On Solving the Second-Order Linear Recurrence Sequence

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***Abstract*—The paper presents a formula for solving the missing terms of a second-order linear recurrence sequence given its first term and last term. The paper also provides a generalization of Natividad (2011).**

***Index Terms*—Second-order linear recurrence sequence, Binet's formula, Fibonacci-like Sequences, missing terms.**

***MSC 2010 Codes* – 11B37, 11B39, 11B50, 11B99.**

I. Introduction

IN [1], Horadam defined a linear recurrence sequence of second order $\{W_n(a, c; p, q)\}$, or simply $\{W_n\}$, as follows:

$$W_{n+1} = pW_n + qW_{n-1}; W_0 = a, W_1 = c$$

where a, c and p, q are arbitrary real numbers for $n > 0$. The Binet's formula for the recurrence sequence $\{W_n\}$ is given by

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

where $A = c - a\beta$, $B = c - a\alpha$ and since the generating function of $\{Wn\}$ is $x^2 = px + q$, it is clear that,

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}$$

and

$$\beta = \frac{p - \sqrt{p^2 + 4q}}{2}$$

Obviously, $\alpha + \beta = p, \alpha - \beta = \sqrt{p^2 + 4q}$ and $\alpha\beta = -q$.

Natividad [2,3] presents a formula for solving the missing terms of a Fibonacci-like sequences and show similar results for solving Pell means. In other words, he provides a formula for solving the missing terms given the first term and the last term of the sequence $\{W_n(a, c; p, q)\}$ for values of $p = 1$ and $q = 1$ and $p = 2$ and $q = 1$.

In this study, we shall provide a formula for solving the missing terms of a second-order linear recurrence sequence $\{W_n(a, c; p, q)\}$ with known values for a and last term b.

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II. Main Result

We start this section with the definition of means of a sequence.

Definition 2.1: If $a, x_1, x_2, x_3, ..., x_{n1}, x_n, b$ is a second-order linear recurrence sequence, satisfying the recurrence relation defined by,

$$W_{n+1} = pW_n + qW_{n-1}$$

then $x_1, x_2, x_3, ..., x_{n1}$, and x_n are called the means of the sequence $\{W_n\}$ between a and b.

Now, suppose we are asked to find the missing terms or the means of the recurrence sequence $\{W_n\}$ given a first term a and a last term b, then it is more convenient for us to solve the problem if we already have a formula for the second term as a function of a, b and the number of missing terms. To do this, we shall prove the theorem given below.

Theorem 2.2: For any real numbers a, b, p and q, the second-order linear recurrence sequence $\{W_n\}$, defined by the recurrence relation $W_{n+1} = pW_n + qW_{n-1}$, can be solve using the formula for the first missing term given by

$$x_1 = \frac{b - W_n(0, 1; p, q)aq}{W_{n+1}(0, 1; p, q)}$$

where n is the number of missing terms and a and b is defined as the first term and the last term of the sequence, respectively.

Proof. Consider the recurrence sequence $\{W_n\}$ defined by the recurrence relation $W_{n+1} = pW_n + qW_{n-1}$ with first term denoted by $W_0 = a$ and last term b. If n is the number of missing terms between a and b then, $b = W_{n+1}$. Now suppose that $c = x_1$, hence the Binets formula for the sequence is given by

$$W_n = \frac{(x_1 - a\beta)\alpha^n - (x_1 - a\alpha)\beta^n}{\alpha - \beta}$$

where $\alpha = \frac{p+\sqrt{p^2+4q}}{2}$ and $\beta = \frac{p-\sqrt{p^2+4q}}{2}$.

Thus,

$$\begin{aligned} b &= \frac{(x_1 - a\beta)\alpha^{n+1} - (x_1 - a\alpha)\beta^{n+1}}{\alpha - \beta} \\ &= \frac{(\alpha^{n+1} - \beta^{n+1})x_1 - (\alpha^{n+1}\beta - \alpha\beta^{n+1})a}{\alpha - \beta} \\ &= \frac{(\alpha^{n+1} - \beta^{n+1})x_1 - (\alpha^n - \beta^n)a\alpha\beta}{\alpha - \beta} \\ &= \frac{(\alpha^{n+1} - \beta^{n+1})x_1}{\alpha - \beta} + \frac{(\alpha^n - \beta^n)aq}{\alpha - \beta} \\ &= W_{n+1}(0,1;p,q)x_1 + W_n(0,1;p,q)aq \end{aligned}$$

It follows that,

$$x_1 = \frac{b - W_n(0,1;p,q)aq}{W_{n+1}(0,1;p,q)}.$$

Now, we show that this formula is valid for any values of n. Suppose there are $k+1$ number of missing terms in the sequence, with a as the first term and b as the last term, then $b = W_{k+2}$, and so

$$\begin{aligned} W_{k+2} &= pW_{k+1} + qW_k \\ &= p\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} + q\frac{\alpha^k - \beta^k}{\alpha - \beta} \\ &= \frac{A\alpha^k(\alpha p + q) - b\beta^k(p\beta + q)}{\alpha - \beta} \end{aligned}$$

Since α and β are roots of the quadratic equation $x^2 = px + q$, then

$$\begin{aligned} b &= \frac{A\alpha^{k+2} - b\beta^{k+2}}{\alpha - \beta} \\ &= \frac{(\alpha^{k+2} - \beta^{k+2})x_1}{\alpha - \beta} + \frac{(\alpha^{k+1} - \beta^{k+1})aq}{\alpha - \beta} \\ &= W_{k+2}(0,1;p,q)x_1 + W_{k+1}(0,1;p,q)aq \end{aligned}$$

By strong mathematical induction, conclusion follows.

Take note that if $p = 1$ and $q = 1$, we have

$$x_1 = \frac{b - F_n a}{F_{n+1}}$$

as shown by Natividad on [1] and for $p = 2$ and $q = 1$, we will obtain

$$x_1 = \frac{b - P_n a}{P_{n+1}}$$

which is exactly the same result found by Natividad on [2].

Now we have a formula for solving the missing terms of any recurrence sequence $\{W_n\}$ satisfying the recurrence relation $W_{n+1} = pW_n + qW_{n-1}$ for any values of p and q given its first term and last term.

III. Applications and Examples

We state some examples and specific applications of our main result presented in Table I and Table II, respectively.

TABLE I
SOME SPECIFIC EXAMPLES.

a	b	p	q	Formula for x_1	n	$a, x_1, x_2, \ldots, x_n, b$
1	13	1	1	$x_1 = \frac{13-F_n}{F_{n+1}}$	5	1,1,2,3,5,8,13
2	71	1	1	$x_1 = \frac{71-2F_n}{F_{n+1}}$	6	2,5,7,12,19,31,40,71
3	227	2	1	$x_1 = \frac{227-3P_n}{P_{n+1}}$	5	3,2,7,16,39,94,227
4	449	2	1	$x_1 = \frac{449-4P_n}{P_{n+1}}$	6	4,1,6,13,32,77,186,449
1	171	1	2	$x_1 = \frac{513-2(2^n-(-1)^n)}{2^{n+1}-(-1)^{n+1}}$	7	1,1,3,5,11,21,43,85,171

TABLE II
FORMULA FOR x_1 FOR SPECIFIC VALUES OF P AND Q.

p	q	$x_1 = \frac{b-W_n(0,1;p,q)aq}{W_{n+1}(0,1;p,q)}$
1	3	$x_1 = \frac{\sqrt{13}b-2\left[\left(\frac{1+\sqrt{13}}{2}\right)^n-\left(\frac{1-\sqrt{13}}{2}\right)^n\right]a}{\left(\frac{1+\sqrt{13}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{13}}{2}\right)^{n+1}}$
3	5	$x_1 = \frac{\sqrt{29}b-5\left[\left(\frac{3+\sqrt{29}}{2}\right)^n-\left(\frac{3-\sqrt{29}}{2}\right)^n\right]a}{\left(\frac{3+\sqrt{29}}{2}\right)^{n+1}-\left(\frac{3-\sqrt{29}}{2}\right)^{n+1}}$

IV. Conclusion

A formula was developed to solve the second-order linear recurrence sequence given its first and last term. Also, a solution to solve the missing terms of the general case of a Fibonnaci-like sequences has been provided in the paper.

References

[1] A. F. Horadam, Basic properties of certain generalized sequence of numbers, *Fibonacci Quarterly,* **3** (1965), 161-176.

[2] Natividad, L.R., Deriving a formula in solving Fibonacci-like sequence, *International Journal of Mathematics and Scientific Computing,* **1** (2011), # 1, 19-21.

[3] Natividad, L.R., On Solving Pell Means, *International Journal of Mathematical Archive,* **12** (2011), # 2, 2736-2739.