On Solving the Second-Order Linear Recurrence Sequence

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Abstract—The paper presents a formula for solving the missing terms of a second-order linear recurrence sequence given its first term and last term. The paper also provides a generalization of Natividad (2011).

Index Terms—Second-order linear recurrence sequence, Binet’s formula, Fibonacci-like Sequences, missing terms.


I. INTRODUCTION

In [1], Horadam defined a linear recurrence sequence of second order \( \{W_n(a, c; p, q)\} \), or simply \( \{W_n\} \), as follows:

\[
W_{n+1} = pW_n + qW_{n-1}; \quad W_0 = a, \quad W_1 = c
\]

where \(a, c\) and \(p, q\) are arbitrary real numbers for \(n > 0\). The Binet’s formula for the recurrence sequence \(\{W_n\}\) is given by

\[
W_n = \frac{Aa^n - B\beta^n}{\alpha - \beta}
\]

where \(A = c - a\beta\), \(B = c - a\alpha\) and since the generating function of \(\{W_n\}\) is \(x^2 = px + q\), it is clear that,

\[
\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}
\]

and

\[
\beta = \frac{p - \sqrt{p^2 + 4q}}{2}
\]

Obviously, \(\alpha + \beta = p, \quad \alpha - \beta = \sqrt{p^2 + 4q}\) and \(\alpha\beta = -q\).

Natividad [2,3] presents a formula for solving the missing terms of a Fibonacci-like sequences and show similar results for solving Pell means. In other words, he provides a formula for solving the missing terms given the first term and the last term of the sequence \(\{W_n(a, c; p, q)\}\) for values of \(p = 1\) and \(q = 1\) and \(p = 2\) and \(q = 1\).

In this study, we shall provide a formula for solving the missing terms of a second-order linear recurrence sequence \(\{W_n(a, c; p, q)\}\) with known values for \(a\) and last term \(b\).

II. MAIN RESULT

We start this section with the definition of means of a sequence.

Definition 2.1: If \(a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n, b\) is a second-order linear recurrence sequence, satisfying the recurrence relation defined by,

\[
W_{n+1} = pW_n + qW_{n-1}
\]

then \(x_1, x_2, x_3, \ldots, x_{n-1}, x_n\) are called the means of the sequence \(\{W_n\}\) between \(a\) and \(b\).

Now, suppose we are asked to find the missing terms or the means of the recurrence sequence \(\{W_n\}\) given a first term \(a\) and a last term \(b\), then it is more convenient for us to solve the problem if we already have a formula for the second term as a function of \(a, b\) and the number of missing terms. To do this, we shall prove the theorem given below.

Theorem 2.2: For any real numbers \(a, b, p\) and \(q\), the second-order linear recurrence sequence \(\{W_n\}\), defined by the recurrence relation \(W_{n+1} = pW_n + qW_{n-1}\), can be solve using the formula for the first missing term given by

\[
x_1 = \frac{b - W_n(0, 1; p, q)aq}{W_{n+1}(0, 1; p, q)}
\]

where \(n\) is the number of missing terms and \(a\) and \(b\) is defined as the first term and the last term of the sequence, respectively.

Proof. Consider the recurrence sequence \(\{W_n\}\) defined by the recurrence relation \(W_{n+1} = pW_n + qW_{n-1}\) with first term denoted by \(W_0 = a\) and last term \(b\). If \(n\) is the number of missing terms between \(a\) and \(b\) then, \(b = W_{n+1}\). Now suppose that \(c = x_1\), hence the Binet’s formula for the sequence is given by

\[
W_n = \frac{(x_1 - a\beta)\alpha^n - (x_1 - a\alpha)\beta^n}{\alpha - \beta}
\]

where \(\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}\) and \(\beta = \frac{p - \sqrt{p^2 + 4q}}{2}\).
Thus,
\[
\begin{align*}
  b &= \frac{(x_1 - a\beta)a^{n+1} - (x_1 - a\alpha)\beta^{n+1}}{\alpha - \beta} \\
  &= \frac{(\alpha^{n+1} - \beta^{n+1})x_1 - (\alpha^{n+1} - \beta^{n+1})a}{\alpha - \beta} \\
  &= \frac{(\alpha^{n+1} - \beta^{n+1})x_1 - (\alpha^n - \beta^n)a\alpha\beta}{\alpha - \beta} \\
  &= \frac{(\alpha^{n+1} - \beta^{n+1})x_1}{\alpha - \beta} + (\alpha^n - \beta^n)aq \\
  &= W_{n+1}(0, 1; p, q)x_1 + W_n(0, 1; p, q)aq
\end{align*}
\]

It follows that,
\[
x_1 = \frac{b - W_n(0, 1; p, q)aq}{W_{n+1}(0, 1; p, q)}.
\]

Now, we show that this formula is valid for any values of \( n \). Suppose there are \( k + 1 \) number of missing terms in the sequence, with \( a \) as the first term and \( b \) as the last term, then \( b = W_{k+2} \), and so
\[
W_{k+2} = pW_{k+1} + qW_k = \frac{p\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} + q\frac{\alpha^k - \beta^k}{\alpha - \beta} = \frac{\alpha^k(p + q) - b\beta^k(p + q)}{\alpha - \beta}.
\]

Since \( \alpha \) and \( \beta \) are roots of the quadratic equation \( x^2 = px + q \), then
\[
b = \frac{\alpha^{k+2} - \beta^{k+2}}{\alpha - \beta} = \frac{(\alpha^{k+2} - \beta^{k+2})x_1}{\alpha - \beta} + (\alpha^{k+1} - \beta^{k+1})aq \\
= W_{k+2}(0, 1; p, q)x_1 + W_{k+1}(0, 1; p, q)aq.
\]

By strong mathematical induction, conclusion follows.

Take note that if \( p = 1 \) and \( q = 1 \), we have
\[
x_1 = \frac{b - F_n a}{F_{n+1}}
\]
as shown by Natividad on [1] and for \( p = 2 \) and \( q = 1 \), we will obtain
\[
x_1 = \frac{b - P_n a}{P_{n+1}}
\]
which is exactly the same result found by Natividad on [2].

Now we have a formula for solving the missing terms of any recurrence sequence \( \{W_n\} \), satisfying the recurrence relation \( W_{n+1} = pW_n + qW_{n-1} \) for any values of \( p \) and \( q \) given its first term and last term.

### III. APPLICATIONS AND EXAMPLES

We state some examples and specific applications of our main result presented in Table I and Table II, respectively.

#### REFERENCES

