# Existence and Uniqueness Solution of Non Linear Kronecker Product Lyapunov System on Time Scales

R. Suryanarayana and Ch. Gopala Rao

***Abstract*- In this paper, existence and uniqueness of solutions to two-point boundary value problem associated with the non-linear kronecker product Lyapunov system on time scales are derived. These generalize the results obtained for two-point boundary value problems involving kronecker products. Bartles-Stewart algorithm and a modified QR-algorithm are presented for solving the system of equations involving kronecker products.**

***Index words*: Kronecker product Lyapunov system, boundary value problems.**

***MSC 2010 Codes*: 34 B15, 34B18.**

## 1. INTRODUCTION

THE nonlinear Kronecker product Lyapunov systems arise in a number of areas of Applied Mathematics such as dynamical programming, optimal filters, filter design, design of receding horizon control strategies for multivariate systems and system engineering.

In 1992, Murty, K.N., Howell, G.W., and Sivasundaram, S [2] established existence and uniqueness of solutions to two-point and multipoint boundary value problems associated with non-linear Lyapunov systems. In 1999, Murty, K.N., Lakshminarasamma, V., [4] obtained the existence and uniqueness of solutions of the kronecker product first order system of differential equations.

In this paper, we consider the two-point boundary value problem associated with the non-linear kronecker product Lyapunov system on time scales

$$(X(t)\otimes Y(t))^{\Delta} = (A(t)\oplus C(t))(X(t)\otimes Y(t)) + (X(\sigma(t))\otimes Y(\sigma(t)))(B(t)\oplus D(t)) + (F_1(t,(X(t)\otimes Y(t)))\otimes(F_2(t,(X(t)\otimes Y(t))) \quad (1.1)$$

$$(M_1\otimes M_2)(X(t_0)\otimes Y(t_0)) + (N_1\otimes N_2)(X(t_1)\otimes Y(t_1)) = \alpha_1\otimes\alpha_2 \quad (1.2)$$

where A(t), B(t), C(t), D(t), X(t) and Y(t) are square matrices of order n.

R Suryanarayana is Associate Professor of Mathematics, GMR Institute of Technology, Rajam, Srikakulam(dist.), A.P, India E-mail Address: rsnarayana@yahoo.co.in; suryanarayana.r@gmrit.org
Ch Gopla Rao is Assistant Professor of Mathematics, GMR Institute of Technology, Rajam, Srikakulam(dist.), A.P, India

We assume that the components of A(t), B(t), C(t), D(t) are continuous functions on $[t_0,t_1]$ and

$$F_1, F_2 : [t_0,t_1]\times R^{n^2\times n^2} \to R^{n^2\times n^2}$$

are continuous and satisfy Lipschitz condition on closed interval $[t_0,t_1]$.

We assume that F(t,0) ≡ 0 on $[t_0,t_1]$. $M_1$, $N_1$, $M_2$, $N_2$, $\alpha_1$ and $\alpha_2$ are constant square matrices of order n.

The Kronecker product and Kronecker sum are denoted by ⊗ and ⊕ respectively. This paper is organized as follows: In section 2, we develop the general solution of the non-linear kronecker product Lyapunov system (1.1) in terms of kronecker product of a fundamental matrix solution. Section 3, deals with two - point boundary value problem associated with (1.1) satisfying the boundary condition matrix (1.2). Here modified Bartles-Stewart algorithm and QR - algorithm are used as a tool to evaluate the constant square matrix $C_1\otimes C_2$ which is embedded in a peculiar form.

## II. SOLUTION OF THE NON-LINEAR KRONECKER PRODUCT LYAPUNOV SYSTEM

In this section, we establish the general solution of the non-linear kronecker product Lyapunov system on time scales (1.1)and (1.2) in terms of a fundamental matrix solution. We now present the following theorems.

**Theorem 2.1:** If $(Y_1(t)\otimes Z_1(t))$ and $(Y_2(t)\otimes Z_2(t))$ are fundamental matrix solutions of

$$(X(t)\otimes Y(t))^{\Delta} = (A(t)\oplus C(t))(X(t)\otimes Y(t)),$$
$$(X(t)\otimes Y(t))^{\Delta} = (B(t)\oplus D(t))^{*}(X(\sigma(t))\otimes Y(\sigma(t)))$$

respectively, then any solution of the homogeneous kronecker product Lyapunov system

$$(X(t)\otimes Y(t))^{\Delta} = (A(t)\oplus C(t))(X(t)\otimes Y(t)) + (X(\sigma(t))\otimes Y(\sigma(t)))(B(t)\oplus D(t)) \quad (2.1)$$

is of the form

$$(X(t)\otimes Y(t)) = (Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^{*}(t)\otimes Z_2^{*}(t)),$$

where $C_1, C_2$ are constant square matrices of order n. $Y_1(t), Z_1(t), Y_2(t)$ and $Z_2(t)$ are fundamental matrix solutions of $X^{\Delta}(t)=A(t)X(t)$, $Y^{\Delta}(t)=C(t)Y(t)$, $X^{\Delta}(t)=B^{*}(t)X(\sigma(t))$, $Y^{\Delta}(t)=D^{*}(t)Y(\sigma(t))$ respectively.

**Proof :** We seek a solution of the homogeneous kronecker product Lyapunov system (2.1) in the form $(X(t)\otimes Y(t))=(Y_1(t)\otimes Z_1(t))(K_1(t)\otimes K_2(t))$, where $K_1(t), K_2(t)$ are square matrices of order n. Then

$$(Y_1(t)\otimes Z_1(t))^{\Delta}(K_1(t)\otimes K_2(t))+(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(t)\otimes K_2(t))^{\Delta}$$
$$=(A(t)\oplus C(t))(Y_1(t)\otimes Z_1(t))(K_1(t)\otimes K_2(t))$$
$$+(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(\sigma(t))\otimes K_2(\sigma(t)))(B(t)\oplus D(t))$$

i.e.,

$(K_1(t)\otimes K_2(t))^{\Delta}=(K_1(\sigma(t))\otimes K_2(\sigma(t)))(B(t)\oplus D(t))$ i.e.,

$(K_1(t)\otimes K_2(t))^{\Delta^*}=(B(t)\oplus D(t))^{*}(K_1(\sigma(t))\otimes K_2(\sigma(t)))^{*}$ Since $(Y_2(t)\otimes Z_2(t))$ is a fundamental matrix solution of $(X(t)\otimes Y(t))^{\Delta}=(B(t)\oplus D(t))^{*}(X(\sigma(t))\otimes Y(\sigma(t)))$, it follows that there exists a constant square matrix $(C_3\otimes C_4)$ such that

$$(K_1(t)\otimes K_2(t))^{*}=(Y_2(t)\otimes Z_2(t))(C_3\otimes C_4)\Leftrightarrow$$
$$(K_1(t)\otimes K_2(t))=(C_3^{*}\otimes C_4^{*})(Y_2^{*}(t)\otimes Z_2^{*})$$

Hence, $X(t)\otimes Y(t)=(Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^{*}(t)\otimes Z_2^{*}(t))$

(Take $C_3^{*}=C_1, C_4^{*}=C_2$)

**Theorem 2.2:** Any solution of the non-linear kronecker product Lyapunov system (1.1) is of the form $X(t)\otimes Y(t)=(Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^{*}(t)\otimes Z_2^{*}(t))+\overline{X}(t)\otimes\overline{Y}(t)$ where $\overline{X}(t)\otimes\overline{Y}(t)$ is a particular solution of (1.1).

**Theorem 2.3 :** A particular solution $(\overline{X}(t)\otimes\overline{Y}(t))$ of the non-linear kronecker product Lyapunov system (1.1) is of the form

$$\overline{X}(t)\otimes\overline{Y}(t)=(Y_1(t)\otimes Z_1(t))\left[\int_{t_0}^{t}(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}F_1(s,(X(s)\otimes Y(s)))\otimes F_2(s,(X(s)\otimes Y(s)))(Y_2^{*}(s)\otimes Z_2^{*}(s))^{-1}\Delta s\right](Y_2^{*}(t)\otimes Z_2^{*}(t)).$$

**Proof :** We seek a particular solution of the kronecker product Lyapunov system (2.1) in the form $\overline{X}(t)\otimes\overline{Y}(t)=(Y_1(t)\otimes Z_1(t))(K_1(t)\otimes K_2(t))(Y_2^{*}(t)\otimes Z_2^{*}(t))$,

where $K_1(t), K_2(t)$ are square matrices of order n. Then substituting $\overline{X}(t)\otimes\overline{Y}(t)$ in (1.1). We get,

$$((Y_1(t)\otimes Z_1(t))(K_1(t)\otimes K_2(t))(Y_2^{*}(t)\otimes Z_2^{*}(t)))^{\Delta}$$
$$=(A(t)\oplus C(t))(Y_1(t)\otimes Z_1(t))(K_1(t)\otimes K_2(t))(Y_2^{*}(t)\otimes Z_2^{*}(t))$$
$$+(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(\sigma(t))\otimes K_2(\sigma(t)))(Y_2^{*}(\sigma(t))\otimes Z_2^{*}(\sigma(t)))(B(t)\oplus D(t))$$
$$+F_1(t,(X(t)\otimes Y(t)))\otimes F_2(t,(X(t)\otimes Y(t))).$$

$$(Y_1(t)\otimes Z_1(t))^{\Delta}(K_1(t)\otimes K_2(t))(Y_2^{*}(t)\otimes Z_2^{*}(t))+$$
$$(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(t)\otimes K_2(t))^{\Delta}(Y_2^{*}(t)\otimes Z_2^{*}(t))$$
$$(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(\sigma(t))\otimes K_2(\sigma(t)))(Y_2^{*}(t)\otimes Z_2^{*}(t))^{\Delta}$$
$$=(A(t)\oplus C(t))(Y_1(t)\otimes Z_1(t))(K_1(t)\otimes K_2(t))(Y_2^{*}(t)\otimes Z_2^{*}(t))+$$
$$(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(\sigma(t))\otimes K_2(\sigma(t)))(Y_2^{*}(\sigma(t))\otimes Z_2^{*}(\sigma(t)))(B(t)\oplus D(t))$$
$$+F_1(t,(X(t)\otimes Y(t)))\otimes F_2(t,(X(t)\otimes Y(t)))$$
$$(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(t)\otimes K_2(t))^{\Delta}(Y_2^{*}(t)\otimes Z_2^{*}(t))=F_1(t,(X(t)\otimes Y(t)))\otimes F_2(t,(X(t)\otimes Y(t)))$$
$$(K_1(t)\otimes K_2(t))^{\Delta}=(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))^{-1}(F_1(t,(X(t)\otimes Y(t)))\otimes F_2(t,(X(t)\otimes Y(t))))(Y_2^{*}(t)\otimes Z_2^{*}(t))^{-1}$$
$$K_1(t)\otimes K_2(t)=\left[\int_{t_0}^{t}(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}F_1(s,(X(s)\otimes Y(s)))\otimes F_2(s,(X(s)\otimes Y(s))).(Y_2^{*}(t)\otimes Z_2^{*})^{-1}\Delta s\right]$$

Hence, a particular solution is given by

$$\overline{X}(t)\otimes\overline{Y}(t)=(Y_1(t)\otimes Z_1(t))\left[\int_{t_0}^{t}(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}F_1(s,(X(s)\otimes Y(s)))\otimes F_2(s,(X(s)\otimes Y(s)))(Y_2^{*}(s)\otimes Z_2^{*}(s))^{-1}\Delta s\right](Y_2^{*}(t)\otimes Z_2^{*}(t)).$$

The general solution of (1.1) is given by

$$(X(t)\otimes Y(t))=(Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^{*}(t)\otimes Z_2^{*}(t))+$$
$$(Y_1(t)\otimes Z_1(t))\left[\int_{t_0}^{t}(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}F_1(s,(X(s)\otimes Y(s)))\otimes F_2(s,(X(s)\otimes Y(s)))(Y_2^{*}(s)\otimes Z_2^{*}(s))^{-1}\Delta s\right](Y_2^{*}(t)\otimes Z_2^{*}(t)).$$

Now we wish to obtain the solution of two-point kronecker product boundary value problem (1.1) and (1.2). Substituting the general solution of (1.1) in the boundary condition (1.2), we get ,

$$(M_1\otimes M_2)(Y_1(t_0)\otimes Z_1(t_0))(C_1\otimes C_2)(Y_2^{*}(t_0)\otimes Z_2^{*}(t_0))$$
$$+(N_1\otimes N_2)(Y_1(t_1)\otimes Z_1(t_1))(C_1\otimes C_2)(Y_2^{*}(t_1)\otimes Z_2^{*}(t_1))$$
$$(Y_2^{*}(s)\otimes Z_2^{*}(s))^{-1}\Delta s(Y_2^{*}(t_1)\otimes Z_2^{*}(t_1))=\alpha_1\otimes\alpha_2 \qquad (2.2)$$

The equation (2.2) is equivalent to

$$(A_0 \otimes B_0)(C_1 \otimes C_2)(A_1 \otimes B_1) + (P_0 \otimes Q_0)(C_1 \otimes C_2)(P_1 \otimes Q_1) = \eta \quad (2.3)$$

where

$A_0 = M_1 Y_1(t_0),\ B_0 = M_2 Z_1(t_0),\ A_1 = Y_2^*(t_0),\ B_1 = Z_2^*(t_0),$

$P_0 = N_1 Y_1(t_1),$

$Q_0 = N_2 Z_1(t_1),\ P_1 = Y_2^*(t_1),\ Q_1 = Z_2^*(t_1)$ and

$$\eta = (\alpha_1 \otimes \alpha_2) - (N_1 \otimes N_2)$$

$$(Y_1(t_1) \otimes Z_1(t_1)) \int_{t_0}^{t_1} (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} F_1(s,(X(s) \otimes Y(s)))$$

$$\otimes F_2(s,(X(s) \otimes Y(s)))\ (Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s (Y_2^*(t_1) \otimes Z_2^*(t_1))$$

$A_0$, $B_0$, $A_1$, $B_1$, $P_0$, $Q_0$, $P_1$ and $Q_1$ are square matrices of order n and $\eta$ is a square matrix of order $n^2$.

## III. ANALYSIS OF THE MATRIX $C_1 \otimes C_2$

In this section, we shall be concerned with the general form of the constant matrix $C_1 \otimes C_2$ satisfying (2.3). Now the matrix equation (2.3) is equivalent to a system of vector equations

$$G\left(\overline{C_1 \otimes C_2}\right) = \overline{\eta} \quad (3.1)$$

where

$G = \left[(A_0 \otimes B_0) \otimes (A_1 \otimes B_1)^* + (P_0 \otimes Q_0) \otimes (P_1 \otimes Q_1)^*\right]$ is a square matrix of order $n^4$, $\overline{C_1 \otimes C_2}$ and $\overline{\eta}$ are $n^4$ column vectors corresponding to the matrices $C_1 \otimes C_2$ and $\eta$. In fact by viewing (2.3) as a system of $n^2$ - scalar equations for the elements of $C_1 \otimes C_2$, (3.1) is exactly the same set of equations written in a vector system. In order to make pronouncements about existence and uniqueness techniques for the solution of (3.1), we need some information about the eigen values of G. We denote the set of all eigen values of the matrix A by $\sigma$ (A), the spectrum of A.

**Case (i) If** $(A_0 \otimes B_0)$ and $(A_1 \otimes B_1)$ are non singular matrices, then (2.3) is equivalent to

$$C_1 \otimes C_2 - (R_0 \otimes S_0)(C_1 \otimes C_2)(R_1 \otimes S_1) = L \quad (3.2)$$

where

$R_0 = A_0^{-1} P_0,\ S_0 = B_0^{-1} Q_0,\ R_1 = P_1 A_1^{-1},\ S_1 = Q_1 B_1^{-1}$ and

$$L = (A_0 \otimes B_0)^{-1}\ \eta\ (A_1 \otimes B_1)^{-1}.$$

Using the result on kronecker product of matrices (3.2), can be written as

$$C_1 \otimes C_2 - (R_0 \otimes S_0)(C_1 \otimes C_2)(R_1 \otimes S_1) = L$$

$$+ (N_1 \otimes N_2)(Y_1(t_1) \otimes Z_1(t_1))$$

$$\int_{t_0}^{t_1} (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} F_1(s,(X(s) \otimes Y(s))) \otimes F_2(s,(X(s) \otimes Y(s)))$$

$$\Leftrightarrow \left[(I \otimes I) - (R_0 \otimes S_0) \otimes (R_1 \otimes S_1)^*\right]\left(\overline{C_1 \otimes C_2}\right) = \overline{L}$$ where

$$\Leftrightarrow I\left(\overline{C_1 \otimes C_2}\right) - (G_1 \otimes G_2)\left(\overline{C_1 \otimes C_2}\right) = \overline{L}$$

$G_1 = R_0 \otimes S_0$, $G_2 = (R_1 \otimes S_1)^*$ and I is an unit matrix of order $n^4$.

Nowputting

$C_1 \otimes C_2 = L + (R_0 \otimes S_0)(C_1 \otimes C_2)(R_1 \otimes S_1)$ in the second term on LHS of (3.2), we get the following equivalent statements

$$(C_1 \otimes C_2) - (R_0 \otimes S_0)[L + (R_0 \otimes S_0)(C_1 \otimes C_2)(R_1 \otimes S_1)](R_1 \otimes S_1) = L$$

$$\Leftrightarrow \left(\overline{C_1 \otimes C_2}\right) - (G_1 \otimes G_2)\left[\overline{L} + (G_1 \otimes G_2)\left(\overline{C_1 \otimes C_2}\right)\right] = \overline{L}$$

$$C_1 \otimes C_2 - (R_0 \otimes S_0)^2 (C_1 \otimes C_2)(R_1 \otimes S_1)^2 = L + (R_0 \otimes S_0) L (R_1 \otimes S_1)$$

$$\Leftrightarrow \left(\overline{C_1 \otimes C_2}\right) - (G_1 \otimes G_2)^2 \left(\overline{C_1 \otimes C_2}\right) = \overline{L} + (G_1 \otimes G_2)\overline{L}$$

... ... ...

$$C_1 \otimes C_2 - (R_0 \otimes S_0)^m (C_1 \otimes C_2)(R_1 \otimes S_1)^m = L + (R_0 \otimes S_0) L (R_1 \otimes S_1) + \ldots$$

$$\ldots \ldots\ldots + (R_0 \otimes S_0)^{m-1} L (R_1 \otimes S_1)^{m-1}$$

$$\Leftrightarrow \left(\overline{C_1 \otimes C_2}\right) - (G_1 \otimes G_2)^m \left(\overline{C_1 \otimes C_2}\right)$$

$$= \overline{L} + (G_1 \otimes G_2)\overline{L} + \ldots + (G_1 \otimes G_2)^{m-1}\overline{L}$$

If the spectral radii of $(R_0 \otimes S_0)$ and $(R_1 \otimes S_1)$ are such that $\rho(R_0 \otimes S_0)\ \rho(R_1 \otimes S_1) < 1$ ,then

$(R_0 \otimes S_0)^m (C_1 \otimes C_2)(R_1 \otimes S_1)^m \to 0$ as $m \to \infty$.

Inthiscase, $(C_1 \otimes C_2) = L + \sum_{m=1}^{\infty} (R_0 \otimes S_0)^m L (R_1 \otimes S_1)^m$

$$= (A_0 \otimes B_0)^{-1} \eta (A_1 \otimes B_1)^{-1} + \sum_{m=1}^{\infty} (R_0 \otimes S_0)^m (A_0 \otimes B_0)^{-1} \eta$$

$$(A_1 \otimes B_1)^{-1} (R_1 \otimes S_1)^m$$

$$= [M_1 Y_1(t_0) \otimes M_2 Z_1(t_0)]^{-1}$$

$$[\alpha_1 \otimes \alpha_2 - (N_1 \otimes N_2)(Y_1(t_1) \otimes Z_1(t_1))$$

$$\int_{t_0}^{t_1} (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} F_1(s,(X(s) \otimes Y(s))) \otimes F_2(s,(X(s) \otimes Y(s)))$$

$$(Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s\ .$$

$$(Y_2^*(t_1) \otimes Z_2^*(t_1))]$$

$$(Y_2^*(t_0) \otimes Z_2^*(t_0))^{-1} + \sum_{m=1}^{\infty} (-1)^m \left[(M_1 Y_1(t_0))^{-1} (N_1 Y_1(t_1)) \otimes\right.$$

$$(M_2Z_1(t_0))^{-1}(N_2Z_1(t_1))]^m[M_1Y_1(t_0)\otimes M_2Z_1(t_0)]^{-1}$$
$$[\alpha_1\otimes\alpha_2-(N_1\otimes N_2)(Y_1(t_1)\otimes Z_1(t_1))$$
$$\int_{t_0}^{t_1}(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}F_1(s,(X(s)\otimes Y(s)))\otimes F_2(s,(X(s)\otimes Y(s)))$$
$$(Y_2^*(s)\otimes Z_2^*(s))^{-1}\Delta s$$
$$(Y_2^*(t_1)\otimes Z_2^*(t_1))]\,(Y_2^*(t_0)\otimes Z_2^*(t_0))^{-1}$$
$$\left[Y_2^*(t_1)Y_2^{*^{-1}}(t_0)\otimes Z_2^*(t_1)Z_2^{*^{-1}}(t_0)\right]^m.$$

Substituting the value of $C_1\otimes C_2$ in the general solution of $(X(t)\otimes Y(t))$ of (1.1) we get,

$$(X(t)\otimes Y(t))=(Y_1(t)\otimes Z_1(t))[M_1Y_1(t_0)\otimes M_2Z_1(t_0)]^{-1}$$
$$\{(\alpha_1\otimes\alpha_2)-(N_1\otimes N_2)(Y_1(t_1)\otimes Z_1(t_1))$$
$$\int_{t_0}^{t_1}(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}F_1(s,(X(s)\otimes Y(s)))\otimes F_2(s,(X(s)\otimes Y(s)))$$
$$(Y_2^*(s)\otimes Z_2^*(s))^{-1}\Delta s\,(Y_2^*(t_1)\otimes Z_2^*(t_1))\}$$
$$(Y_2^*(t_0)\otimes Z_2^*(t_0))^{-1}\,(Y_2^*(t)\otimes Z_2^*(t))$$

$$+(Y_1(t)\otimes Z_1(t))$$
$$\sum_{m=1}^{\infty}(-1)^m[(M_1Y_1(t_0))^{-1}(N_1Y_1(t_1))\otimes(M_2Z_1(t_0))^{-1}(N_2Z_1(t_1))]^m$$
$$[M_1Y_1(t_0)\otimes M_2Z_1(t_0)]^{-1}$$
$$\{(\alpha_1\otimes\alpha_2)-(N_1\otimes N_2)(Y_1(t_1)\otimes Z_1(t_1))$$
$$\int_{t_0}^{t_1}(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}F_1(s,(X(s)\otimes Y(s)))\otimes$$
$$F_2(s,(X(s)\otimes Y(s)))(Y_2^*(s)\otimes Z_2^*(s))^{-1}\Delta s$$
$$(Y_2^*(t_1)\otimes Z_2^*(t_1))\}\,(Y_2^*(t_0)\otimes Z_2^*(t_0))^{-1}$$
$$\left[Y_2^*(t_1)Y_2^{*^{-1}}(t_0)\otimes Z_2^*(t_1)Z_2^{*^{-1}}(t_0)\right]^m(Y_2^*(t)\otimes Z_2^*(t))$$

$$+(Y_1(t_1)\otimes Z_1(t_1))\int_{t_0}^{t}(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}$$
$$F_1(s,(X(s)\otimes Y(s)))\otimes F_2(s,(X(s)\otimes Y(s)))$$
$$(Y_2^*(s)\otimes Z_2^*(s))^{-1}\Delta s(Y_2^*(t)\otimes Z_2^*(t)).$$

In order to obtain a unique solution to the two-point boundary value problem (1.1) and (1.2), we define an operator $H_1$ by

$$\left\|(X(t)\otimes Y(t))^{(i)}-(X(t)\otimes Y(t))^{(i-1)}\right\|$$
$$\le\left\|H_1(X(t)\otimes Y(t))^{(i-1)}-H_1(X(t)\otimes Y(t))^{(i-2)}\right\|$$
$$\le\|Y_1(t)\otimes Z_1(t)\|\;\left\|[M_1Y_1(t_0)\otimes M_2Z_1(t_0)]^{-1}\right\|$$
$$\|N_1Y_1(t_1)\otimes N_2Z_1(t_1)\|$$

$$\int_{t_0}^{t_1}\left\|(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}\right\|\Big\|F_1\left(s,(X(s)\otimes Y(s))^{(i-1)}\right)\otimes F_2\left(s,(X(s)\otimes Y(s))^{(i-1)}\right)$$

$$-F_1\left(s,(X(s)\otimes Y(s))^{(i-2)}\right)\otimes\left(s,(X(s)\otimes Y(s))^{(i-2)}\right)\Big\|$$
$$\left\|(Y_2^*(s)\otimes Z_2^*(s))^{-1}\right\|\Delta s$$

$$\sum_{m=1}^{\infty}\left|(-1)^m\right|\;\left\|[(M_1Y_1(t_0))^{-1}(N_1Y_1(t_1))\otimes(M_2Z_1(t_0))^{-1}(N_2Z_1(t_1))]^m\right\|$$
$$\left\|[M_1Y_1(t_0)\otimes M_2Z_1(t_0)]^{-1}\right\|\|N_1Y_1(t_1)\otimes N_2Z_1(t_1)\|$$
$$\int_{t_0}^{t_1}\left\|(Y_1(s)\otimes Z_1(s))\right\|^{-1}\otimes F_2\left(s,(X(s)\otimes Y(s))^{(i-2)}\right)\Big\|$$
$$\left\|(Y_2^*(s)\otimes Z_2^*(s))^{-1}\right\|ds\left\|(Y_2^*(t_1)\otimes Z_2^*(t_1))\right\|$$

$$\left\|(Y_2^*(t_0)\otimes Z_2^*(t_0))^{-1}\right\|\left\|\left[\left(Y_2^*(t_1)Y_2^{*^{-1}}(t_0)\right)\otimes\left(Z_2^*(t_1)Z_2^{*^{-1}}(t_0)\right)\right]^m\right\|$$
$$\|(Y_2^*(t)\otimes Z_2^*(t))\|$$
$$\|(Y_1(t)\otimes Z_1(t))\|\int_{t_0}^{t}\left\|(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}\right\|$$
$$\Big\|F_1\left(s,(X(s)\otimes Y(s))^{(i-1)}\right)\otimes F_2\left(s,(X(s)\otimes Y(s))^{(i-1)}\right)-$$
$$F_1\left(s,(X(s)\otimes Y(s))^{(i-2)}\right)\otimes F_2\left(s,(X(s)\otimes Y(s))^{(i-2)}\right)\Big\|$$
$$\left\|(Y_2^*(s)\otimes Z_2^*(s))^{-1}\right\|\Delta s\left\|(Y_2^*(t)\otimes Z_2^*(t))\right\|.$$
$$\le[\beta_1(l_1k_2+l_2k_1)+\beta_2(l_1k_2+l_2k_1)+\beta_3(l_1k_2+l_2k_1)]|t_1-t_0|$$

$$\left\|(X(s)\otimes Y(s))^{(i-1)}-(X(s)\otimes Y(s))^{(i-2)}\right\|.$$
$$\le\gamma_1|t_1-t_0|\left\|(X(s)\otimes Y(s))^{(i-1)}-(X(s)\otimes Y(s))^{(i-2)}\right\|$$
$$\le\gamma_1^2|t_1-t_0|^2\left\|(X(s)\otimes Y(s))^{(i-2)}-(X(s)\otimes Y(s))^{(i-3)}\right\|$$
$$\ldots\quad\ldots\quad\ldots$$
$$\le\gamma_1^{(i-1)}|t_1-t_0|^{i-1}\left\|(X(s)\otimes Y(s))^{(1)}-(X(s)\otimes Y(s))^{(0)}\right\|$$

Where $\beta_1=\|Y_1(t_1)\otimes Z_1(t_1)\|\left\|(M_1Y_1(t_0)\otimes M_2Z_1(t_0))^{-1}\right\|$

$$\|N_1Y_1(t_1)\otimes N_2Z_1(t_1)\|$$
$$\left\|(Y_1(s)\otimes Z_1(s))^{-1}\right\|\left\|(Y_2^*(s)\otimes Z_2^*(s))^{-1}\right\|\left\|(Y_2^*(t_1)\otimes Z_2^*(t_1))\right\|$$
$$\left\|(Y_2^*(t_0)\otimes Z_2^*(t_0))^{-1}\right\|\left\|(Y_2^*(t)\otimes Z_2^*(t))\right\|$$

$$\beta_2 = \|Y_1(t) \otimes Z_1(t)\|$$

$$\sum_{m=1}^{\infty} (-1)^m \left\| \left[ (M_1 Y_1(t_0))^{-1} (N_1 Y_1(t_1)) \otimes (M_2 Z_1(t_0))^{-1} (N_2 Z_1(t_1)) \right]^m \right\|$$

$$\left\| [M_1 Y_1(t_0) \otimes M_2 Z_1(t_0)]^{-1} \right\| \left\| N_1 Y_1(t_1) \otimes N_2 Z_1(t_1) \right\|$$

$$\left\| (Y_1(s) \otimes Z_1(s))^{-1} \right\| \left\| (Y_2^*(s) \otimes Z_2^*(s))^{-1} \right\|$$

$$\left\| (Y_2^*(t_1) \otimes Z_2^*(t_1)) \right\| \left\| (Y_2^*(t_0) \otimes Z_2^*(t_0)) \right\|^{-1}$$

$$\left\| \left[ Y_2^*(t_1) Y_2^{*^{-1}}(t_0) \otimes Z_2^*(t_1) Z_2^{*^{-1}}(t_0) \right]^m \right\| \left\| (Y_2^*(t) \otimes Z_2^*(t)) \right\|.$$

$$\beta_3 = \left\| (Y_1(t) \otimes Z_1(t)) \right\| \left\| (Y_1(s) \otimes Z_1(s))^{-1} \right\| \left\| (Y_2^*(s) \otimes Z_2^*(s))^{-1} \right\| \left\| (Y_2^*(t) \otimes Z_2^*(t)) \right\|.$$

and $\gamma_1 = (l_1 k_2 + l_2 k_1)(\beta_1 + \beta_2 + \beta_3)$.

Thus if $\gamma_1 (t_1 - t_0) < 1$, $H_1$ is a contraction operator. Hence by the Banach fixed point theorem, $H_1$ has a unique fixed point and this fixed point is the unique solution of the two-point kronecker product boundary value problem (1.1) and (1.2).

**Case(ii):**Suppose $(A_0 \otimes B_0)$ is invertible. Then the system of equations (2.3) is equivalent to

$$(P \otimes Q)(C_1 \otimes C_2) + (C_1 \otimes C_2)(R \otimes S) = \omega \quad (3.3)$$

where, $P = A_0^{-1} P_0$, $Q = B_0^{-1} Q_0$, $R = A_1 P_1^{-1}$, $S = B_1 Q_1^{-1}$

$$\omega = (A_0 \otimes B_0)^{-1} \eta (P_1 \otimes Q_1)^{-1}$$

one of the most effective methods of solving the matrix equation (3.5) is the Bartles - Stewart algorithm. Key to this technique is the orthogonal reduction of (P ⊗ Q) and (R ⊗ S) to triangular form using QR - algorithm for eigen values. We now give the method of finding general solution to the system (3.3).

Let (P ⊗ Q) , (R ⊗ S) ∈ $R^{n^2 \times n^2}$ be given matrices and define the linear transformation

$\Phi : R^{n^2 \times n^2} \to R^{n^2 \times n^2}$ by

$$\Phi(C_1 \otimes C_2) = (P \otimes Q)(C_1 \otimes C_2) + (C_1 \otimes C_2)(R \otimes S) = \omega \quad (3.4)$$

If$\sigma(P) = \{\lambda_1, \lambda_2, .. \lambda_n\}$, $\rho(Q) = \{\mu_1, \mu_2, .. \mu_n\}$ then

$\sigma(P \otimes Q) = \{\lambda_i \mu_j : i, j = 1, 2..n\}$, *and if*

$\sigma(R) = \{\nu_1, \nu_2, .. \nu_n\}$

$\sigma(S) = \{\rho_1, \rho_2, .. \rho_n\}$ *then* $\sigma(R \otimes S) = \{\nu_i \rho_j : i, j = 1, 2 ... n\}$.

*Then* $(P \otimes Q)(u \otimes v)(\bar{u} \otimes \bar{v})^T$ =

$\lambda\mu (u \otimes v)(\bar{u} \otimes \bar{v})^T$ and $(u \otimes v)(\bar{u} \otimes \bar{v})^T (R \otimes S)$

$= \nu\rho\,(u \otimes v)(\bar{u} \otimes \bar{v})^T$. We get

$$(U_1^{-1} \otimes U_2^{-1})(P \otimes Q)(U_1 \otimes U_2) =$$

$$\text{Diag}\{\lambda_1, \lambda_2, \ldots \lambda_n, \mu_1, \mu_2, \ldots \mu_n\} = D_1 \otimes D_2$$

$$(V_1^{-1} \otimes V_2^{-1})(R \otimes S)(V_1 \otimes V_2)$$

$$= \text{Diag}\{\nu_1, \nu_2, \ldots \nu_n, \rho_1, \rho_2, \ldots \rho_n\} = D_3 \otimes D_4.$$

Then equation (3.4) is equivalent to

$$(U_1^{-1} \otimes U_2^{-1})(P \otimes Q)(U_1 \otimes U_2)$$
$$(U_1^{-1} \otimes U_2^{-1})(C_1 \otimes C_2)(V_1 \otimes V_2) +$$
$$(U_1^{-1} \otimes U_2^{-1})(C_1 \otimes C_2)(V_1 \otimes V_2)(V_1^{-1} \otimes V_2^{-1})$$
$$(R \otimes S)(V_1 \otimes V_2)$$
$$= (U_1^{-1} \otimes U_2^{-1}) \omega (V_1 \otimes V_2)$$

Now to solve the system, we proceed as follows :

**Step 1 :** By using similarity transformations reduce (P⊗Q), (R⊗S) to diagonal form to get $D_1 \otimes D_2 = (U_1^{-1} \otimes U_2^{-1})(P \otimes Q)(U_1 \otimes U_2)$ and $D_3 \otimes D_4 = (V_1^{-1} \otimes V_2^{-1})(R \otimes S)(V_1 \otimes V_2)$

**Step2:** Solve $(U_1 \otimes U_2)\ E = \omega\ (V_1 \otimes V_2)$ for E.

**Step 3 :** Solve the transformed system

$$(D_1 \otimes D_2)(X_1 \otimes Y_1) + (X_1 \otimes Y_1)(D_3 \otimes D_4) = E$$

*for* $(X_1 \otimes Y_1)$

**Step4:**Solvethesystem

$$(C_1 \otimes C_2)(V_1 \otimes V_2) = (U_1 \otimes U_2)(X_1 \otimes Y_1)$$

*for* $(C_1 \otimes C_2)$.

From these above steps we get the solution of the system (6.3.4) is as

$$C_1 \otimes C_2 = (U_1 \otimes U_2)(X_1 \otimes Y_1)(V_1^{-1} \otimes V_2^{-1}),$$

where $(X_{ij} \otimes Y_{ij})_1 = \dfrac{e_{ij}}{\lambda_i \mu_j + \nu_i \rho_j}$,

$E = e_{ij} = (U_1^{-1} \otimes U_2^{-1}) \omega (V_1 \otimes V_2)$. Now by substituting the general form of $C_1 \otimes C_2$ in the general solution (X(t) ⊗Y(t)) of (6.1.1), we have

$$(X(t) \otimes Y(t)) = (Y_1(t) \otimes Z_1(t))(U_1 \otimes U_2)$$
$$(X_1 \otimes Y_1)(V_1^{-1} \otimes V_2^{-1})(Y_2^*(t) \otimes Z_2^*(t))$$
$$+ (Y_1(t) \otimes Z_1(t))$$
$$\int_{t_0}^{t} (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} F_1(s, (X(s) \otimes Y(s))) \otimes F_2(s, (X(s) \otimes Y(s)))$$
$$(Y_2^*(s) \otimes Z_2^*(s))^{-1}\ \Delta s\ (Y_2^*(t) \otimes Z_2^*(t)).$$

In order to obtain a unique solution to the two-point boundary value problem (1.1) and (1.2), we define the operator $H_2$ by

$$(X\otimes Y)^{(i)} = H_2(X\otimes Y)^{(i-1)}$$
$$= (Y_1(t)\otimes Z_1(t))(U_1\otimes U_2)(X_1\otimes Y_1)$$
$$(V_1^{-1}\otimes V_2^{-1})(Y_2^*(t)\otimes Z_2^*(t))$$
$$+(Y_1(t)\otimes Z_1(t))$$
$$\int_{t_0}^{t}(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}F_1(s,(X(s)\otimes Y(s))^{[i-1]})\otimes F_2(s,(X(s)\otimes Y(s))^{[i-1]})$$

$$(Y_2^*(s)\otimes Z_2^*(s))^{-1}\ \Delta s\ (Y_2^*(t)\otimes Z_2^*(t)).$$

Then

$$\left\|(X(t)\otimes Y(t))^{(i)}-(X(t)\otimes Y(t))^{(i-1)}\right\|$$
$$\le \left\|H_2(X(t)\otimes Y(t))^{(i-1)}-H_2(X(t)\otimes Y(t))^{(i-2)}\right\|$$
$$\le \left\|Y_1(t)\otimes Z_1(t)\right\|\int_{t_0}^{t}\left\|(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}\right\|$$
$$\left\|F_1(s,(X(s)\otimes Y(s))^{[i-1]})\otimes F_1(s,(X(s)\otimes Y(s))^{[i-2]})\right.$$

$$\left\|F_2(s,(X(s)\otimes Y(s))^{[i-1]})\right\|+\left\|F_1(s,(X(s)\otimes Y(s))^{[i-2]})\right\|$$
$$\left\|F_2(s,(X(s)\otimes Y(s))^{[i-1]})\right.$$

$$F_2(s,(X(s)\otimes Y(s))^{[i-2]})\Big\|+\left\|(Y_2^*(s)\otimes Z_2^*(s))^{-1}\right\|\Delta s\ \left\|(Y_2^*(t)\otimes Z_2^*(t))\right\|.$$

Let $M = \left\|(Y_1(t)\otimes Z_1(t))\right\|\ \left\|(Y_1(s)\otimes Z_1(s))^{-1}\right\|$,

$$l_1 = \left\|F_1(s,(X(s)\otimes Y(s))^{[i-2]})\right\|$$

and $l_2 = \left\|F_2(s,(X(s)\otimes Y(s)))^{(i-1)}\right\|$

and $K_1$, $K_2$ be Lipchitz constants.

$$\le M(k_1 l_2 + k_2 l_1)\left|(t_1-t_0)\right|$$
$$\left\|(X(t)\otimes Y(t))^{(i-1)}-(X(t)\otimes Y(t))^{(i-2)}\right\|$$

and let

$$\gamma_1 = M(k_1 l_2 + k_2 l_1)$$
$$\le \gamma_1^2\left|t_1-t_0\right|^2\left\|(X(t)\otimes Y(t))^{(i-2)}-(X(t)\otimes Y(t))^{(i-3)}\right\|$$

... ... ..

$$\le \gamma_1^{(i-1)}\left|t_1-t_0\right|^{i-1}\left\|(X(t)\otimes Y(t))^{(1)}-(X(t)\otimes Y(t))^{(0)}\right\|$$

Thus $\gamma_1(t_1-t_0)<1$, $H_2$ is a contraction operator and by Banach fixed point theorem, $H_2$ has a unique fixed point. This fixed point is the unique solution of the two-point boundary value problem (1.1) & (1.2).

## REFERENCES


[1] S. Barnett, Matrix differential equations and Kronecker products, SIAM J. Applied Math, vol. 2, pp. 1-5, 1973.

[2] K.N. Murty, G.W. Howell and S. Sivasundaren, Two (multi) point nonlinear Lyapunov systems – Existence and uniqueness, J. Math. Anal. Appl., vol. 169, no. 2, pp. 505-515, 1992.

[3] K.N. Murty, B.D.C.N. Prasad and K.R. Prasad, A note on matrix Riccati functional differential equation, J. Math. Anal. Appl., vol. 132, no. 1, pp. 50-56, 1998.

[4] K.N. Murthy, V. Lakshmi,Narasimma, On Kronecker product initial value problems – Existence and uniqueness, Applied Math. Computations, 2000.

[5] H. Neudecker, A note on Kronecker matrix products and matrix equation systems, SIAM J. Applied Maths, vol. 17, pp. 603-606, 1969.