On Fourier Series Solution of MHD Falkner-Skan Equation

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Abstract: **In this paper, MHD Falkner-Skan equation has been solved through the expansion in Fourier series and the results are compared with those found in the literature. The results show that the classical expansion in Fourier series delivers a solution with very good accuracy, particularly when the number of terms in the Fourier series is finite.**

Index Terms— **Fourier series, MHD Falkner-Skan equation**

MSC 2010 Codes **—76N20, 42A16**

I. INTRODUCTION

THE analysis of flow fluid in a boundary layer adjacent to the wedge is an essential part in the area of fluid dynamics.

In their pioneering work, Falkner-Skan [1] have analyzed the two-dimensional laminar boundary layer flow passing through a fixed wedge to illustrate the applications of Prandtl's boundary layer theory . The famous Falkner-Skan equation governing MHD boundary layer flow past a stationary wedge is given by [2]:

$$\frac{d^3 f}{d\eta^3}+f\frac{d^2 f}{d\eta^2}+\beta\left[1-\left(\frac{df}{d\eta}\right)^2\right] + M\left(1-\frac{df}{d\eta}\right)=0 \quad 0\le\eta<\infty \tag{1}$$

along with the boundary conditions

$$f(0)=f'(0)=0, \quad f'(+\infty)=1 \tag{2}$$

where f is the dimensionless stream function of η-variable and β is the parameter of the stream wise pressure gradient and M is dimensionless parameter representing the transverse magnetic field, applied normal to the wedge surface. The Falkner-Skan equation constitutes a third order nonlinear two point boundary value problem and, the exploration of the essence of its analytical solutions is still valuable.

When $\beta=M=0$, the equation (1) becomes

$$\frac{\partial^3 f}{\partial\eta^3}+f\frac{\partial^2 f}{\partial\eta^2}=0,$$

i.e., the Blasius equation, which perhaps one of the most famous equations of fluid dynamics and represents the problem of an incompressible fluid that passes on a semi-infinite flat plate. Further, in the case of accelerating flows ($\beta>0$), the velocity profiles have no points of inflection, whereas in the case of decelerated flows ($\beta<0$), physically relevant solutions exist only for $-0.1988\le\beta\le 2$.

The solutions of the Falkner-Skan equation has been studied numerically first by Hartree [3], finding that for values of $\beta>0$, the values of $f''(\eta=0)$ required to satisfy the boundary conditions (2) was unique, however, for $\beta<0$ the Falkner-Skan equation has multiple solutions. Cebeci and Keller [4] have studied this equation using shooting method in which a successive values of $f''(\eta=0)$ are guessed until $f'(\eta\to+\infty)=1$.

On the other hand, Meskyn [5] solved Falkner-Skan equation through analytical approximations. Solutions with high accuracy using finite-differences can be found in [6]. Shi-Jun Liao [7] applied the homotopy analysis method to solve the Falkner-Skan equation. Recently, Rosales and Frederick [8] solved classical Graetz problem by applying Fourier series. More recently, Rosales and Valencia [9] studied the Blasius equation, obtained for $\beta=0$ using Fourier series, and gave very good results for $f''(0)$ compared with a high accuracy numerical value obtained by Boyd [10].

The objective of the present paper is to obtain the solutions magneto-hydro-dynamic (MHD) Falkner-Skan equation through classical Fourier series.

II. ANALYSIS

We postulate the following solution for $f(\eta)$:

$$f(\eta)=a_0+\sum_{i=1}^{\infty}\frac{a_i}{k_i}\cos(k_i\eta) \tag{3}$$

where a_i represents the Fourier coefficients for the velocity, the number of wavelength k_i is given by

$$k_i=\frac{\pi}{2L}(2i-1)$$

The boundary conditions (2) become,

$$\sum_{i=1}^{\infty}a_i\sin(k_iL)=-1, \quad a_0+\sum_{i=1}^{\infty}\frac{a_i}{k_i}=0 \tag{4}$$

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Replacing Eq. (3) in Eq. (1) yield to:

$$\sum_{i=1}^{\infty} a_i k_i^2 \sin(k_i\eta) - \left(a_0 + \sum_{i=1}^{\infty}\frac{a_i}{k_i}\cos(k_i\eta)\right)\sum_{i=1}^{\infty} a_i k_i \cos(k_i\eta) + \beta\left(1 - \sum_{i,j=1}^{\infty} a_i a_j \sin(k_i\eta)\sin(k_j\eta)\right) + M\left[1 + \sum_{i=1}^{\infty} a_i \sin(k_i\eta)\right] = 0 \quad (5)$$

Multiplying Eq. (5) by $\sin(k_p\eta)$ and integrating from $\eta = 0\ to\ L,$ the following system of nonlinear algebraic equations is obtained:

$$k_p^2 a_p \frac{L}{2} - \left(a_0 \sum_{i=1}^{\infty} a_i A_{ip} + \sum_{i,j=1}^{\infty} a_i a_j B_{ij}^p\right) + \beta\left(\frac{1}{k_p} - \sum_{ij=1}^{\infty} a_i a_j C_{ij}^p\right) + M\left[\frac{1}{k_p} + a_i D_{ij}^p\right] = 0 \quad (6)$$

where:

$$A_{ip} = k_i \int_0^L \cos(k_i\eta)\sin(k_p\eta)\,d\eta = \begin{bmatrix} \frac{1}{2} & i = p \\ \frac{k_i}{k_i + k_p} & i + p \text{ even} \\ \frac{-k_i}{k_i - k_p} & i + p \text{ odd} \end{bmatrix}$$

$$B_{ij}^p = \frac{k_j}{k_i}\int_0^L \cos(k_i\eta)\cos(k_j\eta)\sin(k_p\eta)\,d\eta = -\frac{k_j k_p}{2k_i}\left(\frac{1}{(k_i - k_j)^2 - k_p^2} + \frac{1}{(k_i + k_j)^2 - k_p^2}\right)$$

$$C_{ij}^p = \int_0^L \sin(k_i\eta)\sin(k_j\eta)\sin(k_p\eta)\,d\eta = -\frac{k_p}{2}\left(\frac{1}{(k_i - k_j)^2 - k_p^2} + \frac{1}{(k_i + k_j)^2 - k_p^2}\right)$$

and

$$D_{ij}^p = k_i \int_0^L \sin(k_i\eta)\sin(k_p\eta)\,d\eta = \begin{bmatrix} \frac{Lk_i}{2} & i = p \\ \frac{Lk_i}{k_i + k_p} & i + p \text{ even} \\ \frac{-Lk_i}{k_i - k_p} & i + p \text{ odd} \end{bmatrix} \quad (7)$$

If you are Finally, the boundary conditions (4) and the set of Eq. (6) with $p = 1, 2, ...N-2,$ form a set of N non-linear algebraic equations for the coefficients of the Fourier series.

III. NUMERICAL SOLUTION

The velocity field is obtained by means of numerical solution of the system (4) and (6), and this system of equations is approximated by a finite number of terms N. Further, this system of equations was resolved with the usual Newton-Raphson method. The far field condition $(\eta \to \infty)$ is imposed at $L = 7$; the no slip condition being $\eta = 0$ at the wall. The solution is found through an iterative method for coefficients of the Fourier series, defining the vector $(\vec{a})^T = [a_0, a_1, \dots, a_N]$ and $(\vec{G})^T = [G_0, G_1, \dots, G_N]$, where

$$G_0 = a_0 + \sum_{i=1}^{N}\frac{a_i}{k_i} \quad (8)$$

$$G_1 = 1 + \sum_{i=1}^{N} a_i \sin(k_i L) \quad (9)$$

$$G_p = k_{p-2}^2 a_{p-2}\frac{L}{2} - \left(a_0\sum_{i=1}^{N} a_i A_{ip-2} + \sum_{i,j=1}^{N} a_i a_j B_{ij}^{p-2}\right) + \beta\left(\frac{1}{k_{p-2}} - \sum_{i,j=1}^{N} a_i a_j C_{ij}^{p-2}\right) + M\left(\frac{1}{k_{p-2}} + \sum_{i,j=1}^{N} a_i D_{ij}^{p-2}\right),\ p = 3,\dots,N \quad (10)$$

So, the coefficients a_i of the Fourier series may be recursively determined by the following equation

$$(\vec{a})^{(n+1)} = (\vec{a})^{(n)} - J^{-1}\left((\vec{a})^{(n)}\right)\ \vec{G}\left((\vec{a})^{(n)}\right) \quad (11)$$

where J is the Jacobian matrix $J_{ij} = \frac{\partial G_j}{\partial a_j}$ and $(\vec{a})^{(n+1)}$ represents the value of $\vec{a}$ in the $n+1$ iteration. The iterations for $(\vec{a})^{(n+1)}$ from the Eq.(11) may be repeated until $\left\|(\vec{a})^{(n+1)} - (\vec{a})^{(n)}\right\| \le \varepsilon,$ for some prescribed error of tolerance ε . The elements of Jacobian matrix are given by:

$$J = \begin{bmatrix} 1 & \frac{1}{k_1} & . & . & . & . & \frac{1}{k_N} \\ 0 & \sin(k_1 L) & . & . & . & . & \sin(k_N L) \\ \sum_{i=1}^{\infty} a_i A_{i1} & \gamma_{11} & \gamma_{12} & . & . & . & \gamma_{1N-1} \\ \sum_{i=1}^{\infty} a_i A_{i2} & \gamma_{21} & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \sum_{i=1}^{\infty} a_i A_{iN} & \gamma_{N-2,1} & . & . & . & . & \gamma_{N-2,N-1} \end{bmatrix}$$

Where:

$$\gamma_{pr} = k_p^2 a_p \frac{L}{2} a_{pr} - \left(a_0 A_{rp} + \sum_{j=1}^{N} a_j B_{rj}^p + \sum_{i=1}^{N} a_i B_{ir}^p\right) - \beta\left(\sum_{i=1}^{N} a_i C_{ir}^p + \sum_{j=i}^{N} a_j C_{rj}^p\right) - M\left(\sum_{i=1}^{N} a_i D_{ir}^p + \sum_{j=i}^{N} a_j D_{rj}^p\right) \quad (12)$$

In order to verify the accuracy of the proposed method for the velocity field, we have computed our results for $f''(0) = -\sum_{i=1}^{N} a_i k_i$. A simple inspection of Eq. (6) shows that when $p >> 1$, the terms $\frac{\beta}{k_p}$ and $\frac{M}{k_p}$ appearing in it decays slowly when $p \to \infty$. This implies that it will require a finite number of terms in the expression

$f''(0) = -\sum_{i=1}^{N} a_i k_i$ for this to coverage.

IV. Results And Conclusion

In order to verify the accuracy of our present method for different values of the parameter β, L as well as different numbers of terms N considered in series, values of $f''(0)$, viz., velocity gradient at the wall, are compared with the values obtained numerically in literature. Table 1 presents the comparison of $f''(0)$ in the case of $\beta = 0$ (Blasius flow) when $M = 0$ taking $L = 7$ and it converges quickly to 0.469600 in conformity with the literature [2]. Further, Table 1 also contains finite-difference solutions obtained by Asaithambi [6] and the comparisons are found to be excellent agreement, for the range of β viz., $-0.1988 \le \beta \le 2.0$, particularly when N is large.

Table 1: Comparision of velocity gradient $f''(0)$ at the wall when $M = 0.0$, obtained by Fourier series:

	$f''(0) = -\sum_{i=1}^{N} a_i k_i$ L = 8			
	M = 0.0		M = 1.0	
β	**N = 300**	**N = 400**	**N = 300**	**N = 400**
2.0	1.68722	1.68721	1.95614	1.95923
1.0	1.23258	1.23258	1.58312	1.58356
0.5	0.92768	0.92768	1.35702	1.35923
0.0	0.46960	0.46960	1.08794	1.08845
-0.1	0.31927	0.31927	1.02615	1.02756
-0.18	0.12863	0.12863	0.97476	0.97528
-0.1988	0.00521	0.00521	0.96177	0.96256

Table 2 contains the values of $f''(0)$ for different values of β when $M = 0$ and $M = 1.0$ taking L = 8, where the effect of magnetic field ($M \neq 0$) is found to be significant on the velocity gradient on the wall.

Table 2: Values of $f''(0)$ obtained by Fourier series for different values of M and β

	$f''(0) = -\sum_{i=1}^{N} a_i k_i$ L = 7			
β	**Ref [6]**	**N=50**	**N=100**	**N=300**
2.0	1.68722	1.63025	1.65882	1.67776
1.0	1.23258	1.20416	1.21839	1.22785
0.5	0.92768	0.91348	0.92058	0.92531
0.0	0.46960	0.46960	0.46960	0.46960
-0.1	0.31927	0.32211	0.32068	0.31974
-0.18	0.12863	0.13374	0.13119	0.12948
-0.1988	0.00521	0.01095	0.00804	0.00615

Figure 1 shows variations of the velocity function $f'(\eta)$ with the similarity variable η for different values of β in the presence of magnetic field (i.e., when $M = 1.0$), obtained with the expansion in Fourier series.

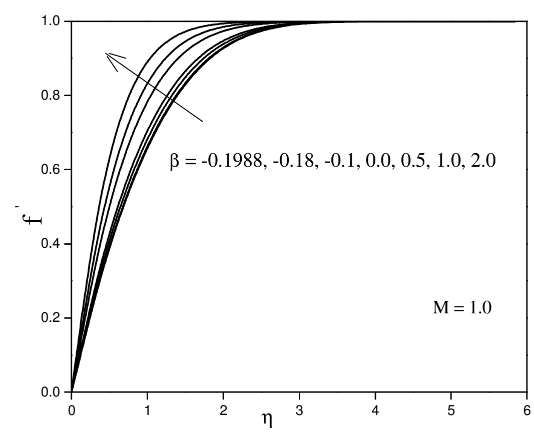

Fig. 1. Variation of velocity profiles with similarlity variable η

It is observed that velocity profiles increase with the increase of pressure gradient parameter β. Also, it appears that these profiles are seen in conformity with the boundary conditions both at the wall $(\eta = 0)$ and at the edge of the boundary layer $(\eta \to \infty)$. Consequently, the method presented in this paper is good and delivers the same values that optimized numerical method with which it is compared.

The present method can be easily extended to more complex flows, for example, boundary layer flows with heat and mass transfer.

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