Relation between Domination Number, Energy of Graph and Rank

M. Kamal Kumar

Abstract— **In the last 40 years, graph theory has seen an explosive growth due to its application in different field of sciences. Perhaps the fasting growing areas within graph theory are Domination theory, energy of graphs etc. Matrix representation of graph &. Rank of matrix is most familiar in graph theory**

In this paper we find few bounds which relate domination number of G, energy of G and rank of the incident matrix of the graph G, and we pose some open problems for further research. The motivation for this paper is to establish a link between these parameters.

Index Terms — **Domination Number, Energy of Graph, Rank of the incidence matrix.**

MSC 2010 Codes — **15A45, 05C50, 05C69.**

I. INTRODUCTION

Claude Berge introduced the theory of Domination in 1958. The inspiration for this concept was drawn from the classical problem of covering chessboard with minimum number of chess pieces.

The most common definition given of a dominating set is that it is a set of vertices $D \subset V$ in a graph G = (V, E) having the property that every vertex $v \in V - D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the cardinality of a smallest dominating set of G.

Eigenvalues and Eigenvectors provide insight into the geometry of the associated linear transformation. Energy of graph originated from theoretical chemistry. The energy of a graph is the sum of the absolute values of the Eigen values of its adjacency matrix. From the pioneering work of Coulson [1] there exists a continuous interest towards the general Mathematical properties of the total π electron energy ε as calculated within the framework of the Huckel Molecular Orbital (HMO) model. These efforts enabled one to get an insight into the dependence of ε on molecular structure. The properties of $\varepsilon(G)$ are discussed in detail in [4], [6], [7].

M. Kamal Kumar, Asst. Professor, CMR Institute Technology, Bangalore, India (phone: 080-5600493; fax: 080-28524630; e-mail:- kamalmvz@gmail.com).

The number of non-zero rows in the row reduced form of a matrix A is called the rank of A denoted rank (A) or $\rho(A)$. Rank of the matrix is the number of linearly independent rows or the number of linearly independent columns. A matrix always represents a linear transformation between two vector spaces. From the rank of the matrix we come to know several properties about this linear transformation. Rank of the Matrix equals the dimension of the linear manifold spanned by vectors $x_1, x_2, x_3, \dots\dots\dots, x_k$.

In this paper we find few bounds which relate domination number of G, energy of G and rank of the incident matrix of the graph G, and pose some open problems for further research. The Motivation for this paper is to establish a link between these parameters.

This is the paper motivated from the paper [10] by CLEMENS BRAND,NORBERT SEIFTER, **Eigenvalues and domination in graphs,** Mathematica Slovaca, Vol. 46 (1996), No. 1, 33–39, where the authors had found the relation between largest Eigen value of the Laplacian Matrix and Domination number.

Theorem-1

Let G be a complete graph without loops and multiple edges, $E(G)$ is the energy of graph G. $I(G)$ is the incident matrix of graph G, $\rho(G) = Rank\, I(G)$ $\gamma(G)$ is the domination number of G then $\gamma(G) = \left\lfloor \frac{E(G)}{Rank\, I(G)} \right\rfloor$

Proof

The proof can be done in two ways

a) Direct method
b) Mathematical Induction

a) Direct Method

TABLE I
COMPLETE GRAPH

G	$\gamma(G)$	$E(G)$	$\rho(G)$	$\Delta(G)$	Eigen values
K_2	1	2	2	-1	-1,1
K_3	1	4	3	2	-1, -1, 2
K_4	1	6	4	-3	-1,-1,-1,3
K_5	1	8	5	4	-1,-1,-1,-1,4

K_6	1	10	6	-5	-1,-1,-1,-1,-1,5
K_7	1	12	7	6	-1,-1,-1,-1,-1,-1,6
K_8	1	14	8	-7	-1,-1,-1,-1,-1,-1,-1,7
K_9	1	16	9	8	-1,-1,-1,-1,-1,-1,-1,-1,8
K_{10}	1	18	10	-9	-1,-1,-1,-1,-1,-1,-1,-1,-1,9
K_{11}	1	20	11	10	-1,-1,-1,-1,-1,-1,-1,-1,-1, -1,10
K_{12}	1	22	12	-11	-1,-1,-1,-1,-1,-1,-1,-1,-1, -1,-1,11
K_{13}	1	24	13	12	-1,-1,-1,-1,-1,-1,-1,-1,-1, -1,-1, -1,12
K_{14}	1	26	14	-13	-1,-1,-1,-1,-1,-1,-1,-1,-1, -1,-1, -1, -1,13
K_{15}	1	28	15	14	-1,-1,-1,-1,-1,-1,-1,-1,-1, -1,-1, -1, -1,-1,14
-	-	-	-	-	- - - - - - - - -
-	-	-	-	-	- - - - - - - - -
K_n	1	$2(n-1)$	n	$(-1)^{n-1}n$	$(n-1)1's \ \&((n-1)$

We know that $\gamma(G)=1$ when G is a complete graph

$$E(K_n)=2(n-1) \quad (1)$$

From the table we get that

$$\frac{E(G)}{Rank\ I(G)}=\frac{2(n-1)}{n}<\frac{2n}{n}<2 \quad (2)$$

$\left\lfloor \frac{E(G)}{Rank\ I(G)} \right\rfloor = \left\lfloor \frac{2(n-1)}{n} \right\rfloor = 1 = \gamma(G)$, when G is a complete graph

b) Mathematical Induction to prove that

$$\gamma(G) = \left\lfloor \frac{E(G)}{Rank\ I(G)} \right\rfloor$$

For K_2, $\gamma(G) = \left\lfloor \frac{E(G)}{Rank\ I(G)} \right\rfloor = \left\lfloor \frac{2}{2} \right\rfloor = 1$ (1)

K_3, $\gamma(G) = \left\lfloor \frac{E(G)}{Rank\ I(G)} \right\rfloor = \left\lfloor \frac{3}{2} \right\rfloor = 1$ (2)

Let LHS=RHS for n=k i.e. $\gamma(G_k) = \left\lfloor \frac{E(G_k)}{Rank\ I(G_k)} \right\rfloor$

To prove that LHS=RHS for n=k+1 i.e.

$$\gamma(G_{k+1}) = \left\lfloor \frac{E(G_{k+1})}{Rank\ I(G_{k+1})} \right\rfloor \quad (3)$$

We know that $\mathrm{E}(G) \le \mathrm{E}(G_{k+1})$,

$$Rank\ I(G_k) \le Rank\ I(G_{k+1}) \quad (4)$$

By inspection $2Rank\ I(G_{k+1}) > E(G_{k+1})$

Hence we conclude LHS=RHS for n=k+1 in turn LHS=RHS for all n

Hence the proof

Theorem-2

Let path P be a connected graph with no loops and multiple edges,

Then $\gamma(P_n) \le \left\lceil \frac{E(P_n)}{Rank\ I(P_n)} \right\rceil + \lfloor E(P_n) - Rank\ I(P_n) \rfloor$

Proof

We check the results for few paths from the Table-2: Path

For n= 2, $1 \le \left\lceil \frac{2}{2} \right\rceil + \lfloor 2-2 \rfloor = 1$ (1)

For n = 3, $1 \le \left\lceil \frac{2.828}{2} \right\rceil + \lfloor 2.828-2 \rfloor = 2$ (2)

For n = 4, $2 \le \left\lceil \frac{4.472}{4} \right\rceil + \lfloor 4.472-4 \rfloor = 2$ (3)

To prove the above result in general we consider a complete graph

If we delete all the extra edges from a complete graph with n vertices in order to get a path P_n , we write the following equations

$$\mathrm{E}(P_n) < \mathrm{E}(K_n)\ ,\ Rank\ \ I(P_n) \le Rank\ \ I(K_n) \quad (4)$$

Therefore from the above equation we can write that

$$\frac{E(P_n)}{Rank\ I(P_n)} \le \left\lceil \frac{E(K_n)}{Rank\ I(K_n)} \right\rceil$$

$$\Rightarrow \frac{E(P_n)}{Rank\ I(P_n)} + k = \left\lceil \frac{E(K_n)}{Rank\ I(K_n)} \right\rceil \quad (5)$$

Where k is a constant which is chosen as $k = \lfloor E(P_n) - Rank\ I(P_n) \rfloor$

Therefore

$$\left\lfloor \frac{E(K_n)}{Rank\ I(K_n)} \right\rfloor \le \left\lceil \frac{E(P_n)}{Rank\ I(P_n)} \right\rceil + \lfloor E(P_n) - Rank\ I(P_n) \rfloor$$

From theorem 1 we can write that

$$\gamma(P_n) \le \left\lceil \frac{E(P_n)}{Rank\ I(P_n)} \right\rceil + \lfloor E(P_n) - Rank\ I(P_n) \rfloor$$

Hence the theorem

TABLE II
PATH

G	$\gamma(G)$	E(G)	$\rho(G$	$\Delta(G)$	Eigen values
P_2	1	2	2	-1	± 1
P_3	1	2.828	2	0	± 1.414,0
P_4	2	4.472	4	1	± 1.618, ± 0.618
P_5	2	5.464	4	0	± 1.732, ± 1,0
P_6	2	6.988	6	-1	± 1.802, ± 1.247, ± 0.445
P_7	3	8.054	6	0	± 1.848, ± 1.414, ± 0,0.765
P_8	3	9.516	8	1	± 1.879, ± 1.532, ± 1, ± 0.347
P_9	3	10.628	8	0	± 1.902, ± 1.618, ± 1.176, ± 0.618,0
P_{10}	4	12.056	10	-1	± 1.919, ± 1.683, ± 1.310, ± 0.831, ± 0.285
P_{11}	4	13.192	10	0	± 1.932, ± 1.732, ± 1.414, ± 1, ± 0.518, 0
P_{12}	4	14.529	12	1	± 1.942, ± 1.771, ± 1.497, ± 1.136, ± 0.709, ± 0.241
P_{13}	5	15.752	12	0	± 1.350, ± 1.802, ± 1.564, ± 1.247, ± 0.868, ± 0.445,0
P_{14}	5	17.132	14	-1	± 1.956, ± 1.827, ± 1.618, ± 1.338, ± 1, ± 0.618, ± 0.209
P_{15}	5	18.306	14	0	± 1.962, ± 1.848, ± 1.663, ± 1.414, ± 1.111, ± 0.765, ± 0.390,0
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-	-	-	-	-	- - - - - - - - -

Theorem-3

Let a cycle C be a connected graph with no loops and multiple edges.

Then $\gamma(C_n) \leq \left\lceil \dfrac{E(C_n)}{Rank\ I(C_n)} \right\rceil + \left\lfloor E(C_n) - Rank\ I(C_n) \right\rfloor$

Proof

We can prove the above theorem similar to theorem-2

TABLE III
CYCLE

G	$\gamma(G)$	E(G)	$\rho(G)$	$\Delta(G)$	Eigen values
C_3	1	4	3	2	-1,-1,2
C_4	2	4	2	0	-2,2,0,0
C_5	2	6.472	5	2	1.618,1.618, ± 0.618,2
C_6	2	8	6	-4	± 2, ± 1, ± 1
C_7	3	8.988	7	2	-1.802,-1.802,-0.445,-0.445, ± 1.247,2
C_8	3	9.656	6	0	± 2, ± 1.414, ± 1.414, 0
C_9	3	11.516	9	2	-1.879,-1.879,-1 0.347,0.347,2,1.532, 1.532,-1
C_{10}	4	12.944	10	-4	± 2, ± 1.616, ± 1.618, ± 0.618, ± 0.618
C_{11}	4	14.206	11	2	-1.919,-1.919, -1.310,-1.310,-0.285, - 0.285, 0.831, 0.831, 2, 1.683, 1.683
C_{12}	4	14.928	10	0	± 2, ± 1, ± 1, ± 1.732, ± 1.732, 0
C_{13}	5	16.562	13	2	-1.942,-1.914, -1.497,-1.497, -0.709,-0.709, 0.241, 0.241, 1.136, 1.136,2,1.771,1.771
C_{14}	5	17.976	14	-4	± 2, ± 1.802, ± 1.802, ± 1.247, ± 1.247, ± 0.445, ± 0.445
C_{15}	5	19.132	15	2	-1.956,-1.618, -1.618,-1,-1,-0.209, -0.209, 0.618, 0.618, 1.338, 1.338, 2,1.827,1.827,-1.956
-	-	-	-	-	- - - - - - - - -
-	-	-	-	-	- - - - - - - - -

Few Open problems

1. It can be shown that for theorem-3 Cycles we can get better results for specific cases of n- number of vertices being odd, even, prime etc.
2. The relation between these parameters can be extended to other classes of graphs and other types of Domination
3. Similar bounds i.e. relation between Domination number, Energy of graph and Rank of the incident matrix could be extended to any simply connected graphs, sub graphs, partitioning of graphs and other related topics.

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