Nonlinear Observer Design for Mechanical Systems

Sundarapandian Vaidyanathan

***Abstract*—This paper investigates the nonlinear observer design for mechanical systems, viz. nonlinear pendulum systems, undamped oscillators and Lienard systems. Explicitly, Sundarapandian's Theorem (2002) for observer design for nonlinear systems is used to solve the problem of local exponential observer design for nonlinear pendulum systems, undamped oscillators and Lienard systems. Numerical examples and simulations of nonlinear observer design for mechanical systems are shown to illustrate the results and validate the proposed observer design for the mechanical systems addressed in this paper.**

***Index Terms*—Nonlinear pendulum, Lienard system, nonlinear observers, stability, nonlinear systems.**

***MSC 2010 Codes* – 34H10, 34D06, 93C10, 93C15.**

I. Introduction

LOCAL observer design for nonlinear control systems is one of the central problems in control systems literature. The problem of designing observers for linear control systems was first introduced by Luenberger ([1], 1966) and that for nonlinear control systems was introduced by Thau ([2], 1973). Over the past three decades, significant attention has been paid in the control systems literature to the construction of observers for nonlinear control systems.

A necessary condition for the existence of a local exponential observer for the nonlinear control systems was obtained by Xia and Gao ([3], 1988). Explicitly, in [3], it was shown that an exponential observer exists for the nonlinear plant only when the linearization of the nonlinear plant is *detetable*.

On the other hand, sufficient conditions for nonlinear observers for nonlinear control systems has been obtained in the control systems literature using an impressive variety of methods. Kou, Elliott and Tarn ([4], 1975) obtained sufficient conditions for the existence of local exponential observers using Lyapunov-like method. In ([5]-[10]), suitable coordinate transformations were found under which a nonlinear control plant is transferred into a *canonical form*, where the observer design is carried out.

In [11], Kazantzis and Kravaris obtained results on nonlinear observer design using Lyapunov auxiliary theorem. In ([12]-[13]), Tsinias derived sufficient Lyapunov-like conditions for the existence of local asymptotic observers for nonlinear systems. A harmonic analysis approach was proposed by Celle *et al.* ([14], 1989) for the synthesis of nonlinear observers for nonlinear control systems.

Necessary and sufficient conditions for the existence of local exponential observers for nonlinear control systems were obtained using differential geometric techniques by Sundarapandian ([15], 2002). Krener and Kang ([16], 2003) introduced a new method for the design of observers for nonlinear control systems using backstepping.

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In this paper, we shall apply Sundarapandian's theorem (2002) for exponential observer design for nonlinear systems to construct nonlinear observers for mechanical systems like nonlinear pendulum systems, undamped oscillators and Lienard systems, which are important models of stable systems in Mechanical Engineering.

This paper is organized as follows. In Section II, we review the definition of nonlinear observers and the results of observers and observability for nonlinear systems. In Section III, we discuss the design of nonlinear observers for nonlinear pendulum systems. In Section IV, we discuss the design of nonlinear observers for Lienard systems. In Section V, we summarize the main results obtained in this paper.

II. Review of Nonlinear Observer Design for Nonlinear Systems

From the observation of certain states of the system considered as *outputs* or *indicators*, it is desired to estimate the states of the whole system as a function of time. This motivates the concept of *state observers* or *state estimators* for nonlinear control systems. Mathematically, observers for nonlinear systems are defined as follows.

Consider the nonlinear system described by

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the *state* and $y \in \mathbb{R}^p$ the *output*. It is assumed that $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are $\mathcal{C}^1$ mappings and that for some $x^\star \in \mathbb{R}^n$, the following hold:

$$f(x^\star) = 0 \quad \text{and} \quad h(x^\star) = 0 \tag{2}$$

We note that the solutions $x^\star$ of the equation

$$f(x) = 0 \tag{3}$$

are called the *equilibrium points* of the plant dynamics

$$\dot{x} = f(x) \tag{4}$$

Definition 2.1: The nonlinear system (1) is called **locally observable** at the equilibrium $x = x^\star$ over a given time interval $[0, T]$, if there exists $\epsilon > 0$ such that for any two different solutions $x(t)$ and $\bar{x}(t)$ of the plant dynamics (4) with

$$|x(t) - x^\star| < \epsilon \quad \text{and} \quad |\bar{x}(t) - x^\star| < \epsilon \quad \text{for} \quad t \in [0, T],$$

the observed functions $h \circ x$ and $h \circ \bar{x}$ are different, *i.e.* there exists some $\tau \in [0, T]$ such that

$$(h \circ x)(\tau) \neq (h \circ \bar{x})(\tau) \qquad \blacksquare$$

For the formulation of a sufficient condition for local observability of the nonlinear system (1), consider the linearization of (1) at the equilibrium $x = x^\star$ given by

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned} \tag{5}$$

where

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=x^\star} \quad \text{and} \quad C = \left[\frac{\partial h}{\partial x}\right]_{x=x^\star}. \tag{6}$$

Theorem 2.2: (Lee and Markus, [17], 1971)

If the observability matrix for the linear system (5) given by

$$\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n, then the nonlinear system (1) is locally observable at $x = x^\star$. ■

Definition 2.3: An $n \times n$ matrix H is called *Hurwitz* if all the eigenvalues of H have strictly negative real parts. ■

Next, the definition of nonlinear observers for the given nonlinear system (5) is given.

Definition 2.4: [15] A $\mathcal{C}^\infty$ dynamical system

$$\dot{z} = g(z, y), \quad (z \in \mathbb{R}^n) \tag{7}$$

is called a local **asymptotic** (respectively, **exponential**) observer for the nonlinear system (1) if the composite system (1) and (7) satisfies the following two requirements:

(i) If $z(0) = x(0)$, then $z(t) = x(t), \forall t \geq 0$.

(ii) There exists a neighbourhood V of the equilibrium $x = x^\star$ of $\mathbb{R}^n$ such that for all $z(0), x(0) \in V$, the error $e(t) = z(t) - x(t)$ decays asymptotically (respectively, exponentially) to zero as $t \to \infty$. ■

Next, we state an important theorem, which provides a simple method for constructing exponential observers for stable nonlinear systems.

Theorem 2.5: (Sundarapandian, [15], 2002)

Suppose that the nonlinear system (1) is Lyapunov stable at the equilibrium $x = x^\star$ and that there exists a matrix K such that $A - KC$ is Hurwitz. Then the dynamical system defined by

$$\dot{z} = f(z) + K[y - h(z)] \tag{8}$$

is a local exponential observer for the nonlinear system (1). ■

Remark 2.6: Suppose that the estimation error e is defined as

$$e = z - x, \tag{9}$$

where z is the state of the observer (8) given by Theorem 2.5. Then the estimation error is governed by the dynamics

$$\dot{e} = f(x + e) - f(x) - K[h(x + e) - h(x)] \tag{10}$$

where the observer gain matrix K is chosen so that $A - KC$ is Hurwitz.

Linearizing the error dynamics (10) at $x = x^\star$, we obtain the linear system

$$\dot{e} = Ee, \quad \text{where} \quad E = A - KC. \tag{11}$$

If (C, A) is observable, *i.e.* if the observability matrix $\mathcal{O}(C, A)$ has full rank, then the eigenvalues of $E = A - KC$ can be arbitrarily assigned in the complex plane. Since the linearization of the error dynamics (10) is governed by the system matrix $E = A - KC$, it follows that when (C, A) is observable, then a local exponential observer of the form (8) can be always constructed so that the transient response of the error decays quickly with any desired speed of convergence. ■

III. Nonlinear Observer Design for the Pendulum Systems

A. Pendulum Models

In this subsection, we discuss the derivation of pendulum models [18] considered for nonlinear observer design.

Consider the simple pendulum model as shown in Figure 1.

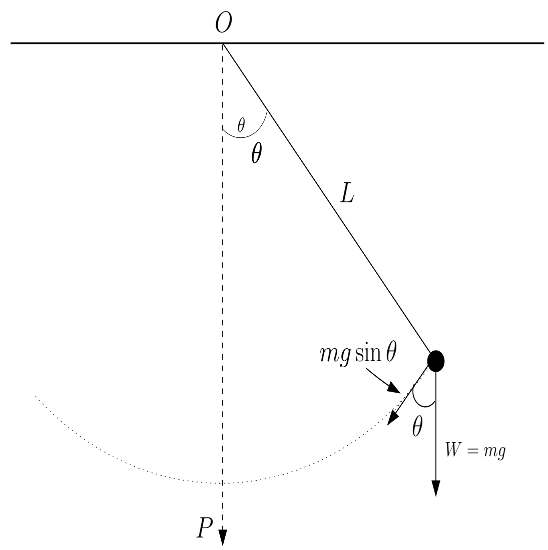

Fig. 1: Simple Pendulum Model

Let m denote the mass of the bob and L the length of the rod. Let θ denote the angle suspended by the rod and the vertical axis through the pivot point. The pendulum is free to swing in the vertical plane and the bob of the pendulum moves in a circle of radius L.

Using Newton's second law of motion, the equation of motion of the pendulum in the tangential direction can be easily obtained as

$$mL\ddot{\theta} = -mg\sin\theta - LQ(\dot{\theta}) \tag{12}$$

where $Q(\dot{\theta})$ is the damping force.

Using the phase variables $x_1 = \theta, x_2 = \dot{\theta}$, the pendulum equation (12) can be written in state-space form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{L}\sin x_1 - \tfrac{1}{m}Q(x_2) \end{aligned} \tag{13}$$

We also suppose that the angular displacement θ is available for measurement. Thus, we consider the output y as

$$y = x_1 \tag{14}$$

In this section, we shall use Sundarapandian's result (Theorem 2.5) for solving the nonlinear observer design problem for the pendulum model described in (13)-(14) for the following cases:

(a) No damping, *i.e.* $Q(x_2) = 0$.
(b) Linear damping, *i.e.* $Q(x_2) = kx_2$.
(c) Quadratic damping, *i.e.* $Q(x_2) = kx_2|x_2|$.

[Here, $k > 0$ is the damping constant.]

B. Observer Design for Pendulums with No Damping

Here, we consider the pendulum model described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L}\sin x_1 \\ y &= x_1 \end{aligned} \tag{15}$$

The system dynamics in (15) has a Lyapunov stable equilibrium at $x = 0$. The state orbits of this dynamics around $x = 0$ are shown in Figure 2 (For simulation, we take $L = 2g$.)

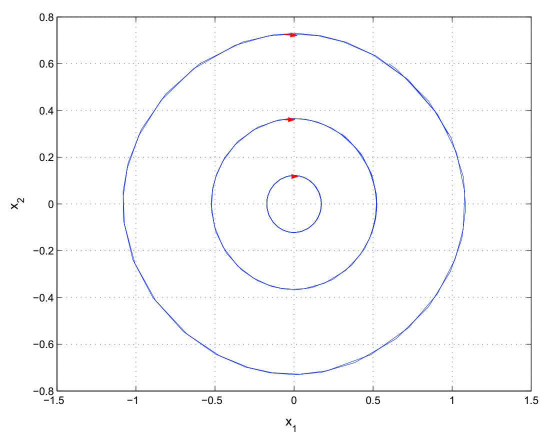

Fig. 2: State Orbits of Pendulum with No Damping

The linearization matrices for the system (15) at $x = 0$ are given by

$$C = [1 \quad 0] \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$$

The observability matrix for this system is

$$\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has full rank. Thus, by Theorem 2.5, we obtain the following result for the pendulum system with no damping.

Theorem 3.1: The pendulum system with no damping described by (15) has a local exponential observer given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{g}{L}\sin z_1 \end{bmatrix} + K\,[y - z_1] \tag{16}$$

where K is an observer gain matrix chosen such that $A - KC$ is Hurwitz. Since (C, A) is observable, a gain matrix K can be found such that the error matrix $E = A - KC$ has arbitrarily assigned set of eigenvalues with negative real parts. ■

Example 3.2: Consider the pendulum model (15) with $L = 2g$. In this case, the plant equations simplify to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.5\sin x_1 \\ y &= x_1 \end{aligned} \tag{17}$$

The system linearization matrices at $x = 0$ are

$$C = [1 \quad 0] \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix}$$

Note that the pair (C, A) is observable.

Using the Ackermann's formula for the observer gain matrix [19], we can choose K so that the error matrix $K = A - KC$ has the eigenvalues $\{-4, -4\}$.

A simple calculation using MATLAB yields

$$K = \begin{bmatrix} 8.0 \\ 15.5 \end{bmatrix}.$$

By Theorem 3.1, a local exponential observer for the pendulum system (17) near the equilibrium $(x_1, x_2) = (0, 0)$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -0.5\sin z_1 \end{bmatrix} + \begin{bmatrix} 8.0 \\ 15.5 \end{bmatrix}[y - z_1]. \tag{18}$$

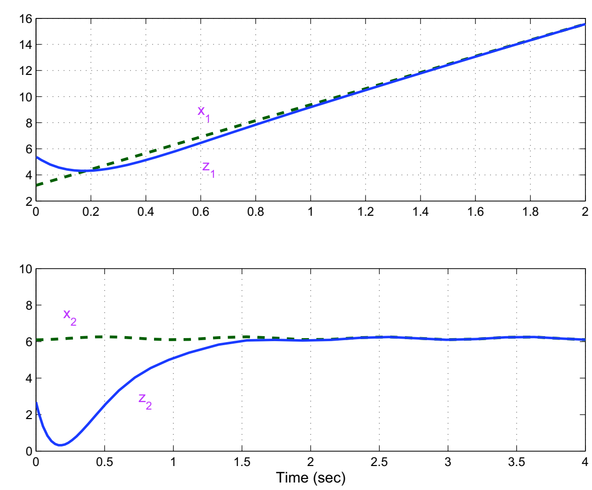

Fig. 3: Exponential Observer for Pendulum with No Damping

Figure 3 depicts the exponential convergence of the observer states z_1 and z_2 of the system (18) to the states x_1 and x_2 of the pendulum plant (17).

For simulation, we have taken the initial conditions as

$$x(0) = \begin{bmatrix} 3.2 \\ 6.1 \end{bmatrix} \quad \text{and} \quad y(0) = \begin{bmatrix} 5.4 \\ 2.7 \end{bmatrix}. \quad ■$$

C. Observer Design for Pendulums with Linear Damping

Here, we consider the pendulum model described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L}\sin x_1 - \frac{k}{m}x_2 \\ y &= x_1 \end{aligned} \tag{19}$$

The linearization matrices for the system (19) at $x = 0$ are given by

$$C = [1 \quad 0] \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{k}{m} \end{bmatrix}$$

The characteristic equation of the matrix A is given by

$$\lambda^2 + \frac{k}{m}\lambda + \frac{g}{L} = 0$$

Since all the coefficients in the above quadratic equation are positive, by Routh's criterion, it is immediate that the eigenvalues of A have strictly negative real parts.

Since A is Hurwitz, we conclude by Lyapunov's first theorem that the origin is an exponentially stable equilibrium for the plant dynamics in (19).

The observability matrix for this system is

$$\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has full rank. Thus, by Theorem 2.5, we obtain the following result for the pendulum system with linear damping.

Theorem 3.3: The pendulum system with linear damping described by (19) has a local exponential observer given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{g}{L}\sin z_1 - \frac{k}{m} z_2 \end{bmatrix} + K\,[\, y - z_1\,] \qquad (20)$$

where K is an observer gain matrix chosen such that $A - KC$ is Hurwitz. Since (C, A) is observable, a gain matrix K can be found such that the error matrix $E = A - KC$ has arbitrarily assigned set of eigenvalues with negative real parts. ■

Example 3.4: Consider the pendulum system (19) with

$$L = 2g, \quad k = 0.2 \quad \text{and} \quad m = 1.$$

In this case, the pendulum system (19) simplifies to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.5\sin x_1 - 0.2x_2 \\ y &= x_1 \end{aligned} \qquad (21)$$

The plant dynamics in (21) is exponentially stable at $x = 0$. The state portrait of (21) is shown in Figure 4.

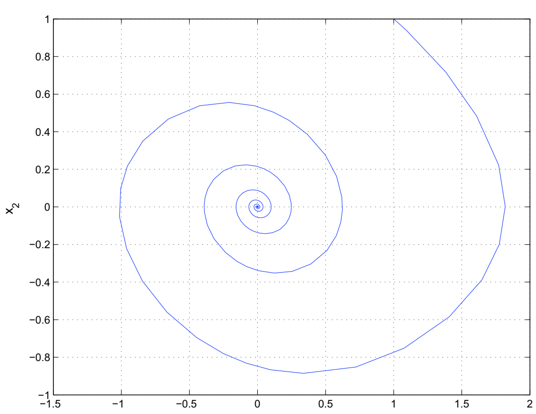

Fig. 4: State Orbits of Pendulum with Linear Damping

The system linearization matrices are

$$C = [1 \quad 0] \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.2 \end{bmatrix}$$

Note that the pair (C, A) is observable.

Using the Ackermann formula for the observer gain matrix [19], we can choose K so that the error matrix $E = A - KC$ has the eigenvalues $\{-4, -4\}$.

A simple calculation using MATLAB yields

$$K = \begin{bmatrix} 7.80 \\ 13.94 \end{bmatrix}$$

By Theorem 3.3, a local exponential observer for the pendulum system (21) near the equilibrium $(x_1, x_2) = (0, 0)$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -0.5\sin z_1 - 0.2z_2 \end{bmatrix} + K\,[y - z_1]\,. \qquad (22)$$

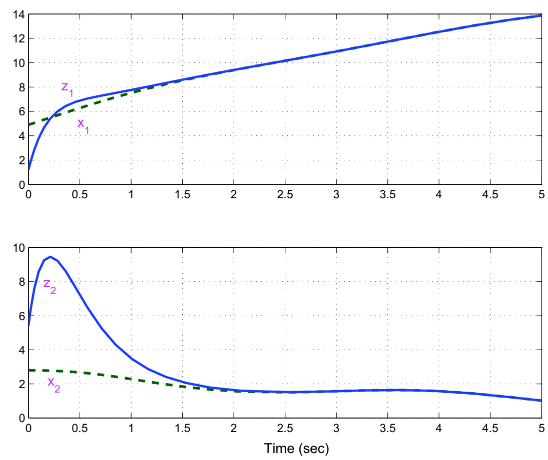

Fig. 5: Exponential Observer for Pendulum with Linear Damping

Figure 5 depicts the exponential convergence of the observer states z_1 and z_2 of the system (22) to the states x_1 and x_2 of the pendulum plant (21).

For simulation, we have taken the initial conditions as

$$x(0) = \begin{bmatrix} 4.9 \\ 2.8 \end{bmatrix} \quad \text{and} \quad y(0) = \begin{bmatrix} 1.2 \\ 5.4 \end{bmatrix}. \quad ■$$

D. Observer Design for Pendulums with Quadratic Damping

Here, we consider the pendulum model described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L}\sin x_1 - \frac{k}{m} x_2 |x_2| \\ y &= x_1 \end{aligned} \qquad (23)$$

Clearly, the pendulum dynamics in (23) has an asymptotically stable equilibrium at $x = 0$.

The linearization matrices for the system (23) at $x = 0$ are given by

$$C = [1 \quad 0] \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$$

The observability matrix for this system is

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has full rank. Thus, by Theorem 2.5, we obtain the following result for the pendulum system with linear damping.

Theorem 3.5: The pendulum system with linear damping described by (23) has a local exponential observer given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{g}{L}\sin z_1 - \frac{k}{m} z_2|z_2| \end{bmatrix} + K\left[y - z_1\right] \quad (24)$$

where K is an observer gain matrix chosen such that $A - KC$ is Hurwitz. Since (C, A) is observable, a gain matrix K can be found such that the error matrix $E = A - KC$ has arbitrarily assigned set of eigenvalues with negative real parts. ■

Example 3.6: Consider the pendulum system (23) with

$$L = 2g, \quad k = 0.2 \text{ and } m = 1.$$

In this case, the pendulum system (19) simplifies to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.5\sin x_1 - 0.2x_2|x_2| \\ y &= x_1 \end{aligned} \quad (25)$$

The plant dynamics in (25) is asymptotically stable at $x = 0$. The state portrait of (25) is shown in Figure 6.

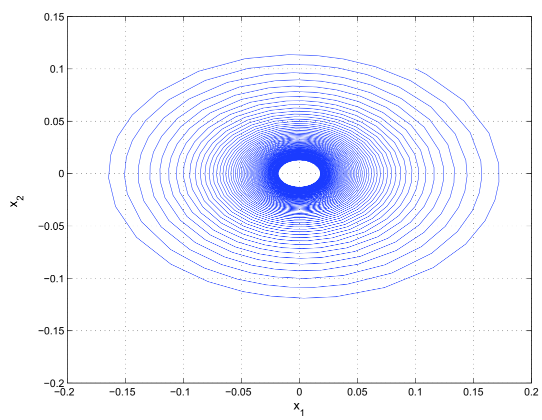

Fig. 6: State Orbits of Pendulum with Quadratic Damping

The system linearization matrices are

$$C = [\,1 \quad 0\,] \text{ and } A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix}$$

Note that the pair (C, A) is observable.

Using the Ackermann formula for the observer gain matrix [19], we can choose K so that the error matrix $E = A - KC$ has the eigenvalues $\{-4, -4\}$.

A simple calculation using MATLAB yields

$$K = \begin{bmatrix} 8.0 \\ 15.5 \end{bmatrix}$$

By Theorem 3.5, a local exponential observer for the pendulum system (25) near the equilibrium $(x_1, x_2) = (0, 0)$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -0.5\sin z_1 - 0.2z_2|z_2| \end{bmatrix} + K\left[y - z_1\right]. \quad (26)$$

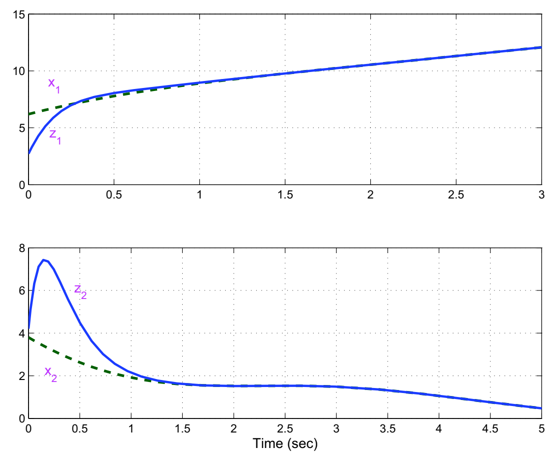

Fig. 7: Exponential Observer for Pendulum with Quadratic Damping

Figure 7 depicts the exponential convergence of the observer states z_1 and z_2 of the system (26) to the states x_1 and x_2 of the pendulum plant (25).

For simulation, we have taken the initial conditions as

$$x(0) = \begin{bmatrix} 6.2 \\ 3.8 \end{bmatrix} \text{ and } y(0) = \begin{bmatrix} 2.7 \\ 4.2 \end{bmatrix}. \ \blacksquare$$

IV. Nonlinear Observer Design for the Lienard Systems

A. System Description

In this section, we discuss the model and stability result for the Lienard equation [20], which is a classical example of an asymptotically stable system in Mechanical Engineering.

The Lienard equation is described by the second-order differential equation

$$\ddot{u} + \alpha(u)\dot{u} + \beta(u) = 0 \quad (27)$$

where u is the displacement of a moving object. Here, $\alpha(u)\dot{u}$ is a frictional force that is linear in velocity and $\beta(u)$ is the restoring force.

Throughout this section, we shall assume that the functions α, β are $\mathcal{C}^1$ on $-\infty < u < \infty$ and that the functions α, β satisfy the following two assumptions:

(H1) $\alpha(u) > 0$ for $u \neq 0$.
(H2) $u\beta(u) > 0$ for $u \neq 0$.

For our analysis and discussion, it is convenient to express the second-order differential equation (27) as a system of two first-order differential equations. This is carried out by defining the phase variables

$$\begin{aligned} x_1 &= u \\ x_2 &= \dot{u} \end{aligned} \quad (28)$$

Using the phase variables in Eq. (28), we can express the Lienard equation (27) in the system form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha(x_1)x_2 - \beta(x_1) \end{aligned} \tag{29}$$

Next, we establish a theorem on the stability of the Lienard system (29). This is a known result in Lyapunov stability theory [20].

Theorem 4.1: The Lienard system (29) has an asymptotically stable equilibrium at $x = 0$.

Proof: We consider the energy function as a candidate Lyapunov function, *viz.*

$$V(x_1, x_2) = \int_0^{x_1} \beta(\tau)\, d\tau + \frac{1}{2} x_2^2 \tag{30}$$

We shall establish the asymptotic stability of the equilibrium $x = 0$ by showing that V is a Lyapunov function for the system (29).

First, we note that V is a positive definite function on $\mathbb{R}^2$.

Next, differentiating V along the state trajectories of (29), we obtain

$$\dot{V}(x) = -\alpha(x_1)x_2^2 \leq 0 \tag{31}$$

which shows that $\dot{V}$ is a negative semi-definite function on $\mathbb{R}^2$.

Thus, by Lyapunov stability theory [21], it follows that $x = 0$ is a Lyapunov stable equilibrium of the Lienard system (29).

Next, by LaSalle's invariance principle [21], we know that the solutions of the Lienard system (29) approach asymptotically to the largest invariant set S contained in the set

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 \ : \ \dot{V}(x_1, x_2) = 0 \right\}.$$

Note that $\dot{V} = 0$ if either $\alpha(x_1) = 0$ or $x_2 = 0$.

(A) Suppose that $\alpha(x_1) = 0$. By assumption (H1), it is immediate that $x_1 = 0$. If $x_1(t) \equiv 0$, then $\dot{x}_1(t) \equiv 0$ implying that $x_2(t) \equiv 0$. Thus, $(x_1(t), x_2(t)) \equiv$. Hence, Case (A) corresponds to the trivial solution at the origin.

(B) Suppose that $x_2 = 0$. If $x_2(t) \equiv 0$, then $\dot{x}_2 \equiv 0$. Since

$$\dot{x}_2 = -\alpha(x_1)x_2 - \beta(x_1)$$

which yields $\beta(x_1) \equiv 0$. Hence, it is immediate that $x_1(t) \equiv 0$.

Thus, $(x_1(t), x_2(t)) \equiv$. Hence, Case (B) corresponds to the trivial solution at the origin.

Combining Cases (A) and (B), we conclude that $S = \{(0, 0)\}$.

Thus, by LaSalle's Invariance Principle, all solutions of the Lienard system (29) approach asymptotically the set S or equivalently that the equilibrium $x = 0$ of the Lienard system (29) is locally asymptotically stable.

Hence, we have shown that the Lienard system (29) is asymptotically stable at the equilibrium $x = 0$. ■

Example 4.2: Consider the second-order differential equation described by

$$\ddot{u} + a\dot{u} + u^3 = 0, \quad (a > 0) \tag{32}$$

Comparing (32) with the Lienard equation (27), we get

$$\alpha(u) = a \ \text{ and } \ \beta(u) = u^3 \tag{33}$$

Clearly, $\alpha(u) = a > 0$ and $u\beta(u) = u^4 > 0$ for $u \neq 0$.

Thus, it is immediate that (32) is indeed a Lienard's equation. Next, we express this as a system by defining the state variables as

$$x_1 = u \ \text{ and } \ x_2 = \dot{u} \tag{34}$$

Hence, we obtain the Lienard system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - x_1^3 \end{aligned} \tag{35}$$

By Theorem 4.1, it follows that $x = 0$ is an asymptotically stable equilibrium of the system (35).

The state orbits of the Van der Pol system (35) are depicted in Figure 8. (For numerical simulations, we take $a = 2$.)

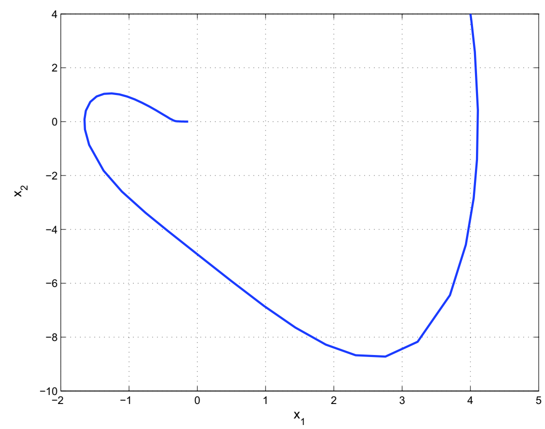

Fig. 8: State Orbits of the Lienard System (35)

From Figure 8, it is evident that the system (35) is asymptotically stable at $x = 0$. ■

Example 4.3: (*Van der Vol's Equation*)

Consider the Van der Pol's equation given by

$$\ddot{u} + \epsilon(1 - u^2)\dot{u} + u = 0. \quad (\epsilon > 0) \tag{36}$$

Van der Pol's equation was the fruitful result of the Dutch electrical engineer, Balthazar Van der Pol during the 1920s and 1930s.

Comparing (36) with the Lienard equation (27), we get

$$\alpha(u) = \epsilon(1 - u^2) \ \text{ and } \ \beta(u) = u \tag{37}$$

Clearly,

$$\alpha(u) = \epsilon(1 - u^2) > 0 \ \text{ for } \ |u| < 1$$

and

$$u\beta(u) = u^2 > 0 \ \text{ for } \ u \neq 0.$$

Thus, it is immediate that (36) is indeed a Lienard's equation. Next, we express this as a system by defining the state variables as

$$x_1 = u \ \text{ and } \ x_2 = \dot{u} \tag{38}$$

Hence, we obtain the Lienard system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \epsilon(1 - x_1^2)x_2 \end{aligned} \tag{39}$$

By Theorem 4.1, it follows that $x = 0$ is an asymptotically stable equilibrium of the system (39).

The state orbits of the Lienard system (39) are depicted in Figure 9. (For numerical simulations, we take $\epsilon = 1$.)

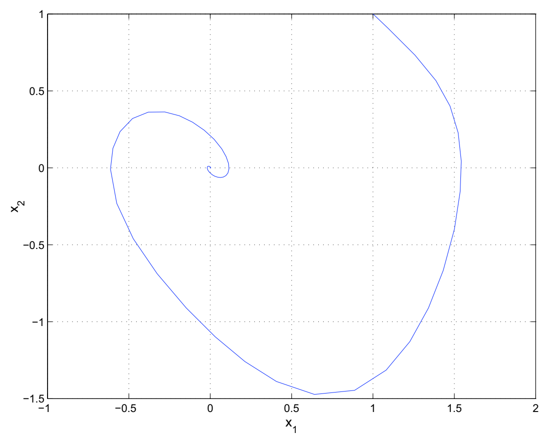

Fig. 9: State Orbits of the Van der Pol System (39)

This can be proved rigorously by applying Lyapunov stability theory and LaSalle's invariance principle.

Consider the quadratic Lyapunov function

$$V(x_1, x_2) = x_1^2 + x_2^2$$

which is clearly positive definite on $\mathbb{R}^2$.

Differentiating V along the trajectories of the Vanderpol system (39), we get

$$\dot{V} = -\epsilon(1 - x_1^2)x_2^2 \le 0 \quad \text{for } |x_1| < 0$$

which shows that $\dot{V}$ is a locally negative semi-definite function in $\mathbb{R}^2$.

Hence, by Lyapunov stability theory, it is immediate that the system (39) is Lyapunov stable at $x = 0$.

By LaSalle's invariance principle, we know that the trajectories of (39) approach asymptotically to the largest invariant set S contained in

$$\Omega = \{x \in \ R^2 \ : \ \dot{V} = 0\}$$

In the region $|x_1| < 1$, we know that $\dot{V} = 0$ if and only if $x_2 = 0$. If $x_2(t) \equiv 0$, then $\dot{x_2}(t) \equiv 0$.

Since

$$\dot{x_2} = -x_1 - \epsilon(1 - x_1^2)x_2,$$

it is immediate that $x_1(t) \equiv 0$.

Thus, $S = \{(0, 0)\}$. Hence, by LaSalle's invariance principle, the Van der Pol system (39) is asymptotically stable at $x = 0$. ■

B. Nonlinear Observer Design for the Lienard System

The Lienard system is described by the planar dynamics

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha(x_1)x_2 - \beta(x_1) \end{aligned} \tag{40}$$

where α, β are $\mathcal{C}^1$ functions satisfying the assumptions (**H1**) and (**H2**) stated in the previous subsection.

We suppose that the displacement u is available for measurement, *i.e.* the output function for the Lienard system (40) is given by

$$y = x_1 \tag{41}$$

Combining the state equation (40) with the output equation (41), we obtain the Lienard system as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha(x_1)x_2 - \beta(x_1) \\ y &= x_1 \end{aligned} \tag{42}$$

By Theorem 4.1, the Lienard system is asymptotically stable at the equilibrium $x = 0$.

Thus, we can apply Sundarapandian's theorem (2002) to construct nonlinear observers for the Lienard system given by (42).

Linearizing the Lienard system (42) at $x = 0$, we obtain the system matrices

$$A = \begin{bmatrix} 0 & 1 \\ -r & \star \end{bmatrix} \quad \text{and} \quad C = [\,1 \quad 0\,]$$

where $r = \dot{\beta}(0)$.

Thus, the observability matrix is obtained as

$$\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has full rank.

Thus, by Kalman's rank condition [19], the pair (C, A) is observable.

Thus, it follows that we can find an observer gain matrix K such that the eigenvalues of the error matrix $E = A - KC$ is Hurwitz.

Hence, by Theorem 2.5 (Sundarapandian, 2002), we obtain the following result.

Theorem 4.4: A local exponential observer for the Lienard system (42) is described by the dynamics

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\alpha(z_1)z_2 - \beta(z_1) \end{bmatrix} + K\,[\,y - z_1\,] \tag{43}$$

where K is an observer gain matrix chosen such that $A - KC$ is Hurwitz. Since (C, A) is observable, a gain matrix K can be always found that the error matrix $E = A - KC$ has arbitrarily assigned set of eigenvalues with negative real parts. ■

Example 4.5: Here, we describe the construction of local exponential observer for the Lienard system described in Example 4.2 with $a = 2$.

Thus, we consider the Lienard system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_2 - x_1^3 \\ y &= x_1 \end{aligned} \quad (44)$$

The nonlinear system (44) has the linearization pair

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad C = [1 \quad 0].$$

Clearly, the pair (C, A) is observable.

Using the Ackermann formula for the observer gain matrix [19], we can choose the gain matrix K so that the error matrix

$$E = A - KC$$

has the stable eigenvalues $\{-4, -4\}$.

A simple calculation using MATLAB yields

$$K = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

By Theorem 4.4, a local exponential observer for the Lienard system (44) near the equilibrium $x = 0$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -2z_2 - z_1^3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} [y - z_1] \quad (45)$$

For simulation, we take the initial conditions as

$$x(0) = \begin{bmatrix} 3.0 \\ 2.5 \end{bmatrix} \quad \text{and} \quad z(0) = \begin{bmatrix} 5.2 \\ 1.4 \end{bmatrix}.$$

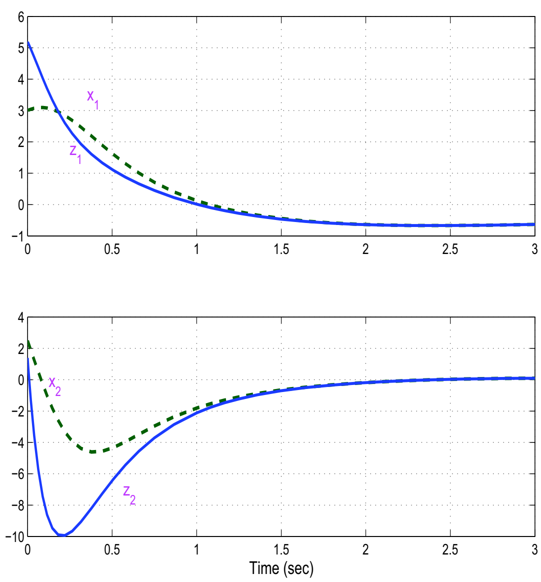

Fig. 10: Observer for the Lienard System (44)

Figure 10 depicts the exponential convergence of the states of the observer system (45) to the the states of the Lienard system (44). ■

Example 4.6: Here, we describe the construction of local exponential observer for the Van der Pol system described in Example 4.3 with $\epsilon = 1$.

Thus, we consider the Lienard system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - (1 - x_1^2)x_2 \\ y &= x_1 \end{aligned} \quad (46)$$

The nonlinear system (46) has the linearization pair

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad C = [1 \quad 0].$$

Clearly, the pair (C, A) is observable.

Using the Ackermann formula for the observer gain matrix [19], we can choose the gain matrix K so that the error matrix

$$E = A - KC$$

has the stable eigenvalues $\{-4, -4\}$.

A simple calculation using MATLAB yields

$$K = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

By Theorem 4.4, a local exponential observer for the Van der Pol system (46) near the equilibrium $x = 0$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -z_1 - (1 - z_1^2)z_2 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \end{bmatrix} [y - z_1] \quad (47)$$

For simulation, we take the initial conditions as

$$x(0) = \begin{bmatrix} 1.2 \\ 0.4 \end{bmatrix} \quad \text{and} \quad z(0) = \begin{bmatrix} 2.6 \\ 1.8 \end{bmatrix}.$$

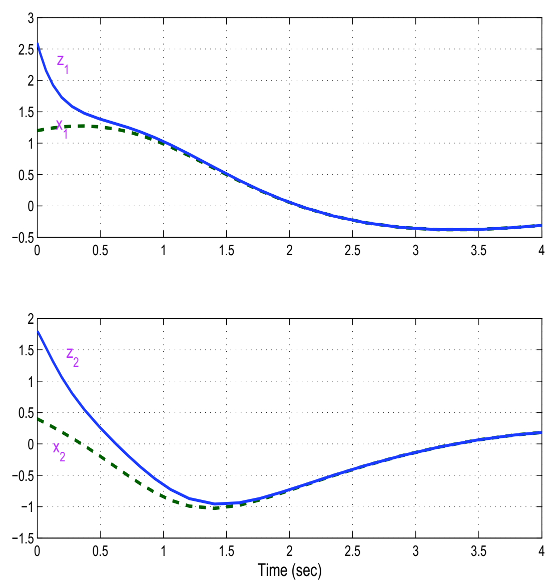

Fig. 11: Observer for the Van der Pol System (46)

Figure 11 depicts the exponential convergence of the states of the observer system (47) to the the states of the Lienard system (46). ■

V. Conclusion

In this paper, we described an overview of recent results on the nonlinear observer design and observability of nonlinear dynamical systems. As applications of nonlinear observer design (Sundarapandian, 2002), we discussed the construction of local exponential observers for classical mechanical systems, viz. pendulum systems and Lienard systems. Pendulum systems are classical examples of stable nonlinear systems. In this paper, we described the construction of local exponential observers for pendulum systems for three important cases, viz. (a) no damping, (b) linear damping and (c) quadratic damping. Numerical simulations have been shown in detail to illustrate the construction of local exponential observers for the pendulum systems for all three cases of damping. Next, we established a stability result for the Lienard system using the concept of energy function and LaSalle's Invariance Principle. Explicitly, we showed that the Lienard system has an asymptotically stable equilibrium at the origin. Next, we applied Sundarapandian's theorem (2002) on nonlinear observer design to construct local exponential observers for Lienard systems. Numerical examples have been worked out in detail for the construction of local exponential observers for the Lienard systems.

References

[1] D. G. Luenberger, "Observers for multivariable linear systems," *IEEE Transactions on Automatic Control,* vol. 2, pp. 190–197, 1966.

[2] F. E. Thau, "Observing the states of nonlinear dynamical systems," *Internat. J. Control,* vol. 18, pp. 471–479, 1973.

[3] X. H. Xia and W. B. Gao, "On exponential observers for nonlinear systems," *Systems and Control Letters,* vol. 11, pp. 319-325, 1988.

[4] S. R. Kou, D. L. Elliott and T. J. Tarn, "Exponential observers for nonlinear time-varying systems," *Internat. J. Control,* vol. 29, pp. 204–216, 1975.

[5] D. Bestla and M. Zeitz, "Canonical form observer design for nonlinear time-varying systems," *Internat. J. Control,* vol. 38, pp. 419–431, 1983.

[6] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Systems and Control Letters,* vol. 3, pp. 47–52, 1983.

[7] A. J. Krener and W. Respondek, "Nonlinear observers with linearizable error dynamics," *SIAM J. Control and Optimization,* vol. 23, pp. 197–216, 1985.

[8] X. H. Xia and W. B. Gao, "Nonlinear observer design by canonical form," *Internat. J. Control,* vol. 47, pp. 1081-1100, 1988.

[9] J. P. Gauthier, H. Hammouri and I. Kupka, "Observers for nonlinear systems," *Proc. of the 30th IEEE Conference on Decision and Control,* vol. 2, pp. 1483-1489, 1991.

[10] J. P. Gauthier, H. Hammouri and S. Othman, "A simple observer for nonlinear systems – Applications to bioreactors," *IEEE Transctions on Automatic Control,* vol. 37, pp. 875–880, 1992.

[11] N. Kazantzis and C. Kravarais, "Nonlinear observer design using Lyapunov auxiliary theorem," *Systems and Control Letters,* vol. 34, pp. 241–247, 1998.

[12] J. Tsinias, "Observer design for nonlinear systems," *Systems and Control Letters,* vol. 13, pp. 135–142, 1989.

[13] J. Tsinias, "Further results on the Observer problem," *Systems and Control Letters,* vol. 14, pp. 411–418, 1990.

[14] F. Celle, J. P. Gauthier, D. Kazakos and G. Salle, "Synthesis of nonlinear observers: A harmonic analysis approach," *Math. Systems Theory,* vol. 22, pp. 291–322, 1989.

[15] V. Sundarapandian, "Local observer design for nonlinear systems," *Mathematical and Computer Modelling,* vol. 35, pp. 25–36, 2002.

[16] A. J.Krener and W. Kang, "Locally convergent nonlinear observers," *SIAM J. Control Optimization,* vol. 42, pp. 155–177, 2003.

[17] E. B. Lee and L. Markus, *Foundations of Optimal Control Theory,* New York: Wiley, 1971.

[18] H. K. Khalil, *Nonlinear Systems,* 3rd edition, New Jersey: Prentice Hall, 1992.

[19] I. J. Nagrath and M. Gopal, *Control Systems Engineering,* 3rd edition, New Delhi: New Age International Ltd., 2002.

[20] F. Brauer and J. A. Nohel, *Qualitative Theory of Differential Equations,* New York: Dover Publications, 1969.

[21] W. Hahn, *The Stability of Motion,* Berlin: Springer, 1967.