Complementary Tree Domination Number and Chromatic Number of Graphs

S. Muthammai, P. Vidhya

***Abstract*— For any graph G = (V, E) a subset D ⊆ V is a dominating set if every vertex in V – D is adjacent to at least one vertex in D. A dominating set D is said to be a complementary tree dominating set if the induced subgraph <V – D> is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number and is denoted by $\gamma_{ctd}(G)$. In this paper, we find an upper bound for the sum of the complementary tree domination number and chromatic number and characterize the corresponding extremal graphs.**

***Index Terms*—domination number, complementary tree domination**

***MSC 2010 Codes* — 05C69**

I. Introduction

By a graph G = (V, E) we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretical terms, we refer Harary [1] and for terms related to domination we refer Haynes et al. [2].

A subset D of V is said to be a dominating set in G if every vertex in V – D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. The concept of complementary tree domination was introduced by Muthammai, Bhanumathi and Vidhya [3]. A dominating set D is called a complementary tree dominating set if the induced subgraph <V – D> is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G, denoted by $\gamma_{ctd}(G)$ and such a set D is called a γ_{ctd} set. The minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour is the chromatic number $\chi(G)$.

In this paper, we obtain sharp upper bound for the sum of the complementary tree domination number and chromatic number and characterize the corresponding extremal graphs. We use the following previous results.

Theorem 1.1 [1]

For any connected graph G, $\chi(G) \leq \Delta(G) + 1$.

Theorem 1.2 [3]

For any connected graph G with $p \geq 2$, $\gamma_{ctd}(G) \leq p - 1$.

Theorem 1.3 [3]

Let G be a connected graph with $p \geq 2$. $\gamma_{ctd}(G) = p - 1$ if and only if G is a star on p vertices.

Theorem 1.4 [3]

Let G be a connected graph containing a cycle. Then $\gamma_{ctd}(G) = p - 2$ if and only if G is isomorphic to one of the following graphs. C_p, K_p or G is the graph obtained by attaching pendant edges at atleast one of the vertices of a complete graph.

Theorem 1.5

Let T be a tree with p vertices which is not a star. Then $\gamma_{ctd}(T) = p - 2$ if and only if T is a path or T is obtained by attaching pendant edges at atleast one of the end vertices.

II. Main Results

In the following, we give an upper bound for the sum of the complementary tree domination number and chromatic number.

Theorem 2.1

For any connected graph G, $\gamma_{ctd}(G) + \chi(G) \leq 2p - 1$, ($p \geq 2$). The equality holds if and only if $G \cong K_2$.

Proof

$$\gamma_{ctd}(G) + \chi(G) \leq p - 1 + \Delta(G) + 1 \leq p + p - 1 = 2p - 1$$

If $\gamma_{ctd}(G) + \chi(G) = 2p - 1$, then the only possible case is $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p$. $\chi(G) = p$ implies $G \cong K_p$. But for K_p ($p \geq 3$), $\gamma_{ctd}(K_p) = p - 2$.

Hence, p = 2 and $G \cong K_2$.

Theorem 2.2

For any connected graph G, $\gamma_{ctd}(G) + \chi(G) = 2p - 2$ ($p \geq 3$) if and only if $G \cong P_3$ or K_p, $p \geq 4$.

Proof

If G is P_3 or K_p, then $\gamma_{ctd}(G) + \chi(G) = 2p - 2$.
Conversely, assume $\gamma_{ctd}(G) + \chi(G) = 2p - 2$.

This is possible only if
(i) $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 1$ (or)
(ii) $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p$

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Case (i) $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 1$.
$\gamma_{ctd}(G) = p - 1$ if and only if G is a star $K_{1,p-1}$ on p vertices ($p \geq 3$). But for $K_{1,p-1}$, $\chi(K_{1,p-1}) = 2$
$\chi(G) = p - 1$ implies $p = 3$ and hence $G \cong P_3$, a path on three vertices.

Case (ii) $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p$
$\chi(G) = p$ implies that $G \cong K_p$ and $\gamma_{ctd}(G) = p - 2$.
Hence $G \cong K_p$, $p \geq 3$

From Cases (i) and (ii), $G \cong P_3$ or K_p, $p \geq 3$.

Theorem 2.3

For any connected graph G, $\gamma_{ctd}(G) + \chi(G) = 2p - 3$ ($p \geq 4$) if and only if G is a star on four vertices or G is the graph obtained by adding a pendant edge at exactly one vertex of K_{p-1}.

Proof

If G is the graph stated in the theorem,
$\gamma_{ctd}(G) + \chi(G) = 2p - 3$.
Conversely, assume $\gamma_{ctd}(G) + \chi(G) = 2p - 3$.
This is possible only if
(i) $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 2$
(ii) $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p - 1$ (or)
(iii) $\gamma_{ctd}(G) = p - 3$ and $\chi(G) = p$

Case (i) $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 2$
$\gamma_{ctd}(G) = p - 1$ if and only if G is a star $K_{1,p-1}$ on p vertices.
But $\chi(K_{1,p-1}) = 2$ and $\chi(G) = p - 2$ implies $p = 4$.
Therefore $G \cong K_{1,3}$, a star on 4 vertices.

Case (ii) $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p - 1$
$\chi(G) = p - 1$ implies G contains a clique K_{p-1} on $p - 1$ vertices.
Let x be the vertex other than the vertices of K_{p-1} and let u_1, u_2, …, u_{p-1} be the vertices of K_{p-1}. Since G is connected, v is adjacent to u_i, for some i, i = 1, 2, …, p −1. x is not adjacent to all the u_i, since otherwise G will contain a clique on p vertices.

Subcase (i) Let x be adjacent to exactly one u_i ($1 \leq i \leq p-1$), say u_1. Then $D = \{x, u_3, u_4, \ldots, u_{p-1}\}$ is a minimal ctd set.
$V - D = \{u_1, u_2\}$ and $\langle V - D\rangle \cong K_2$ and hence $\gamma_{ctd}(G) = p - 2$
i.e., G is the graph obtained by adding a pendant edge at exactly one vertex of K_{p-1}.

Subcase (ii) Let x be adjacent to more than one u_i ($1 \leq i \leq p-2$). Let k be the largest integer such that x is adjacent to u_k ($2 \leq k \leq p-2$). Then $D = \{u_2, u_3, \ldots, u_{p-2}\}$ is a minimal ctd set, $V - D = \{x, u_1, u_{p-1}\}$ and $\langle V - D\rangle \cong P_3$, a path on three vertices.
Hence $\gamma_{ctd}(G) = p - 3$ which is a contradiction, since $\gamma_{ctd}(G) = p - 2$.

Case (iii) $\gamma_{ctd}(G) = p - 3$ and $\chi(G) = p$.
$\chi(G) = p$ implies that, $G \cong K_p$. But for K_p, $\gamma_{ctd}(G) = p - 2$, which is a contradiction. Hence, no graph exists.

Therefore, by Case (i), (ii) and (iii), G is one of the following graphs. G is a star on 4 vertices (or) G is the graph obtained by adding a pendant edge at exactly one vertex of K_{p-1}.

Theorem 2.4

For any connected graph G, on p vertices,
$\gamma_{ctd}(G) + \chi(G) = 2p - 4$ ($p \geq 5$) if and only if G is one of the following graphs.
(a) G is a star on 5 vertices.
(b) G is a cycle on 4 (or) 5 vertices.
(c) G is the graph obtained by attaching exactly two pendant edges at one vertex or two vertices of K_{p-2}.
(d) G is the graph obtained by joining a new vertex to j ($2 \leq j \leq p-2$) vertices of K_{p-1}.

Proof

If G is the graph given in the theorem, then
$\gamma_{ctd}(G) + \chi(G) = 2p-4$.
Conversely, assume that $\gamma_{ctd}(G) + \chi(G) = 2p - 4$
This is possible only if
(i) $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 3$
(ii) $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p - 2$
(iii) $\gamma_{ctd}(G) = p - 3$ and $\chi(G) = p - 1$
(iv) $\gamma_{ctd}(G) = p - 4$ and $\chi(G) = p$.

Case (i) $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 3$
By Theorem 1.3, $\gamma_{ctd}(G) = p - 1$ if and only if G is the star $K_{1,p-1}$ on p vertices.
But for $K_{1,p-1}$, $\chi(K_{1,p-1}) = 2 = p - 3$ and hence $p = 5$.
Therefore $G \cong K_{1,4}$.

Case (ii) $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p - 2$
By Theorem 1.4 and Theorem 1.5, $\gamma_{ctd}(G) = p - 2$ if and only if G is one of the following graphs.
(a) C_p,
(b) K_p
(c) G is the graph obtained by attaching pendant edges at atleast one of the vertices of a complete graph.
(d) G is a path (not a star)
(e) G is obtained by attaching pendant edges at least one of the end vertices (not a star).

Subcase (i) $G \cong C_p$, a cycle on p vertices

$$\chi(C_p) = \begin{cases} 2 & \text{if p is even} \\ 3 & \text{if p is odd} \end{cases}$$

$\chi(G) = p - 2$ gives p = 4 (or) 5
Hence $G \cong C_4$ (or) C_5

Subcase (ii) $G \cong K_p$
$\chi(K_p) = p$. But $\chi(G) = p - 2$. Hence $G \not\cong K_p$.
Subcase (iii) G is the graph obtained by attaching pendant edges at atleast one of the vertices of a complete graph.

$\chi(G) = p - 2$ implies G contains a clique K_{p-2} on $p - 2$ vertices.
Let $S = \{x, y\} \subseteq V(G) - V(K_{p-2})$.
Let $V(K_{p-2}) = \{u_1, u_2, \ldots, u_{p-2}\}$.
Hence, it is clear that x and y are not adjacent and x is adjacent to some u; and y is adjacent to some u_j of K_{p-2}.

Assume, x is adjacent to u_1 and y is adjacent to u_2 ($u_1 \neq u_2$).
Then $D = \{x, y, u_3, \ldots, u_{n-2}\}$ is a minimal ctd set.
$V - D = \{u_1, u_2\}$ and $<V - D> \cong K_2$.
Hence $\gamma_{ctd}(G) = p - 2$

We can conclude that, G is the graph obtained by attaching exactly two pendant edges at the two distinct vertices of the complete graph K_{p-2}.

If both x and y are adjacent to some u_1, then $D = \{x, y, u_3, \ldots, u_{p-2}\}$ is a minimal ctd set.
$V - D = \{u_1, u_2\}$, $<V - D> \cong K_2$
Hence, $\gamma_{ctd}(G) = p - 2$.

That is, G is the graph obtained by attaching two pendant edges at exactly one vertex of K_{p-2}.

Case (iii) $\gamma_{ctd}(G) = p - 3$ and $\chi(G) = p - 1$.
$\chi(G) = p - 1$ implies G contains a clique K_{p-1} on $p - 1$ vertices.
Let $x \in V(G) - V(K_{p-1})$ and
$V(K_{p-1}) = \{u_1, u_2, \ldots, u_{p-1}\}$
Since G is connected, x is adjacent to u_i for some i, $i = 1, 2, \ldots, p - 1$.
x is not adjacent to all the u_i, since otherwise G will contain a clique on p vertices.

Subcase (i) Let x be adjacent to exactly one u_i ($1 \leq i \leq p - 1$).
Then as in subcase (i) of Theorem 2.3, $\gamma_{ctd}(G) = p - 2$, which is a contradiction. Hence $\gamma_{ctd}(G) = p - 1$.

Subcase (ii) Let x be adjacent to more than one u_i ($i \leq p - 2$)
Then as in subcase (ii) of Theorem 2.3, $\gamma_{ctd}(G) = p - 3$.
Hence, G is the graph obtained by joining a new vertex to j ($2 \leq j \leq p - 2$) vertices of K_{p-1}.

Case (iv) $\gamma_{ctd}(G) = p - 4$ and $\chi(G) = p$.
This is not possible since $G \cong K_p$.
From Cases (i), (ii), (iii), (iv) G is one of the following graphs.
(a) a star on 5 vertices
(b) a cycle on 4 (or) 5 vertices
(c) G is the graph obtained by attaching exactly two pendant edges at one vertex or two vertices of K_{p-2}.
(d) G is the graph obtained by joining a new vertex to j ($2 \leq j \leq p - 2$) vertices of K_{p-1}.

Remark 2.5.
Let G be connected graph on p vertices.
From Theorem 2.2 and Theorem 2.4, we see that if G is one of the following graphs, then $\gamma_{ctd}(G) = \chi(G)$
(i) G is a path on three vertices.
(ii) G is a cycle on four or five vertices.
(iii) G is the graph obtained by attaching exactly two pendant edges at one vertex or two vertices of K_{p-2}, $p \geq 4$, where K_{p-2} is the complete graph on $p - 2$ vertices.

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