# 3-Equitable Labeling for Some Star and Bistar Related Graphs

S.K. Vaidya and N.H. Shah

***Abstract*—In this paper we prove that the splitting graphs of $K_{1,n}$ and $B_{n,n}$ are 3-equitable graphs. We also show that the shadow graph of $B_{n,n}$ is a 3-equitable graph. Further we prove that square graph of $B_{n,n}$ is 3-equitable for $n \equiv 0(mod\ 3)$ and $n \equiv 1(mod\ 3)$ and not 3-equitable for $n \equiv 2(mod\ 3)$.**

***Index Terms*—3-equitable labeling, star, bistar, splitting graph, shadow graph, square graph.**

***MSC 2010 Codes* - 05C78.**

## I. Introduction

In this paper we consider simple, finite, connected and undirected graph $G = (V(G), E(G))$ with order $p$ and size $q$. For all standard terminology and notations we follow West [9]. We will give brief summary of definitions which are useful for the present investigations.

**Definition 1.1 :** If the vertices of the graph are assigned values subject to certain condition(s) then is known as *graph labeling*.

A detailed study on applications of graph labeling is reported in Bloom and Golomb [3]. According to Beineke and Hegde [2] graph labeling serves as a frontier between number theory and structure of graphs. For an extensive survey on graph labeling and bibliographic references we refer to Gallian [4].

**Definition 1.2 :** Let $G = (V(G), E(G))$ be a graph. A mapping $f : V(G) \rightarrow$ {0,1,2} is called *ternary vertex labeling* of $G$ and $f(v)$ is called the $label$ of the vertex $v$ of $G$ under $f$.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1, 2\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ and $v_f(2)$ be the number of vertices of $G$ having labels 0,1 and 2 respectively under $f$ and let $e_f(0)$,$e_f(1)$ and $e_f(2)$ be the number of edges having labels 0,1 and 2 respectively under $f^*$.

**Definition 1.3 :** A ternary vertex labeling of a graph $G$ is called a *3-equitable labeling* if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph $G$ is *3-equitable* if it admits 3-equitable labeling.

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The concept of 3-equitable labeling was introduced by Cahit [1] and he proved that an Eulerian graph with number of edges congruent to $3(mod\ 6)$ is not 3-equitable. In the same paper he proved that $C_n$ is 3-equitable if and only if $n \not\equiv 3(mod\ 6)$ and all caterpillars are 3-equitable. Cahit [1] claimed to prove that $W_n$ is 3-equitable if and only if $n \not\equiv 3(mod\ 6)$ but Youssef [8] proved that $W_n$ is 3-equitable for all $n \geq 4$. The 3-equitable labeling in the context of vertex duplication is discussed by Vaidya et al. [5] while same authors in [6] have investigated 3-equitable labling for some shell related graphs. Vaidya et al. [7] have discussed 3-equitablity of graphs in the context of some graph operations and proved that the shadow and middle graph of cycle $C_n$, path $P_n$ are 3-equitable.

Generally there are three types of problems that can be considered in this area.

1) How 3-equitability is affected under various graph operations?
2) Construct new families of 3-equitable graph by investigating suitable labeling.
3) Given a graph theoretic property P, characterize the class of graphs with property P that are 3-equitable.

The problems of second type are largely discussed while the problems of first and third types are rarely discussed and they are of great importance also. The present work is aimed to discuss the problems of first kind.

**Definition 1.4 :** The *splitting graph* of a graph $G$ is obtained by adding to each vertex $v$ a new vertex $v'$ such that $v'$ is adjacent to every vertex that is adjacent to $v$ in $G$, i.e. $N(v) = N(v')$. The resultant graph is denoted by $S'(G)$.

**Definition 1.5:** The *shadow graph* $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbours of the corresponding vertex $u''$ in $G''$.

**Definition 1.6:** For a simple connected graph $G$ the *square of graph* $G$ is denoted by $G^2$ and defined as the graph with the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance 1 or 2 apart in $G$.

## II. Main Results

**Theorem 2.1 :** $S'(K_{1,n})$ is 3-equitable graph.

***Proof :*** Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $v$ be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, u_3, \ldots, u_n$ are added vertices corresponding to $v, v_1, v_2, v_3, \ldots, v_n$ to obtain $S'(K_{1,n})$. Let $G$ be the graph $S'(K_{1,n})$ then $|V(G)| = 2n+2$ and $|E(G)| = 3n$. To define $f : V(G) \to \{0, 1, 2\}$ we consider following three cases.

**Case 1:** $n \equiv 0(mod\ 3)$

$$\begin{array}{ll} f(v) = 2, & \\ f(u) = 0, & \\ f(v_i) = 0; & 1 \le i \le \frac{n}{3} + 1 \\ f\left(v_{\frac{n}{3}+1+i}\right) = 1; & 1 \le i \le \frac{n}{3} - 1 \\ f\left(v_{\frac{2n}{3}+i}\right) = 2; & 1 \le i \le \frac{n}{3} \\ f(u_i) = 0; & 1 \le i \le \frac{n}{3} - 1 \\ f\left(u_{\frac{n}{3}-1+i}\right) = 1; & 1 \le i \le \frac{n}{3} + 2 \\ f\left(u_{\frac{2n}{3}+1+i}\right) = 2; & 1 \le i \le \frac{n}{3} - 1 \end{array}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(1) = \frac{2n}{3} + 1 = v_f(2) + 1$$
$$e_f(0) = e_f(1) = e_f(2) = n$$

**Case 2:** $n \equiv 1(mod\ 3)$

Since $n \equiv 1(mod\ 3)$, $n = 3k + 1$ some $k \in \mathbb{N}$.

$$\begin{array}{ll} f(v) = 2, & \\ f(u) = 0, & \\ f(v_i) = 0; & 1 \le i \le k + 1 \\ f(v_{k+1+i}) = 1; & 1 \le i \le k - 1 \\ f(v_{2k+i}) = 2; & 1 \le i \le k + 1 \\ f(u_i) = 0; & 1 \le i \le k - 1 \\ f(u_{k-1+i}) = 1; & 1 \le i \le k + 3 \\ f(u_{2k+2+i}) = 2; & 1 \le i \le k - 1 \end{array}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(2) = \frac{2n+1}{3} = v_f(1) - 1$$
$$e_f(0) = e_f(1) = e_f(2) = n$$

**Case 3:** $n \equiv 2(mod\ 3)$

Since $n \equiv 2(mod\ 3)$, $n = 3k + 2$ some $k \in \mathbb{N}$.

$$\begin{array}{ll} f(v) = 2, & \\ f(u) = 0, & \\ f(v_i) = 0; & 1 \le i \le k + 1 \\ f(v_{k+1+i}) = 1; & 1 \le i \le k \\ f(v_{2k+1+i}) = 2; & 1 \le i \le k + 1 \\ f(u_i) = 0; & 1 \le i \le k \\ f(u_{k+i}) = 1; & 1 \le i \le k + 2 \\ f(u_{2k+2+i}) = 2; & 1 \le i \le k \end{array}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(2) = v_f(1) = \left\lfloor \frac{2(n+1)}{3} \right\rfloor$$

$$e_f(0) = e_f(1) = e_f(2) = n$$

Thus in each case we have $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$, for all $0 \le i, j \le 2$.
Hence $S'(K_{1,n})$ is a 3-equitable graph.

**Illustration 2.2 :** 3-equitable labeling of the graph $S'(K_{1,7})$ is shown in Fig. 1.

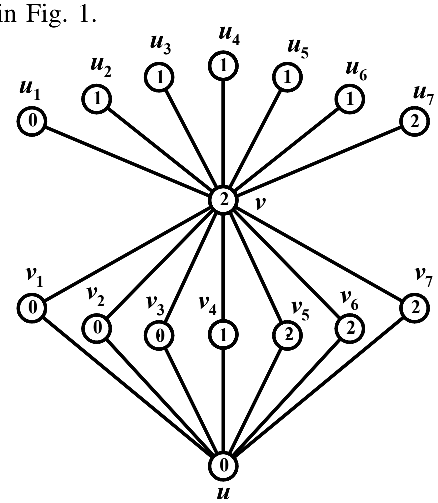


**Figure 1**

**Theorem 2.3 :** $S'(B_{n,n})$ is 3-equitable graph.

***Proof :*** Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \le i \le n\}$ where $u_i, v_i$ are pendant vertices. In order to obtain $S'(B_{n,n})$, add $u', v', u'_i, v'_i$ vertices corresponding to $u, v, u_i, v_i$ where, $1 \le i \le n$. If $G = S'(B_{n,n})$ then $|V(G)| = 4(n+1)$ and $|E(G)| = 6n+3$. To define $f : V(G) \to \{0, 1, 2\}$ we consider following four cases.

**Case 1:** $n = 2, 5$

The graphs $S'(B_{2,2})$ and $S'(B_{5,5})$ are to be dealt separately and their 3-equitable labeling is shown is Fig. 2. and Fig. 3.

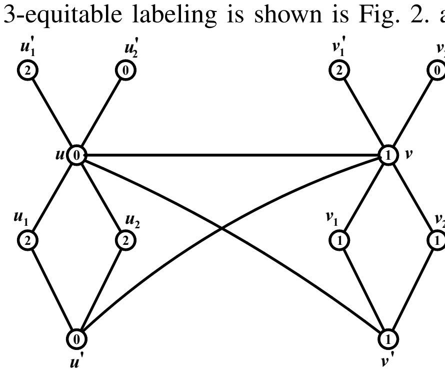


**Figure 2**

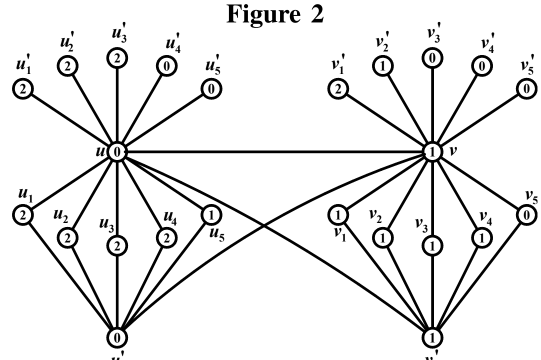


**Figure 3**

**Case 2:** $n \equiv 0(mod\ 3)$
Since $n \equiv 0(mod\ 3)$, $n = 3k$ some $k \in \mathbb{N}$.

$$\begin{array}{ll}
f(u) = 0, & \\
f(u') = 0, & \\
f(v) = 1, & \\
f(v') = 1, & \\
f(u_i) = 2; & 1 \leq i \leq n \\
f(u'_1) = 2; & \\
f(u'_{1+i}) = 1; & 1 \leq i \leq k \\
f(u'_{k+1+i}) = 0; & 1 \leq i \leq n-k-1 \\
f(v_i) = 0; & 1 \leq i \leq 2(k-1) \\
f(v_{2k-2+i}) = 1; & 1 \leq i \leq n-2k+2 \\
f(v'_i) = 2; & 1 \leq i \leq k \\
f(v'_{k+i}) = 1; & 1 \leq i \leq 2k-2 \\
f(v'_i) = 0; & i = n, n-1
\end{array}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(2) = 4k+1 = v_f(1) - 1$$
$$e_f(0) = e_f(1) = e_f(2) = 2n+1$$

**Case 3:** $n \equiv 1(mod\ 3)$
Since $n \equiv 1(mod\ 3)$, $n = 3k+1$ some $k \in \mathbb{N}$.

$$\begin{array}{ll}
f(u) = 0, & \\
f(u') = 0, & \\
f(v) = 1, & \\
f(v') = 1, & \\
f(u_i) = 2; & 1 \leq i \leq n \\
f(u'_1) = 2; & \\
f(u'_{1+i}) = 1; & 1 \leq i \leq k \\
f(u'_{k+1+i}) = 0; & 1 \leq i \leq 2k \\
f(v_i) = 1; & 1 \leq i \leq k+2 \\
f(v_{k+2+i}) = 0; & 1 \leq i \leq 2k-1 \\
f(v'_i) = 2; & 1 \leq i \leq k \\
f(v'_{k+i}) = 1; & 1 \leq i \leq 2k-1 \\
f(v'_i) = 0; & i = n, n-1
\end{array}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(1) = 4k+3 = v_f(2) + 1$$
$$e_f(0) = e_f(1) = e_f(2) = 2n+1$$

**Case 4:** $n \equiv 2(mod\ 3)$
Since $n \equiv 2(mod\ 3)$, $n = 3k+2$ some $k \in \mathbb{N} - \{1\}$.

$$\begin{array}{ll}
f(u) = 0, & \\
f(u') = 0, & \\
f(v) = 1, & \\
f(v') = 1, & \\
f(u_i) = 2; & 1 \leq i \leq n \\
f(u'_1) = 2; & \\
f(u'_{1+i}) = 1; & 1 \leq i \leq k+1 \\
f(u'_{k+2+i}) = 0; & 1 \leq i \leq 2k \\
f(v_i) = 1; & 1 \leq i \leq k+4 \\
f(v_{k+4+i}) = 0; & 1 \leq i \leq 2k-2 \\
f(v'_i) = 2; & 1 \leq i \leq k+1 \\
f(v'_{k+1+i}) = 1; & 1 \leq i \leq 2k-3 \\
f(v'_i) = 0; & n-3 \leq i \leq n
\end{array}$$

In view of the above labeling patten we have

$$v_f(0) = v_f(1) = 4(k+1) = v_f(2)$$
$$e_f(0) = e_f(1) = e_f(2) = 2n+1$$

Thus in each case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, for all $0 \leq i, j \leq 2$.
Hence $S'(B_{n,n})$ is a 3-equitable graph.

**Illustration 2.4 :** 3-equitable labeling of the graph $S'(B_{6,6})$ is shown in Fig. 4.

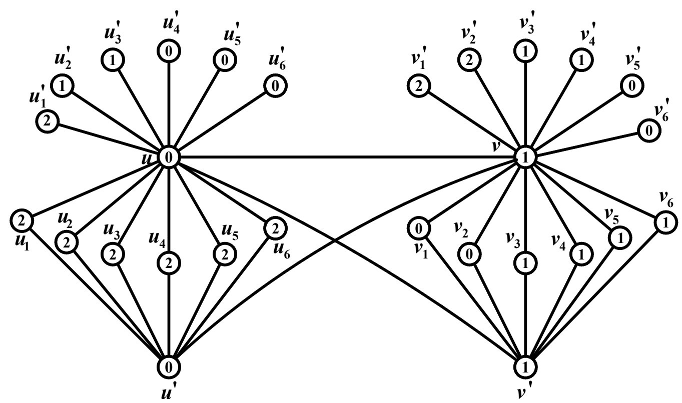


**Figure 4**

**Theorem 2.5 :** $D_2(B_{n,n})$ is 3-equitable graph.
***Proof :*** Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ and $\{u', v', u'_i, v'_i, 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$. Let $G$ be the graph $D_2(B_{n,n})$ then $|V(G)| = 4(n+1)$ and $|E(G)| = 4(2n+1)$. To define $f : V(G) \rightarrow \{0, 1, 2\}$ we consider following three cases.
**Case 1:** $n \equiv 0(mod\ 3)$
Since $n \equiv 0(mod\ 3)$, $n = 3k$ some $k \in \mathbb{N}$.

$$\begin{array}{ll}
f(u) = 0, & \\
f(u') = 2, & \\
f(v) = 0, & \\
f(v') = 2, & \\
f(u_i) = 0; & 1 \leq i \leq 2k+1 \\
f(u_{2k+1+i}) = 1; & 1 \leq i \leq k-1 \\
f(u'_i) = 1; & 1 \leq i \leq 2(k-1) \\
f\left(u'_{2(k-1)+i}\right) = 2; & 1 \leq i \leq k+2 \\
f(v_i) = 0; & 1 \leq i \leq 2k-1 \\
f(v_{2k-1+i}) = 1; & 1 \leq i \leq k+1 \\
f(v'_i) = 1; & 1 \leq i \leq 3 \\
f(v'_{3+i}) = 2; & 1 \leq i \leq n-3
\end{array}$$

$$v_f(1) = v_f(2) = \frac{4(n+1)-1}{3} = v_f(0) - 1$$
$$e_f(0) = e_f(2) = \frac{8n+3}{3} = e_f(1) - 1$$

**Case 2:** $n \equiv 1(mod\ 3)$
Since $n \equiv 1(mod\ 3)$, $n = 3k+1$ some $k \in \mathbb{N}$.

$$\begin{array}{ll} f(u)=0, & \\ f(u')=2, & \\ f(v)=0, & \\ f(v')=2, & \\ f(u_i)=0; & 1\le i\le n \\ f(u'_i)=2; & 1\le i\le n \\ f(v_i)=0; & 1\le i\le k \\ f(v_{k+i})=1; & 1\le i\le n-k \\ f(v'_i)=1; & 1\le i\le 2k+1 \\ f\left(v'_{2k+1+i}\right)=2; & 1\le i\le k \end{array}$$

$$v_f(0)=v_f(2)=\frac{4(n+1)+1}{3}=v_f(1)+1$$
$$e_f(0)=e_f(2)=e_f(1)=\frac{8n+4}{3}$$

**Case 3:** $n\equiv 2(mod\ 3)$

Since $n\equiv 2(mod\ 3)$, $n=3k+2$ some $k\in\mathbb{N}\cup\{0\}$.

$$\begin{array}{ll} f(u)=0, & \\ f(u')=0, & \\ f(v)=0, & \\ f(v')=1, & \\ f(u_i)=0; & 1\le i\le k+1 \\ f(u_{k+1+i})=1; & 1\le i\le k \\ f(u_{2k+1+i})=2; & 1\le i\le k+1 \\ f(u'_i)=2; & 1\le i\le n \\ f(v_i)=0; & 1\le i\le n-2 \\ f(v_{n-1})=1; & \\ f(v_n)=2; & \\ f(v'_i)=1; & 1\le i\le n \end{array}$$

$$v_f(0)=v_f(1)=v_f(2)=\frac{4(n+1)}{3}$$
$$e_f(0)=e_f(2)=\frac{8n+4+1}{3}=e_f(1)+1$$

Thus in each case we have $|v_f(i)-v_f(j)|\le 1$ and $|e_f(i)-e_f(j)|\le 1$, for all $0\le i,j\le 2$.

Hence $D_2(B_{n,n})$ is a 3-equitable graph.

**Illustration 2.6 :** 3-equitable labeling of the graph $D_2(B_{5,5})$ is shown in Fig. 5.

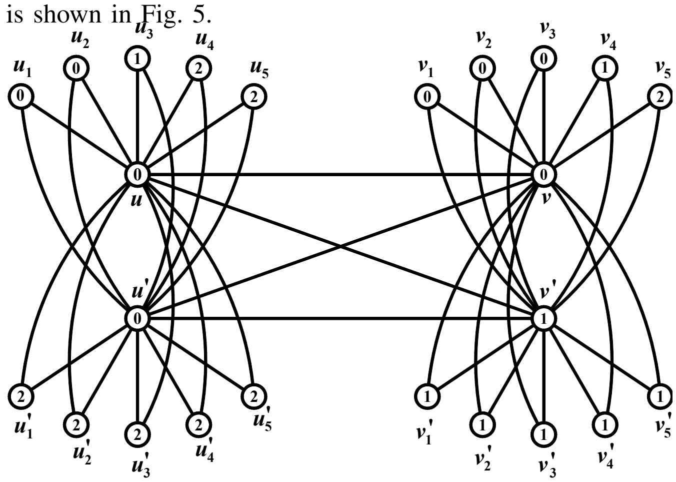

**Figure 5**

**Theorem 2.7 :** $B^2_{n,n}$ is 3-equitable graph for $n\equiv 0(mod\ 3)$ and $n\equiv 1(mod\ 3)$.

***Proof :*** Consider $B_{n,n}$ with vertex set $\{u,v,u_i,v_i,1\le i\le n\}$ where $u_i,v_i$ are pendant vertices. Let $G$ be the graph $B^2_{n,n}$ then $|V(G)|=2n+2$ and $|E(G)|=4n+1$. To define $f:V(G)\to\{0,1,2\}$, we consider following two cases.

**Case 1:** $n\equiv 0(mod\ 3)$

Since $n\equiv 0(mod\ 3)$, $n=3k$ some $k\in\mathbb{N}$.

$$\begin{array}{ll} f(u)=2, & \\ f(v)=0, & \\ f(u_i)=1; & 1\le i\le k \\ f(u_{k+i})=2; & 1\le i\le 2k \\ f(v_i)=0; & 1\le i\le 2k \\ f(v_{2k+i})=1; & 1\le i\le k \end{array}$$

$$v_f(0)=v_f(2)=\frac{2n+3}{3}=v_f(1)+1$$
$$e_f(0)=e_f(1)=\frac{4n}{3}=e_f(2)-1$$

**Case 2:** $n\equiv 1(mod\ 3)$

Since $n\equiv 1(mod\ 3)$, $n=3k+1$ some $k\in\mathbb{N}$.

$$\begin{array}{ll} f(u)=2, & \\ f(v)=0, & \\ f(u_i)=1; & 1\le i\le k \\ f(u_{k+i})=2; & 1\le i\le 2k+1 \\ f(v_i)=0; & 1\le i\le 2k \\ f(v_{2k+i})=1; & 1\le i\le k+1 \end{array}$$

$$v_f(0)=v_f(1)=\frac{2n+1}{3}=v_f(2)-1$$
$$e_f(1)=e_f(2)=\frac{4n+2}{3}=e_f(0)+1$$

Thus in both the cases we have $|v_f(i)-v_f(j)|\le 1$ and $|e_f(i)-e_f(j)|\le 1$, for all $0\le i,j\le 2$.

Hence $B^2_{n,n}$ is a 3-equitable graph for $n\equiv 0(mod\ 3)$ and $n\equiv 1(mod\ 3)$.

**Illustration 2.8 :** 3-equitable labeling of the graph $B^2_{7,7}$ is shown in Fig. 6.

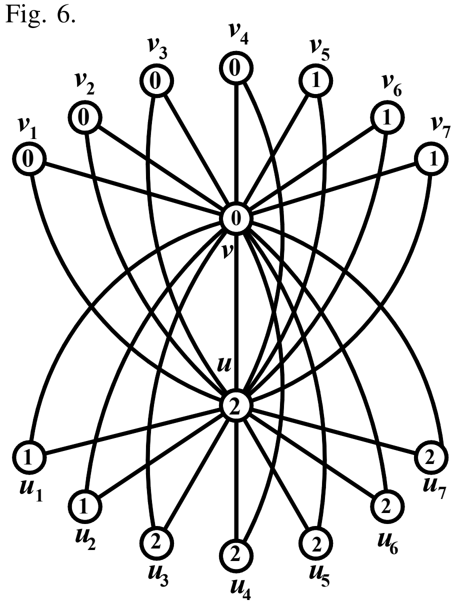

**Figure 6**

**Theorem 2.9 :** $B^2_{n,n}$ is not a 3-equitable graph for $n \equiv 2(mod\ 3)$.

***Proof :*** Let $G$ be the graph $B^2_{n,n}$ then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$. Here $n \equiv 2(mod\ 3)$ therefore $n = 3k_1 + 2$ for some $k_1 \in \mathbb{N}$. Hence $|V(G)| = 3k$ and $|E(G)| = 6k - 3$ where $k = 2k_1 + 2$ . So if $B^2_{n,n}$ is 3-equitable then we must have $v_f(0) = v_f(1) = v_f(2) = k$ and $e_f(0) = e_f(1) = e_f(2) = 2k - 1$.

In $B^2_{n,n}$, note that each $u_i$ and $v_i$ $(1 \leq i \leq n)$ are adjacent to $u$ and $v$ both moreover $u$ and $v$ are adjacent vertices. It is obvious that any edge will have label 1 if it is incident to one vertex with label 1. Following Table 1. shows all possible assignments of vertex label. From the Table 1 (column 6) we can observe that the edge condition violates in all the possible assignments. Hence $B^2_{n,n}$ is not a 3-equitable graph for $n \equiv 2(mod\ 3)$.

## III. Concluding Remarks

The graphs $K_{1,n}$ and $B_{n,n}$ are 3-equitable being caterpillars while we show that the splitting graphs of $K_{1,n}$ and $B_{n,n}$ also admit 3-equitable labeling. Thus 3-equitability remains invariant for the splitting graphs of $K_{1,n}$ and $B_{n,n}$. It is also invariant for shadow graph of $B_{n,n}$. Moreover we prove that the square graph of $B_{n,n}$ is 3-equitable for $n \equiv 0(mod\ 3)$ and $n \equiv 1(mod\ 3)$ while not 3-equitable for $n \equiv 2(mod\ 3)$. To investigate similar results for other graph families and for various graph operations is a potential area of research.

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TABLE I

| $u$ | $v$ | $u_i$'s and $v_i$'s | | | $e_f(1)$ |
|---|---|---|---|---|---|
| vertex label | vertex label | $v'_f(0)$ | $v'_f(1)$ | $v'_f(2)$ | |
| 0 | 0 | $k-2$ | $k$ | $k$ | $2k \neq 2k-1$ |
| 0 | 1 | $k-1$ | $k-1$ | $k$ | $3k-1 \neq 2k-1$ |
| 0 | 2 | $k-1$ | $k$ | $k-1$ | $2k \neq 2k-1$ |
| 1 | 1 | $k$ | $k-2$ | $k$ | $4k \neq 2k-1$ |
| 1 | 2 | $k$ | $k-1$ | $k-1$ | $3k-1 \neq 2k-1$ |
| 2 | 2 | $k$ | $k$ | $k-2$ | $2k \neq 2k-1$ |

Where $v'_f(j) =$ number of vertices having label $j$ for $u_i$ and $v_i$ where $1 \leq i \leq n$ and $0 \leq j \leq 2$ .