A MODEL OF A THREE SPECIES ECOSYSTEM WITH MUTUALISM BETWEEN THE PREDATORS

K.S. Reddy and N.C. Pattabhiramacharyulu

Abstract: **This paper deals with the study of a three species ecosystem consisting of a prey (S_1), and two predators (S_2) and (S_3) surviving on the same prey (S_1). The mathematical model equations constitute a set of three first order non-linear simultaneous differential equations in N_1, N_2 and N_3, and the population densities of S_1, S_2 and S_3 respectively. The equilibrium points of the model are identified. Local and global stabilities are discussed using Routh-Hurwitz criteria and Lyapunov function respectively. We have derived the solution analytically by quasi linearization and graphically by Runge-Kutta method using Matlab software.**

Index Terms **Prey predator, equilibrium points, local and global stability,**

MSC 2010 Codes **—93A30, 00A71**

I. INTRODUCTION

A study on Mathematical Modelling of ecosystems was initiated by Lotka [1] and Volterra [2]. Several mathematicians and ecologists contributed to the growth of this area of knowledge have been extensively reported in the treatises of Meyer [3], Cushing [4], Paul Conlinvaux [5], Freedman [6], Kapur [7, 8].

The ecological interactions can be broadly classified as prey-predation, competition, neutralism, mutualism and so on. Mutualism between two species of organisms benefits both the species. A model on mutualism with harvesting was studied by B. Ravindra Reddy .et. al. [9].Local stability analysis for a two-species ecological mutualism model has been presented by B. Ravindra Reddy et al [10]. Recently, the stability analysis of a three species ecosystem consisting of a prey, and two neutral predators [11], and a prey, predator and super predator [12], were carried out by present authors.

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The present investigation is an analytical study of three species system comprising of two predators (S_2) and (S_3) which are in mutualism with each other and preying on the same prey (S_1).The equilibrium points of the model are identified and criteria for the local and global asymptotic stability of the states have been derived.

II. MATHEMATICAL MODEL

The model equations for a three species multi-system are given by a set of three non-linear ordinary differential equations as

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 - \alpha_{13} N_1 N_3$$
$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^2 + \alpha_{21} N_1 N_2 + \alpha_{23} N_2 N_3$$
$$\frac{dN_3}{dt} = a_3 N_3 - \alpha_{33} N_3^2 + \alpha_{31} N_1 N_3 + \alpha_{32} N_2 N_3 \qquad (2.1)$$

with the following notation.

N_i (t): Population density of the species S_i at time t ,i=1,2,3.

a_i : The natural growth rates of S_i, i = 1,2,3

α_{ii}: The rate of decrease of S_i due to its own insufficient resources i = 1,2,3.

α_{12} :The rate of decrease of the prey (S_1) due to inhibition by the predator (S_2),

α_{13} :The rate of decrease of the prey (S_1) due to inhibition by the predator (S_3),

α_{21} :The rate of increase of the predator (S_2) due to its successful attacks on the prey (S_1),

α_{23} :The rate of increase of the predator (S_2) due to its successful attacks on the predator (S_3),

α_{31} :The rate of increase of the predator (S_3) due to its successful attacks on the prey (S_1),

α_{32} :The rate of increase of the predator (S_3) due to its successful attacks on the predator (S_2),

$K_i = a_i / \alpha_{ii}$: Carrying capacities of S_i , i = 1, 2, 3.

Further the variables N_1, N_2 and N_3 are non-negative and the parameters a_i, K_i, α_{ij} , i=1,2,3, j=1,2,3, are assumed to be non-negative constants.

III. EXISTENCE OF EQUILIBRIUM POINTS

The four possible equilibrium points are

(i) $E_1(0,0,0)$ (In the absence of all the species)

(ii) $E_2(\overline{N}_1,\overline{N}_2,0)$ (In the absence of second predator)

(iii) $E_3\left(N_1^{\phi},0,N_3^{\phi}\right)$ (In the absence of first predator)

(iv) $E_4(N_1^*,N_2^*,N_3^*)$ (The interior equilibrium)

Case (i): $E_1(0,0,0)$ i.e. the population is extinct and this state always exists.

Case (ii): If $\overline{N}_1$ and $\overline{N}_2$ are the positive solutions of

$$a_1N_1-\alpha_{11}N_1^2-\alpha_{12}N_1N_2=0$$

$$a_2\mathrm{N}_2-\alpha_{22}\mathrm{N}_2^2+\alpha_{21}\mathrm{N}_1\mathrm{N}_2=0$$

Then

$$\overline{N}_1=\frac{a_1\alpha_{22}-a_2\alpha_{12}}{\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}},\ \overline{N}_2=\frac{a_2\alpha_{11}+a_1\alpha_{21}}{\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}}$$

$\overline{N}_1$ is positive provided the following inequality holds

$$a_1\alpha_{22}>a_2\alpha_{12}$$

Case (iii): If N_1^{ϕ} and N_3^{ϕ} are the positive solutions of

$$a_1N_1-\alpha_{11}N_1^2-\alpha_{13}N_1N_3=0$$

$$a_3\mathrm{N}_3-\alpha_{33}\mathrm{N}_3^2+\alpha_{31}\mathrm{N}_1\mathrm{N}_3=0$$

Then

$$N_1^{\phi}=\frac{a_1\alpha_{33}-a_3\alpha_{13}}{\alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31}}\ \ N_3^{\phi}=\frac{a_3\alpha_{11}+a_1\alpha_{31}}{\alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31}}$$

N_1^{ϕ} is positive provided the following inequality holds

$$a_1\alpha_{33}>a_3\alpha_{13}$$

Case (iv): If N , N_2^* and N_3^* are the positive solutions of

$$a_1N_1-\alpha_{11}N_1^2-\alpha_{12}N_1N_2-\alpha_{13}N_1N_3=0$$

$$a_2\mathrm{N}_2-\alpha_{22}\mathrm{N}_2^2+\alpha_{21}\mathrm{N}_1\mathrm{N}_2+\alpha_{23}\mathrm{N}_2\mathrm{N}_3=0$$

$$a_3\mathrm{N}_3-\alpha_{33}\mathrm{N}_3^2+\alpha_{31}\mathrm{N}_1\mathrm{N}_3+\alpha_{32}\mathrm{N}_2\mathrm{N}_3=0$$

Then

$$N=\frac{\rho}{D}\ ,\ N_2^*=\frac{\rho_2}{D}\ ,\ N_3^*=\frac{\rho_3}{D}$$

$$\rho_1=a_1\left(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32}\right)-a_2\left(\alpha_{12}\alpha_{33}+\alpha_{13}\alpha_{32}\right)-a_3\left(\alpha_{12}\alpha_{23}+\alpha_{13}\alpha_{22}\right)$$

$$\rho_2=a_1\left(\alpha_{21}\alpha_{33}+\alpha_{31}\alpha_{23}\right)+a_2\left(\alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31}\right)+a_3\left(\alpha_{11}\alpha_{23}-\alpha_{13}\alpha_{21}\right)$$

$$\rho_3=a_1\left(\alpha_{21}\alpha_{32}+\alpha_{22}\alpha_{31}\right)+a_2\left(\alpha_{11}\alpha_{32}-\alpha_{12}\alpha_{31}\right)+a_3\left(\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}\right)$$

$$D=\alpha_{11}\left(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32}\right)+\alpha_{12}\left(\alpha_{21}\alpha_{33}+\alpha_{31}\alpha_{23}\right)+\alpha_{13}\left(\alpha_{31}\alpha_{32}+\alpha_{31}\alpha_{22}\right)$$

For N , N_2^* and N_3^* to be positive if the following inequalities hold

$$\left\{a_2\left(\alpha_{12}\alpha_{33}+\alpha_{13}\alpha_{32}\right)+a_3\left(\alpha_{12}\alpha_{23}+\alpha_{13}\alpha_{22}\right)\right\}<a_1\left(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32}\right)$$

$\alpha_{11}\alpha_{23}>\alpha_{13}\alpha_{21}$, $\alpha_{11}\alpha_{32}>\alpha_{12}\alpha_{31}$ and $\alpha_{22}\alpha_{33}>\alpha_{23}\alpha_{32}$

IV. LOCAL STABILITY OF INTERIOR EQUILIBRIUM POINT

The variational matrix of the system (2.1) is

$$V=\begin{bmatrix}-\alpha_{11}\overline{N}_1 & -\alpha_{12}\overline{N}_1 & -\alpha_{13}\overline{N}_1\\ \alpha_{21}\overline{N}_2 & -\alpha_{22}\overline{N}_2 & \alpha_{23}\overline{N}_2\\ \alpha_{31}\overline{N}_3 & \alpha_{32}\overline{N}_3 & -\alpha_{33}\overline{N}_3\end{bmatrix}$$

Therefore the characteristic equation of interior equilibrium state is

$$\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0 \qquad (4.1)$$

where

$$b_1=\alpha_{11}N_1^*+\alpha_{22}N_2^*+\alpha_{33}N_3^*$$

$$b_2=(\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21})N_1^*N_2^*+(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})N_2^*N_3^*+(\alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31})N_1^*N_3^*$$

$$b_3=(\alpha_{11}\alpha_{22}\alpha_{33}+\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{13}\alpha_{31}\alpha_{22}+\alpha_{12}\alpha_{31}\alpha_{23}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32})N_1^*N_2^*N_3^*$$

According to Routh-Hurwitz's criteria, the necessary and sufficient conditions for local stability of co-existent points are $b_1>0, b_3>0$ and $b_3(b_1b_2-b_3)>0$. Clearly b_1 and b_3 are positive, and after some algebraic deductions, it can be verified that $b_3(b_1b_2-b_3)>0$

Hence the system is locally asymptotically stable i.e., all Eigen values $(\lambda_1,\lambda_2,\lambda_3\ say)$ are negative or negative real parts of complex numbers.

The solution of the perturbed equations is:

$$u_1=A_1e^{\lambda_1t}+B_1e^{\lambda_2t}+C_1e^{\lambda_3t} \qquad (4.2)$$

$$u_2=A_2e^{\lambda_1t}+B_2e^{\lambda_2t}+C_2e^{\lambda_3t} \qquad (4.3)$$

$$u_3=A_3e^{\lambda_1t}+B_3e^{\lambda_2t}+C_3e^{\lambda_3t} \qquad (4.4)$$

where

$$A_1=\frac{u_{10}\lambda_1^2+T_1\lambda_1+U_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},\ B_1=\frac{u_{10}\lambda_2^2+T_1\lambda_2+U_1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}$$

$$C_1=\frac{u_{10}\lambda_3^2+T_1\lambda_3+U_1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)},\ A_2=\frac{u_{20}\lambda_1^2+T_2\lambda_1+U_2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$$

$$B_2=\frac{u_{20}\lambda_2^2+T_2\lambda_2+U_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\ \ C_2=\frac{u_{20}\lambda_3^2+T_2\lambda_3+U_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$$A_3 = \frac{u_{30}\lambda_1^2 + T_3\lambda_1 + U_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},\ B_3 = \frac{u_{30}\lambda_2^2 + T_3\lambda_2 + U_3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}$$

$$C_3 = \frac{u_{30}\lambda_3^2 + T_3\lambda_3 + U_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$$T_1 = u_{10}\alpha_{22}N_2^* + u_{10}\alpha_{33}N_3^* - u_{20}\alpha_{12}N_1^* + u_{30}\alpha_{13}N_1^*$$

$$T_2 = u_{20}(\alpha_{11}N_1^* + \alpha_{33}N_3^*) + (u_{30}\alpha_{23} + u_{10}\alpha_{21})N_2^*$$

$$T_3 = u_{30}(\alpha_{11}N_1^* + \alpha_{22}N_2^*) + u_{20}\alpha_{32}N_3^* + u_{10}\alpha_{31}N_3^*$$

$$U_1 = u_{10}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})N_2^*N_3^* + u_{20}(\alpha_{13}\alpha_{32} - \alpha_{12}\alpha_{33})N_1^*N_3^* + u_{30}(\alpha_{13}\alpha_{22} - \alpha_{12}\alpha_{23})N_1^*N_2^*$$

$$U_2 = u_{20}(\alpha_{11}\alpha_{33} + \alpha_{31}\alpha_{13})N_1^*N_3^* + u_{30}(\alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21})N_1^*N_2^* + u_{10}(\alpha_{21}\alpha_{33} + \alpha_{31}\alpha_{23})N_2^*N_3^*$$

$$U_3 = u_{30}(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})N_1^*N_2^* + u_{20}(\alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31})N_1^*N_3^* + u_{10}(\alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{22})N_2^*N_3^*$$

u_{10}, u_{20} and u_{30} are initial values of u_1, u_2 and u_3 respectively

V.GLOBAL STABILITY

Theorem : The Equilibrium point $E_4(N_1^*, N_2^*, N_3^*)$ is globally asymptotically stable.

Proof: Let us consider the following Lyapunov function

$$V(N_1,N_2,N_3) = N_1 - N_1^* - N_1^*\ln\left[\frac{N_1}{N_1^*}\right] + d_1^*\left\{N_2 - N_2^* - N_2^*\ln\left[\frac{N_2}{N_2^*}\right]\right\} + d_2^*\left\{N_3 - N_3^* - N_3^*\ln\left[\frac{N_3}{N_3^*}\right]\right\} \quad (5.1)$$

Differentiating V w.r.to ‘t’ we get

$$\frac{dV}{dt} = \left(\frac{N_1-N_1^*}{N_1}\right)\frac{dN_1}{dt} + d_1^*\left(\frac{N_2-N_2^*}{N_2}\right)\frac{dN_2}{dt} + d_2^*\left(\frac{N_3-N_3^*}{N_3}\right)\frac{dN_3}{dt} \quad (5.2)$$

Choosing $d_1^* = \frac{\alpha_{12}}{\alpha_{21}}$, $d_2^* = \frac{\alpha_{13}}{\alpha_{31}}$ and with some algebraic manipulation yields

$$\frac{dV}{dt} < -\alpha_{11}(N_1 - N_1^*)^2 - \left(\frac{\alpha_{12}}{\alpha_{21}}\alpha_{22} - \frac{1}{2}\left(\frac{\alpha_{12}}{\alpha_{21}}\alpha_{23} + \alpha_{32}\frac{\alpha_{13}}{\alpha_{31}}\right)\right)(N_2 - N_2^*)^2 - \left(\frac{\alpha_{13}}{\alpha_{31}}\alpha_{33} - \frac{1}{2}\left(\frac{\alpha_{12}}{\alpha_{21}}\alpha_{23} + \alpha_{32}\frac{\alpha_{13}}{\alpha_{31}}\right)\right)(N_3 - N_3^*)^2 < 0 \quad (5.3)$$

Provided $\frac{\alpha_{12}}{\alpha_{21}}\alpha_{22} > \frac{1}{2}\left(\frac{\alpha_{12}}{\alpha_{21}}\alpha_{23} + \alpha_{32}\frac{\alpha_{13}}{\alpha_{31}}\right)$ and

$$\frac{\alpha_{13}}{\alpha_{31}}\alpha_{33} > \frac{1}{2}\left(\frac{\alpha_{12}}{\alpha_{21}}\alpha_{23} + \alpha_{32}\frac{\alpha_{13}}{\alpha_{31}}\right)$$

Therefore, $E_4(N_1^*, N_2^*, N_3^*)$ is globally asymptotically stable.

VI. NUMERICAL EXAMPLE

(1).Let a_1=8, α_{11}=0.05, α_{12}=0.6, α_{13}=0.7, a_2=1, α_{21}=0.7, α_{22}=0.17, α_{23}=0.13, a_3=1.5, α_{31}=0.15, α_{32}=0.18, α_{33}=0.4, N_1=10, N_2=10 and N_3=10

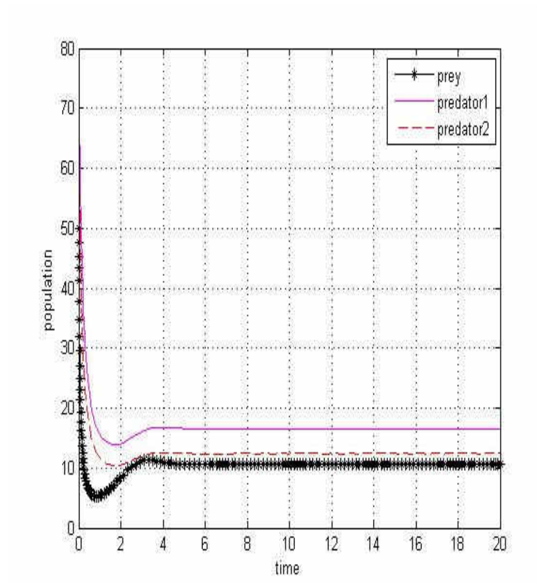

Figure (1)

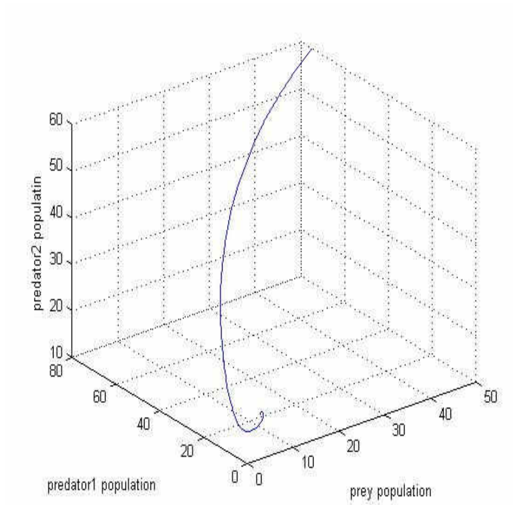

Figure (2)

(2) Let a_1=3, α_{11}=0.01, α_{12}=0.1, α_{13}=0.1, a_2=1, α_{21}=0.1, α_{22}=0.2, α_{23}=0.1, a_3=1, α_{31}=0.1, α_{32}=0.1, α_{33}=0.3, N_1=50, N_2=75 and N_3=60

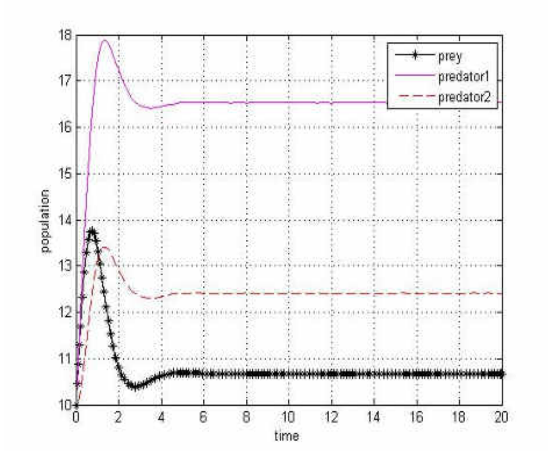

Figure (3)

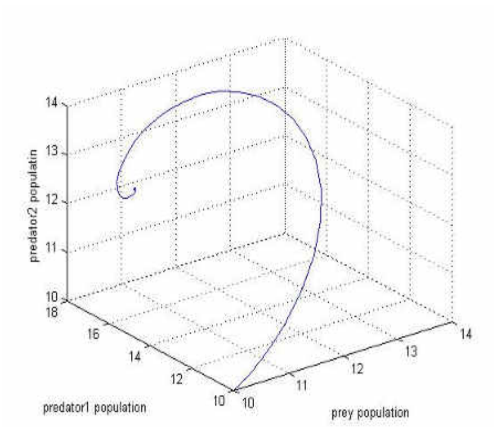

Figure (4)

Figures (1,3) represent variations in the growth rate of the populations against time.

Figures (2,4) represent Phase-space trajectories of the species.

VII. CONCLUSIONS:

A mathematical model of a syn eco-symbiotic population with a prey and two predators has been analyzed with mutualism between the predators. The interior equilibrium is proved to be existing under certain conditions and is globally asymptotically stable. The numerical simulations given in figures (1 to 4) support the stability of the system which is been analyzed by well-defined mathematical models.

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