^{1-F(fx,fy,ct)} \phi(p)dp = \int_0^{1-F(gx,gy,t)} \phi(p)dp$$

where

$$1 - F(fx, fy, ct) = \{\frac{d(x,y)}{(3ct + d(x,y))}\}$$

and

$$1 - F(gx, gy, t) = \{\frac{d(x,y)}{(2t + d(x,y))}\}$$

Thus all the hypothesis of Theorem 3.2 are satisfied with $\phi(t) = t$ for $t > 0, \phi(0) = 0$ and $c \in [\frac{2}{3}, 1)$ and 0 is the unique common fixed point of f and g . □

Next we prove a theorem for a weakly compatible maps satisfying a contractive condition of integral type as follows :

Theorem 2.3 Let $(X, F, \triangle)$ a Menger space. Suppose f and g are weakly compatible self maps of X satisfying $(2.5), (2.7), (2.8)$ and the following condition:

(2.9) if any one of $f(X)$ or $g(X)$ is complete, Then f and g have a unique common fixed point.

Proof. From proof of Theorem 2.2 we conclude that $\{y_n\}$ is a Cauchy sequence in X and from (2.9) either $f(X)$ or $g(X)$ is complete, for definiteness assume that $g(X)$ is complete. Note that the sequence $\{y_{2n}\}$ is contained in $g(X)$ so has a limit in $g(X)$. Call it z. Let $u \in g^{-1}z$. Then $gu = z$. Now we show that $fu = z$. From (2.7), we have
$\int_0^{1-F(fx_n,fu,t)} \phi(p)dp = \int_0^{1-F(gx_n,gu,\frac{t}{c})} \phi(p)dp$, for each $x, y \in X, c \in [0,1)$.

Letting $\lim_{n\to\infty}$ and using Lebesgue dominated convergence theorem, it follows in view of (2.8) that $fu = z$.

Since f and g are weakly compatible, it follows that $fz = fgu = gfu = gz$.

Now we show that z is a common fixed point of f and g.

From (2.7) , we obtain
$\int_0^{1-F(fz,z,t)} \phi(p)dp = \int_0^{1-F(fz,fu,t)} \phi(p)dp$
$\leq \int_0^{1-F(gz,gu,\frac{t}{c})} \phi(p)dp =$
$\int_0^{1-F(fz,z,\frac{t}{c})} \phi(p)dp,$
which is a contradiction, since $c \in [0, 1)$, this implies $fz = z = gz$ and therefore , z is a common fixed point of f and g.

For Uniqueness: Suppose that $w(\neq z)$ is also another common fixed points of f and g. Then from (2.7) , we have
$\int_0^{1-F(z,w,t)} \phi(p)dp = \int_0^{1-F(fz,fw,t)} \phi(p)dp$
$\leq \int_0^{1-F(gz,gw,\frac{t}{c})} \phi(p)dp = \int_0^{1-F(z,w,\frac{t}{c})} \phi(p)dp$
which implies that $z = w$, and hence uniqueness follows. □

Example 2.2 Let $X = [3, 22]$ and d be usual metric on X. let $f, g : X \to X$ be defined by

$$f(x) = \begin{cases} 3 & : \text{if } x = 3 \\ 8 & : \text{if } 3 < x \leq 7 \\ 3 & : \text{if } x > 7 \end{cases}$$

$$g(x) = \begin{cases} 3 & : \text{if } x = 3 \\ 10 & : \text{if } 3 < x \leq 7 \\ \frac{x+2}{3} & : \text{if } x > 7 \end{cases}$$

Define $\psi, \varphi : [0, \infty) \to [0, \infty)$ where $\psi(t) = (t+1)^{t+1} - 1$ and $\varphi(t) = \psi^{'}(t)$.

Also define

$$F(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$ and for each $t > 0$.

Moreover, $fX = \{3\} \cup \{8\}$, $gX = [3, 8] \cup \{10\}$. Hence $fX \subseteq gX$.

To see that f and g are non-compatible maps, consider the sequence

$\{x_n = 7 + \frac{1}{n}, n \geq 1\}$ in X. Then

$\lim_{n\to\infty} fx_n = 3, \lim_{n\to\infty} gx_n = 3, \lim_{n\to\infty} fgx_n = 8$ and $\lim_{n\to\infty} gfx_n = 3$.

Hence f and g are non-compatible maps.But they are weakly compatible maps since they commute at coincidence point at $x = 3$. Thus f and g satisfy all the conditions of the Theorem 2.3 and have a unique common fixed point at $x = 3$.

III. Universal Weakly Compatible Maps

Now we shall define any kind of weakly compatible maps in probabilistic metric spaces as follows:

Definition 3.1. A pair of self-mappings (f, g) of a Menger space (X, F, Δ) is said to be Universal weakly compatible maps if there exists a sequence $\{x_n\} \in X$ such that $F(fx_n, gx_n, t) = 1$ for some $t \in X$. □

Example 3.2. Let $X = [0, +\infty)$. Define $S, T : X \to X$ by $Tx = \frac{x}{2}$ and $Sx = \frac{3x}{4}$, for all $x \in X$.
Consider the sequence $x_n = \frac{1}{n}$.

Clearly $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0$. Then S and T are Universal weakly compatible maps. □

Example 3.3. Let $X = [\frac{2}{3}, +\infty)$. Define $f, g : X \to X$ by $gx = \frac{x+1}{3}$ and $fx = \frac{2x+1}{3}$, for all $x \in X$.
Suppose that f and g are Universal weakly compatible maps. Then, there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$$

for some $z \in X$.

Therefore,

$$\lim_{n\to\infty} x_n = 3z - 1 \text{ and } \lim_{n\to\infty} x_n = \frac{3z - 1}{2}$$

Thus,

$$z = \frac{1}{3},$$

which is a contradiction, since $\frac{1}{3}$ is not contained in X. Hence f and g are not Universal weakly compatible maps. □

Example (3.4). Let $X = [0, 1]$ be equipped with the usual metric $d(x, y) = |x - y|$.

Define

$$F(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$ and for each $t > 0$.

Hence (X, M, Δ) is a Menger space.

Also define

$$f(x) = \begin{cases} 1 - x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{3}{4} & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

Consider the sequence

$$\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}$$

We have

$$\lim_{n\to\infty} f(\frac{1}{2} - \frac{1}{n}) = \frac{1}{2} = \lim_{n\to\infty} g(\frac{1}{2})$$

Thus, the pair (f, g) are Universal weakly compatible maps.

Further, f and g are weakly compatible since $x = \frac{1}{2}$ is their unique coincidence point and

$$fg(\frac{1}{2}) = f(\frac{1}{2}) = g(\frac{1}{2}) = gf(\frac{1}{2})$$

We further observe that

$$d(fg(\frac{1}{2} - \frac{1}{n}), gf(\frac{1}{2} - \frac{1}{n})) \neq 0$$

showing that

$$\lim_{n\to\infty} F(fgx_n, gfx_n, t) \neq 1$$

Therefore, the pair (f, g) is non-compatible. □

Example 3.5. Let $X = \mathbf{R}^+$ be equipped with the usual metric

$$d(x, y) = |x - y|$$

Define

$$F(x,y,t)=\frac{t}{t+d(x,y)}$$

for all $x,y\in X$ and for each $t>0$. Hence (X,M,Δ) is a Menger space.

Also define $f,g:X\to X$ by $fx=0$, if $0<x\leq 1$ and $fx=1$, if $x>1$ or $x=0$; and $gx=[x]$, the greatest integer that is less than or equal to x, for all $x\in X$.

Consider a sequence $\{x_n\}=\{1+\frac{1}{n}\}, n\geq 2$ in $(1,2)$, then we have

$$\lim_{n\to\infty} fx_n=1=\lim_{n\to\infty} gx_n$$

Similarly for the sequence $\{y_n\}=\{1-\frac{1}{n}\}n\geq 2$ in $(0,1)$, we have

$$\lim_{n\to\infty} fy_n=0=\lim_{n\to\infty} gy_n$$

Thus the pair (f,g) are Universal weakly compatible maps. However, f and g are not weakly compatible as as each $u_1\in(0,1)$ and $u_2\in(1,2)$ are coincidence points of f and g, where they do not commute. Moreover, they commute at $x=0,1,2...$ but none of these points are coincidence points of f and g. Further, we note that pair (f,g) is non compatible maps. □

Now we prove a theorem for a pair of weakly compatible maps along with the notion of Universal weakly compatible maps.

Theorem 3.1. Let (X,F,Δ) be a Menger space. Suppose f and g are weak compatible self-maps of X satisfying $(2.5)(2.7),(2.8)$ and the following:

(3.1) f,g are Universal weakly compatible maps
(3.2) $f(X)$ or $g(X)$ is a closed subset X.

Then f and g have a coincidence point. Moreover f and g have a unique common fixed point.

Proof. Since f and g be Universal weakly compatible maps,therefore, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=u\in X$. Since either $f(X)$ or $g(X)$ is a closed subspace of X, for definiteness we assume that $g(X)$ is a closed subset of X.

Further, note that the sequence $\{y_{2n}\}$ which is contained in $g(X)$, so there is a limit in $g(X)$. Call it be u such that $u=ga$. Therefore,

$$\lim_{n\to\infty} fx_n=u=ga=\lim_{n\to\infty} gx_n$$

for some $a\in X$.

This implies $u=ga\in gX$. Now we show that $u=fa=ga$. From (2.7), we have

$$\int_0^{1-F(a,x_n,ct)}\phi(p)dp=\int_0^{1-F(ga,gx_n,t)}\phi(p)dp$$

Letting $\lim_{n\to\infty}$ and using Lebesgue dominated convergence theorem and $c\in[0,1)$, it follows in view of (2.8) that

$$F(fa,fa,ct)\leq F(ga,fa,t)$$

This implies that $u=ga=fa$. Thus a is the coincidence point of f and g. Since f and g are weakly compatible, therefore, $fu=fga=gfa=gu$.

Now we show that $fu=u$. Now from (2.7),we have

$$\int_0^{1-F(fu,fa,ct)}\phi(p)dp=\int_0^{1-F(gu,ga,t)}\phi(p)dp$$

which in turns implies that $fu=u$. Hence u is the unique common fixed point of f and g.

For Uniqueness: Suppose that $w(\neq z)$ is also another fixed point of f and g. From (2.7), we have
$\int_0^{1-F(z,w,t)}\phi(t)dt=\int_0^{1-F(fz,fw,t)}\phi(t)dt$
$<\int_0^{1-F(gz,gw,\frac{t}{c})}\phi(t)dt=\int_0^{1-F(z,w,\frac{t}{c})}\phi(t)dt$
since $c\in[0,1)$, therefore $z=w$ and so uniqueness follows.

Example 3.6. Let $X=[0,1]$ be equipped with the usual metric space.

Define

$$F(x,y,t)=\frac{t}{t+d(x,y)}$$

for all $x,y\in X$ and for each $t>0$. Hence (X,M,Δ) is a Menger space.

Also define $f,g:X\to X$ by $fx=\frac{x}{3}$, and $gx=\frac{x}{2}$, for all $x\in X$.

Clearly $f(X)=[0,\frac{1}{3}]\subset gX=[0,\frac{1}{2}]$.

Moreover, ϕ defined by $\phi(t)=t$ for $t>0$ is a Lebesgue -integral mapping which is summable (with finite integral) on each compact subset of $\mathbf{R}^+$, non-negative, and such that for each $\epsilon>0$,

$$\int_0^{\epsilon}\phi(t)dt>0$$

Now,

$$\int_0^{1-F(fx,fy,ct)}\phi(p)dp=\int_0^{1-F(gx,gy,t)}\phi(p)dp$$

where

$$1-F(fx,fy,ct)=\{\frac{d(x,y)}{(3ct+d(x,y))}\}$$

and

$$1-F(gx,gy,t)=\{\frac{d(x,y)}{(2t+d(x,y))}\}$$

Thus all the hypothesis of theorem 3.1 are satisfied with $\phi(t)=t$ for $t>0$,$\phi(0)=0$ and $c\in[\frac{2}{3},1)$ and 0 is the unique common fixed point of f and g. □

IV. Variants of R-weakly commuting mappings

In 2007, Kohali and Vashistha [6] introduced the notions of R-weakly commuting mappings of type (i), R-weakly commuting mappings type (ii) and R-weakly commuting mappings type (iii) in probabilistic metric spaces as follow:

Definition 4.1. A pair of self-mappings (f,g) of a Menger space (X,F,Δ) is said to be

(i) R-weakly commuting mappings of type (i) if there exists some $R>0$ such that $F(gfx,ffx,t)\geq F(fx,gx,\frac{t}{R})$,
(ii) R-weakly commuting mappings of type (ii) if there exists some $R>0$ such that $F(fgx,ggx,t)\geq F(fx,gx,\frac{t}{R})$,

(iii) R-weakly commuting mappings of type (iii) if there exists some $R > 0$ such that $F(ffx, ggx, t) \geq F(fx, gx, \frac{t}{R})$,for all $x \in X$ and $t > 0$.

Theorem 4.1. Theorem 2.3 (or Theorem 3.1) remains true if weakly compatible property is replaced by any one (retining the rest of hypothesis) of the following:

(a) R-weakly commuting property
(b) R-weakly commuting property of type (i)
(c) R-weakly commuting property of type (ii)
(d) R-weakly commuting property of type (iii)
(e) weakly commuting property.

Proof. Since all the conditions of Theorem 2.3 (or Theorem 3.1) are satisfied, then the existence of coincidence points for both the pairs is insured. Let x be an arbitrary point of coincidence for the pair (f, g), then using R-weak commutativity one gets $F(fgx, gfx, t) \geq F(fx, gx, \frac{t}{R}) = 1$ which amounts to say that $fgx = gfx$. Thus the pair (f, g) is weakly compatible.
Now applying Theorem 2.3 (or Theorem 3.1), one concludes that f and g have a unique common fixed point . In case (f, g) is an R-weakly commuting pair of type (ii), then $F(fgx, ggx, t) \geq F(fx, gx, \frac{t}{R}) = 1$ which amounts to say that $fgx = ggx$.

Now

$$F(fgx, gfx, t) \geq \Delta(F(fgx, ggx, \frac{t}{R})$$

and

$$F(ggx, gfx, \frac{t}{R})) = \Delta(1, 1) = 1$$

yielding thereby

$$fgx = gfx$$

Similarly, if pair is R-weakly commuting mappings of type (i) or type (iii) or weakly commuting, then pair (f, g) also commutes at their points of coincidence. Now in view of Theorem 2.3 (or Theorem 3.1) , in all the cases f and g have a unique common fixed point. This completes the proof. □

As an application of Theorem 2.3 (or Theorem 3.1) we prove a common fixed point theorem for two finite families of mappings which runs as follows:

Theorem 4.2. Let $\{f_1, f_2 ... f_m\}$ and $\{g_1, g_2, ..., g_n\}$ be two finite families of self-mappings of a Menger space (X, F, Δ) such that $f = f_1 f_2 ... f_m, g = g_1 g_2 ... g_n$, satisfy conditions $(2.5), (2.7)$ and (2.8) and the following:

$g_m(X)$ is a closed subspace of X.

Then f and g have a point of coincidence.

Moreover, if $f_i f_j = f_j f_i$ and $g_k g_l = g_l g_k$ for all $i, j \in I_1 = \{1, 2, ..., m\}, k, l \in I_2 = \{1, 2, ..., n\}$, then (for all $i \in I_1, k \in I_2), f_i$ and g_k have a common fixed point.

Proof. The conclusion is immediate i.e., f and g have a point of coincidence as f, and g satisfy all the conditions of Theorem 2.3 (or Theorem 3.1). Now appealing to component wise commutativity of various pairs, one can immediately prove that $fg = gf$ hence, obviously pairs (f, g) is coincidentally commuting. Note that all the conditions of Theorem 2.3 (or Theorem 3.1) are satisfied ensuring the existence of a unique common fixed point say z. Now one need to show that z remains the fixed point of all the component maps. For this consider
$f(f_i z) = ((f_1 f_2 ... f_m) f_i) z = (f_1 f_2 ... f_{m-1})((f_m f_i) z) = (f_1 ... f_{m-1})(f_i f_m z)$
$= (f_1 ... f_{m-2})(f_{m-1} f_i(f_m z)) = (f_1 ... f_{m-2})(f_i f_{m-1}(f_m z))$
$= \ldots = f_1 f_i(f_2 f_3 f_4 ... f_m z) = f_i f_1(f_2 f_3 ... f_m z)$
$= f_i(f_z) = f_i z.$

Similarly, one can show that

$$f(g_k z) = g_k(fz) = g_k z, f(g_k z) = g_k(gz) = gkz$$

and

$$g(f_i z) = f_i(gz) = f_i z$$

which show that (for all I and k) $f_i z$ and $g_k z$ are other fixed points of the pair $(f, g)w$. Now appealing to the uniqueness of common fixed points of both pairs separately, we get $z = f_i z = g_k z$,which shows that z is a common fixed point of f_i, g_k for all i and k.

Corollary 4.1. Let f and g be two self-mappings of a Menger space (X, F, Δ) such that f_m and g_n satisfy the conditions $(2.5), (2.7)$ and (2.8). If one of $f_m(X)$ or $g_n(X)$ is a complete subspace of X, then f and g have a unique common fixed point provided (f, g) commute.

V. Conclusion

Thus we can conclude that Universal weakly compatible maps does not imply weak compatibility. Here, we notice that weakly compatible and any kind of weakly maps are independent to each other, see a paper on metric space by H.K.Pathak,Rosana Rodriguez- Lopez and R .K .Verma [17].

Acknowledgment

The authors would like to thanks Dr. Lakshman Chaturvedi, Vice-Chancellor of GGV,BSP for his inspiration.

References

[1] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci,* vol. 29, no. 9, 531-536, (2002).
[2] Y.J. Cho, P.P. Murthy and M. Stojakovic, Compatible mappings of type (A) and common fixed points in Menger Spaces, *Comm.Korean Math. Soc.*, vol. 7, no. 2, 325-339, (1992).
[3] G. Jungck, Commuting maps and fixed points, *Am. Math. Mon.,* vol. 83, 261-263, (1976).
[4] G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. and Math. Sci.,* vol. 9, 771-779, (1986).
[5] G. Jungck, Common fixed points for non continuous non self mappings on metric spaces, *Far East J. Math. Sci.,* vol. 4, no. 2, 199-212, (1996).
[6] J.K. Kohli and S.Vashistha, Common fixed point theorems in probabilistic metric spaces, *Acta Math. Hungar,* vol. 115, nos. 1-2, 37-47, (2007).
[7] S. Kumar, R. Chugh and R. Kumar, Fixed point theorem for pair of mappings satisfying a general contractive condition of integral type, *Soochow Journal of Mathematics,* vol. 33, no. 2, 181-185, (2007).
[8] S. Kumar, Common fixed point theorems for expansion mappings in various spaces, *Acta Mathematica Hungrica,* vol. 118, nos. 1-2, 9-28, (2008).

[9] S.N. Mishra, Common fixed points of compatible mappings in PM-spaces, *Math. Japan,* vol. 36, 283-289, (1991).

[10] K. Menger, Statistical metrics, *Proc. Nat. Acad. Sci.* vol. 28, 537-555, (1942).

[11] P.P. Murthy, S.S. Chang and Y.J. Cho, Fixed point theorems for compatible mappings of type (P), *Indian J. Maths.,* vol. 36, no. 2, 151-156, (1994).

[12] P.P. Murthy, Important tools and possible applications of metric fixed point theory, *Nonlinear Analysis,* vol. 47, 3479-3490, (2001).

[13] H.K. Pathak, Y.J. Cho, S.M. Kang and B.S. Lee, Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming, *Le Mathematiche,* vol. 50, 15-33, (1995).

[14] H.K. Pathak, Y.J. Cho, S.S. Chang and S.M. Kang, Compatible mappings of type (P) and fixed point theorems in metric spaces and probabilistic metric spaces, *Novi Sad J. Math.,* vol. 26, 87-109, (1996).

[15] H.K. Pathak, S.M. Kang and J.H. Baek, Weak compatible mappings of type (A) and common fixed points, *Kyungpook Maths J,* vol. 35, 345-349, (1995).

[16] H.K. Pathak, Y.J. Cho and S.M. Kang, Remarks on R-Weakly commuting mappings and common fixed point theorems, *Bull. Korean Math. Soc.* vol. 34, no. 2, 247-257, (1997).

[17] H.K. Pathak, R. Rodriguez-Lopez and R.K. Verma, A common fixed point theorem using implicit relation and E.A property in metric spaces, *Filomat,* vol. 21, no. 2, 211-234, (2007).

[18] R.P. Pant, Common fixed points of non commuting mappings, *J. Math. Anal. Appl.,* vol. 188, 436-440, (1994).

[19] B.E. Rhoades, Two fixed point theorems for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.,* vol. 63, 4007-4013, (2003).

[20] B. Schweizer and A. Sklar, *Probabilistic metric spaces,* North Holland series in Probability and Applied Math., Amsterdam, vol. 5, (1983).

[21] M. Stojakovic, Common fixed point theorems in complete metric and probabilistic metric spaces, *Bull. Austral. Math. Soc,* vol. 36, 73-78, (1987).

[22] V.M. Seghal and A.T. Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces, *Math. Systems Theory,* vol. 6, 97-102, (1972).

[23] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, *Pub. Inst. Math.,* vol. 32, no. 46, 149-153, (1982).

[24] P. Vijayaraju, B.E. Rhoades and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.,* vol. 15, 2359-2364, (2005).