One-Dimensional Solute Transport for Uniform and Varying Pulse Type Input Point Source with Temporally Dependent Coefficients in Longitudinal Semi-Infinite Homogeneous Porous Domain

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***Abstract* – Analytical solutions are obtained for one-dimensional advection-diffusion equation with variable time dependent dispersion in longitudinal semi-infinite homogeneous porous medium. The solute dispersion parameter is considered temporally dependent while the seepage velocity uniform. The first order decay and zero-order production terms are considered inversely proportional to the dispersion coefficient. Retardation factor is also considered in the present paper. Analytical solutions are obtained for uniform pulse type and varying pulse type input point source. Porous domain is initially not solute free i.e. porous domain is not clean. The Laplace transformation technique is employed to get the analytical solutions of the present problem. The numerical examples are given to illustrate the various results.**

***Index Terms* - Advection, Aquifer, Diffusion, First-order Decay, Pulse type condition, Retardation Factor, Zero-order Production.**

***MSC 2010 Codes* – 35K57, 76R50**

I. INTRODUCTION

ADVECTION-DIFFUSION problem equation describes the solute transport due to combined effect of diffusion and convection in porous medium. It is a second order partial differential equation of parabolic type. This equation is valid only when the transport of a solute are in Fickian regime, however rate of solute spread linearly with time and the dispersive flux becomes to the concentration gradient [1]. Advection-dispersion equation is applicable in many disciplines like groundwater hydrology, chemical engineering bio sciences, environmental sciences and petroleum engineering. It helps understand the contaminant or pollutants concentration distribution behavior through an open medium like air, rivers, lakes or porous medium like aquifers, underground oil reservoirs. Many authors proposed analytical solutions in finite / infinite domain to simulate solute transport in porous media. A number of analytical solutions describing solute moving through one-dimensional media, considering adsorption, first-order decay and zero-order production, are obtained [2]-[12].

Reference [13] obtained analytical solutions of the solute transport equation with rate-limited desorption and decay. Reference [14] presented analytical solutions of one-dimensional convective dispersive solute transport equation under a various conditions. Reference [15] presented mathematical solutions to describe leaching and degradation of pesticides in a specific type of column experiment, which is frequently required for the official registration of pesticides. Reference [16] presented a new analytical solution for solute diffusion in a semi-infinite two-layer porous medium for arbitrary boundary and initial conditions using the Green's functions approach in the Laplace domain. All these studies advection dispersion equations solved analytically for non-reactive and reactive solutes, subject to various initial and boundary conditions.

In recent years, the advection-dispersion equation has been used in a variety of problems related to hydro-environment research. Reference [17] applied to solve the generalized integral transform technique (GITT) to the one-dimensional advection-dispersion equation (ADE) in heterogeneous porous media coupled with either linear or nonlinear sorption or decay. Reference [18] described a soil water model of one-dimensional flow using the Runge-Kutta Gill algorithm, which shows satisfactory results when compared with published data. Reference [19] considered the parallel plate and cylindrical geometry to model contaminant transport in a main fracture surrounded by a two-dimensional rock matrix. The transformation group theoretical approach is applied to present some new analytical solutions for the advection-dispersion equation governing these models. The application of this approach reduces the number of independent variables, and consequently the governing equation is reduced to ordinary differential equations. Reference [20] developed a methodology for estimating the temporally varying virus inactivation rate coefficient from the experimental virus inactivation data. Reference [21] solved an analytical technique to the Eulerian advection–diffusion equation for non-stationary conditions and passive contaminant within the

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Planetary Boundary Layer is analyzed. Reference [22] presented a range of analytical solutions to the combined transient water and solute transport for horizontal flow and adopted the concept of a scale and time dependent dispersivity used for contaminant transport in aquifers. Reference [23] presented an analytical solution for the non-stationary two-dimensional advection-diffusion equation to simulate the pollutant dispersion using Laplace transformation technique.

Reference [24] presented a formal exact solution of the linear advection–diffusion transport equation with constant coefficients for both transient and steady-state. Reference [25]-[28] obtained analytical solutions for temporally and spatially dependent solute dispersion in one and two dimensional semi-infinite media for continuous and pulse type input conditions. References [29], [30] obtained analytical solutions of advection-dispersion equation in semi-infinite longitudinal porous domain.

In the present paper, the analytical solutions of a one-dimensional advection-dispersion equation with temporally dependent dispersion coefficient are determined using Laplace Transformation Technique. The medium is considered semi-infinite homogeneous in longitudinal direction. The solute dispersion parameter is considered temporally dependent and flow velocity uniform. The first order decay and zero order production terms are taken into account and they are inversely proportional to the dispersion coefficients. In each case the domain is initially the not solute free. Two different types point source are taken. First one uniform pulse type and second one is for varying pulse type input point source. The solutions in all possible combinations of increasing or decreasing temporally dependence are compared with each other and discussed based on the solution with the help of graph.

II. METHODOLOGY

In this study we assume that solute transport is in horizontal direction and described by one-dimensional advection-dispersion equation. Laplace transformation technique which defined by equation (1) is used to get the analytical solutions. The Laplace transformation may be defined as;

If $f(x,t)$ is a any function defined in $a \le x \le b$ and $t > 0$, then its Laplace transform with respect to t is denoted by $L\{f(x,t)\} = F(x,p)$ and is defined by;

$$L\{f(x,t)\} = F(x,p) = \int_0^\infty e^{-pt} f(x,t)dt\,, \quad p > 0 \qquad (1)$$

where p is called the transform variable, which is a complex variable.

The inverse Laplace transform is denoted by $L^{-1}\{F(x,p)\} = f(x,t)$ and defined by the complex variable;

$$L^{-1}\{F(x,p)\} = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-pt} F(x,p)dp\,, \quad c > 0 \qquad (2)$$

III. ADVECTION-DISPERSION EQUATION

Solute transport through a medium is described by a partial differential equation of parabolic type. It is derived on the principle of conservation of mass and Fick's laws of diffusion. This equation is usually known as advection-dispersion equation. In one space dimension the linear advection-dispersion equation may be written as

$$\frac{\partial c}{\partial t} + \frac{1-n}{n}\frac{\partial F}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial c}{\partial x} - u(x,t)c\right) - \gamma(x,t)c + \mu(x,t) \qquad (3)$$

where c is the solute concentration at position x at time t in liquid phase and F is the solute concentration, and n is the porosity of the medium. $D(x,t)$ represent the solute dispersion parameter and is called the dispersion coefficient if it is uniform and steady, $u(x,t)$ is the velocity of the medium transporting the solute particles. If the medium is porous, it satisfies the Darcy's law. $\gamma(x,t)$ represent the first-order decay (The change in concentration of radioactive material in an aquifer due to radioactive decay is expressed by the following expression $C_t = C_i e^{-dt}$, where C_t is the ratio nuclide concentration in the aquifer at time t, C_i is the initial radionuclide concentration in the aquifer, d is the retardation decay constant and t is time after radioactive decay starts. The solute can decay in the liquid phase and or the solid phase. Its decay is commonly expressed by a first order decay reaction) and $\mu(x,t)$ represent the zero-order production. In Eq. (1), the effect of molecular diffusion is not considered because that the mechanical dispersion mostly dominate the hydrodynamic dispersion process during solute transport.

Reference [31] considered two cases, namely,

$$F = K_1 c + K_2 \qquad (4)$$

and
$$\frac{\partial F}{\partial t} = K_1 c - K_2 F \qquad (5)$$

respectively, equilibrium and non-equilibrium relationship between the concentrations in the two phases where K_1 and K_2 are empirical constant. For simplicity, the former relationship is adopted in the present analysis. Using Eq. (4) in Eq. (3) one may obtains,

$$\frac{\partial c}{\partial t} + \frac{1-n}{n}\frac{\partial (K_1 c + K_2)}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial c}{\partial x} - u(x,t)c\right) - \gamma(x,t)c + \mu(x,t) \qquad (6)$$

or $$R_d \frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial c}{\partial x} - u(x,t)c \right) - \gamma(x,t)c + \mu(x,t) \tag{7}$$

where $R_d = \left(1 + \frac{1-n}{n}K_1\right)$ is retardation factor.

IV. ANALYTICAL SOLUTIONS

Let us write $D(x,t)$, $u(x,t)$, $\gamma(x,t)$ and $\mu(x,t)$ in Eq. (3) as

$$D(x,t) = D_0 f_1(x,t)\,;\; u(x,t) = u_0 f_2(x,t) \tag{8}$$

and the first order decay $\gamma(x,t)$ and zero order production $\mu(x,t)$ terms which is inversely proportional to the dispersion coefficient i.e.

$$\gamma(x,t) = \gamma_0 / f_1(x,t)\,;\; \mu(x,t) = \mu_0 / f_1(x,t) \tag{9}$$

respectively, where D_0 (L^2T^{-1}), u_0 (LT^{-1}), γ_0 (T^{-1}) and μ_0 $(ML^{-3}T^{-1})$ are constants. Eq. (7) is re-written as

$$R_d \frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D_0 f_1(x,t)\frac{\partial c}{\partial x} - u_0 f_2(x,t)c \right) - \gamma_0 c / f_1(x,t) + \mu_0 / f_1(x,t) \tag{10}$$

Let us introduce a new independent variable, X by a transformation ([25], [27])

$$X = \int \frac{dx}{f_1(x,t)} \quad \text{or} \quad \frac{dX}{dx} = \frac{1}{f_1(x,t)} \tag{11}$$

Applying the transformation of Eq. (11) on the partial differential equation, Eq. (10) becomes

$$f_1(x,t) R_d \frac{\partial c}{\partial t} = D_0 \frac{\partial^2 c}{\partial X^2} - u_0 \frac{\partial}{\partial X}\{f_2(x,t)c\} - \gamma_0 c + \mu_0 \tag{12}$$

Now this partial differential equation is solved analytically using the initial and boundary conditions of temporally dependent dispersion along a uniform flow for two cases: former one is for uniform pulse type input point source and latter case is for varying pulse type input point source. The domain in each case is a semi-infinite longitudinal medium but it is not solute free at the initial stage.

A. *Temporally dependent dispersion along a uniform flow*

The solute dispersion parameter is supposed to be time dependent. So in Eq. (8) we consider

$$f_1(x,t) = f(mt) \text{ and } f_2(x,t) = 1 \tag{13}$$

where m (T^{-1}) is a unsteady parameter, whose dimension is inverse of that of the time variable t. $f(mt)$ is chosen such that for $m=0$ or $t=0$, $f(mt)=1$. Thus $f(mt)$ is an expression in the non-dimensional variable mt. $m=0$ corresponds to the temporally independent dispersion. In view of the expressions in Eq. (12), the constants D_0 and u_0 in Eq. (8) may be referred to as the initial dispersion coefficient (L^2T^{-1}) and the uniform velocity (LT^{-1}), respectively. Further from Eq. (11) we have

$$X = \int \frac{dx}{f(mt)} \quad \text{or} \quad \frac{dX}{dx} = \frac{1}{f(mt)} \tag{14}$$

The partial differential equation Eq. (12) will become

$$f(mt) R_d \frac{\partial c}{\partial t} = D_0 \frac{\partial^2 c}{\partial X^2} - u_0 \frac{\partial c}{\partial X} - \gamma_0 c + \mu_0 \tag{15}$$

Further another independent variable, T is introduced using the following transformation [32],

$$T = \int_0^t \frac{dt}{f(mt)} \tag{16}$$

As a result, Eq. (15) reduces to a partial differential equation with constant coefficients, we have

$$R_d \frac{\partial c}{\partial T} = D_0 \frac{\partial^2 c}{\partial X^2} - u_0 \frac{\partial c}{\partial X} - \gamma_0 c + \mu_0 \tag{17}$$

The dimension of variable X defined by Eq. (14) and that of variable T defined by Eq. (16) remain those of x and t, respectively and are referred to as the new space variable and the new time variable, respectively. The new time variable obtained from Eq. (16) satisfies the conditions $T=0$ for $t=0$ and $T=t$ for $m=0$. The first condition ensures that the nature of the initial condition does not change in the new time variable domain. The second condition refers to the temporally independent dispersion and Eq. (10) reduces to one with constant coefficients.

B. *Uniform pulse type input point source condition*

The medium though which the solute dispersion occurs is supposed to be of semi-infinite extent along the longitudinal direction. In case of uniform input point source assume that, initially the region of interest is not free from the pollution at the beginning of the source. The source of the pollutant's solute is considered point source and is supposed to be situated at the origin of the domain. It is of the uniform nature of pulse type. Its means, the input concentration at the origin remains uniform up to certain time period till the source is present, and just after the source is eliminated becomes zero. For example, smoke coming out from a chimney of a factory; particulate particles coming out of a volcano; the sewage outlet of a municipal area or effluent outlet of a factory or industry in a surface water medium; infiltrations of wastes from garbage disposal sites, septic tanks, mines, discharge from surface water bodies polluted due to industrial and municipal influents, and reaching the ground water level, particularly with rainwater.

The second boundary condition is of flux type of homogeneous nature at the infinity is assumed zero. Thus the analytical solution, $c(x,t)$ of the advection-dispersion equation Eq. (7) is desired with respect to the following initial and boundary conditions are as follows

$$c(x,t) = C_i \quad ; \; t=0 \,, \; x \ge 0 \tag{18}$$

$$c(x,t) = \begin{cases} C_0 \; ; \; 0 < t \le t_0 \\ 0 \quad ; \; t > t_0 \end{cases} , \; x = 0 \tag{19}$$

$$\frac{\partial c(x,t)}{\partial x} = 0 \; ; \; x \to \infty \,, \; t \ge 0 \,. \tag{20}$$

The initial and boundary conditions in Eqs. (18-20) may be written in terms of new independent variable as

$$c(X,T) = C_i \; ; \; T = 0 \,, \; X \ge 0 \tag{21}$$

$$c(X,T) = \begin{cases} C_0 \; ; \; 0 < T \le T_0 \\ 0 \quad ; \; T > T_0 \end{cases} , \; X = 0 \; ; \tag{22}$$

$$\frac{\partial c(X,T)}{\partial X} = 0 \; ; \; X \to \infty \,, \; T \ge 0 \,. \tag{23}$$

Now Laplace transform which are defined in Eq. (1) may be used to get the analytical solution. But to use it more conveniently the convection term, first order decay and zero order production in the Eq. (17) is eliminated by using a transformation

$$c(X,T) = K(X,T) \exp\left\{ \frac{u_0}{2D_0} X - \frac{1}{R_d}\left(\frac{u_0^2}{4D_0} + \gamma_0 \right) T \right\} + \frac{\mu_0}{\gamma_0} \tag{24}$$

We get the initial and boundary value problem in terms of new dependent variable $K(X,T)$ as

$$R_d \frac{\partial c}{\partial T} = D_0 \frac{\partial^2 c}{\partial X^2} \tag{25}$$

$$K(X,T) = \left(C_i - \frac{\mu_0}{\gamma_0} \right) \exp\left\{ -\frac{u_0}{2D_0} X \right\} ; \; T = 0 \,, \; X \ge 0 \tag{26}$$

$$K(X,T) = \begin{cases} \left(C_0 - \dfrac{\mu_0}{\gamma_0} \right) \exp(\alpha^2 T) \; ; \; 0 < T \le T_0 \\ -\dfrac{\mu_0}{\gamma_0} \exp(\alpha^2 T) \; ; \; T > T_0 \end{cases} ; \; X = 0 \tag{27}$$

$$\frac{\partial K}{\partial X} + \frac{u_0}{2D_0} K = 0 \; ; \; X \to \infty \,, \; T \ge 0 \,. \tag{28}$$

where $\alpha^2 = \dfrac{1}{R_d}\left(\dfrac{u_0^2}{4D_0} + \gamma_0 \right)$.

Applying the Laplace transformation in set of above initial and boundary value problem reduces to an ordinary differential equation of second order boundary value problem, which comprises of following three equations are

$$\frac{d^2 \bar{K}}{dX^2} - \frac{pR_d}{D_0} \bar{K} = -\frac{R_d}{D_0}\left(C_i - \frac{\mu_0}{\gamma_0} \right) \exp\left\{ -\frac{u_0}{2D_0} X \right\} \tag{29}$$

$$\bar{K}(X,p) = \frac{C_0}{(p-\alpha^2)}[1 - \exp\{-(p-\alpha^2)\}T_0] - \frac{\mu_0}{\gamma_0 (p-\alpha^2)} \; ; \; X = 0 \,, \tag{30}$$

$$\frac{d\bar{K}}{dX} + \frac{u_0}{2D_0} \bar{K} = 0 \; ; \; X \to \infty \,. \tag{31}$$

where p is Laplace parameter is defined in Eq. (1).

Thus, the general solution of ordinary differential equation (29) may be written as

$$\bar{K}(X,p) = c_1 \exp\left\{ -\sqrt{\frac{pR_d}{D_0}} X \right\} + c_2 \exp\left\{ \sqrt{\frac{pR_d}{D_0}} X \right\} - \left(C_i - \frac{\mu_0}{\gamma_0} \right) \frac{\exp\left\{ -\frac{u_0}{2D_0} X \right\}}{(p-\beta^2)} \tag{32}$$

where $\beta^2 = \dfrac{u_0^2}{4R_d D_0}$.

Now, using boundary conditions Eq. (30) and Eq. (31) in general solution Eq. (32) for eliminating the arbitrary constant c_1 and c_2, we get

$$c_1 = \frac{C_0}{(p-\alpha^2)}[1-\exp\{-(p-\alpha^2)\}T_0]$$

$$-\frac{\mu_0}{\gamma_0}\frac{1}{(p-\alpha^2)} + \left(C_i - \frac{\mu_0}{\gamma_0}\right)\frac{1}{(p-\beta^2)} \text{ at } X=0$$

and $c_2 = 0$ at $X \to \infty$.

Thus, the particular solution in the Laplace domain may be written as

$$\bar{K}(X,p) = \frac{C_0}{(p-\alpha^2)}[1-\exp\{-(p-\alpha^2)T_0\}]\exp\left\{-\sqrt{\frac{pR_d}{D_0}}X\right\}$$

$$-\frac{\mu_0}{\gamma_0(p-\alpha^2)}\exp\left\{-\sqrt{\frac{pR_d}{D_0}}X\right\}$$

$$+\left(C_i - \frac{\mu_0}{\gamma_0}\right)\frac{\exp\left\{-\sqrt{\frac{pR_d}{D_0}}X\right\}}{(p-\beta^2)}$$

$$-\left(C_i - \frac{\mu_0}{\gamma_0}\right)\frac{\exp\left\{-\frac{u_0}{2D_0}X\right\}}{(p-\beta^2)} \tag{33}$$

Applying inverse Laplace transform, the analytical solution of advection-dispersion equation for uniform pulse type input point source may be written in terms of $c(x,T)$ by using back transformations Eq. (24), Eq. (16) and Eq. (14) as

$$c(x,T) = \frac{\mu_0}{\gamma_0} + \left(C_0 - \frac{\mu_0}{\gamma_0}\right)F_1(x,T)$$

$$+\left(C_i - \frac{\mu_0}{\gamma_0}\right)F_2(x,T)\ ;\ 0 < T \le T_0 \tag{34a}$$

$$c(x,T) = \frac{\mu_0}{\gamma_0} + \left(C_0 - \frac{\mu_0}{\gamma_0}\right)F_1(x,T) - C_0F_1(x,T-T_0)$$

$$+\left(C_i - \frac{\mu_0}{\gamma_0}\right)F_2(x,T)\ ;\ T > T_0 \tag{34b}$$

where

$$F_1(x,T) = \frac{1}{2}\exp\left[\left(\frac{\{u_0 - (u_0^2 + 4\gamma_0 D_0)^{1/2}\}}{2D_0}\right)\frac{x}{f(mt)}\right]$$

$$\times \mathrm{e}rfc\left[\frac{\left\{R_d x/f(mt) - \left(u_0^2 + 4\gamma_0 D_0\right)^{1/2} T\right\}}{2\sqrt{D_0 R_d T}}\right]$$

$$+\frac{1}{2}\exp\left[\left(\frac{\{u_0 + (u_0^2 + 4\gamma_0 D_0)^{1/2}\}}{2D_0}\right)\frac{x}{f(mt)}\right]$$

$$\times \mathrm{e}rfc\left[\frac{\left\{R_d x/f(mt) - \left(u_0^2 + 4\gamma_0 D_0\right)^{1/2} T\right\}}{2\sqrt{D_0 R_d T}}\right],$$

$$F_2(x,T) = \exp\left\{-\frac{\gamma_0 T}{R_d}\right\}\left[1 - \frac{1}{2}\mathrm{e}rfc\left\{\frac{R_d x/f(mt) - u_0 T}{2\sqrt{D_0 R_d T}}\right\}\right.$$

$$\left.-\frac{1}{2}\exp\left\{\frac{u_0}{D_0}X\right\}\mathrm{e}rfc\left\{\frac{R_d x/f(mt) + u_0 T}{2\sqrt{D_0 R_d T}}\right\}\right]$$

and $T = \int_0^t \frac{dt}{f(mt)}$.

C. Varying pulse type input point source condition

In the former case the input concentration at the origin of the one-dimensional longitudinal domain is considered to be uniform up to a certain time period beyond which it is assumed as zero. On the surface water bodies like river, lake, let the pollutants reach at a point (treated as origin) uniformly and is transported down the stream. As soon as the pollutants or its source is diverted away from reaching the river the input concentration at the origin becomes zero. But at a point of an aquifer the pollutants reach due to infiltration from a point source occurring on the surface. It may happen that the source hence the input concentration increases in a certain time domain. Once it is eliminated, the input starts decreasing instead of becoming zero at once. This scenario may be defined by a mixed type input condition which is

$$-D(x,t)\frac{\partial C}{\partial x} + u(x,t)C = \begin{cases} u_0 C_0\ ;\ 0 < t \le t_0 \\ 0\ ;\ t > t_0 \end{cases}\ ;\ x = 0 \tag{35}$$

Using Eq. (8), Eq. (13), Eq. (14) and Eq. (16) the above condition may be written in (X,T) domain as

$$-D_0\frac{\partial C}{\partial X} + u_0 C = \begin{cases} u_0 C_0\ ;\ 0 < T \le T_0 \\ 0\ ;\ T > T_0 \end{cases}\ ;\ X = 0 \tag{36}$$

Now, Eq. (36) reduces by the applying the transformation Eq. (24), into

$$-D_0\frac{\partial K}{\partial X}+\frac{u_0}{2}K=\begin{cases}\left(u_0C_0-\dfrac{u_0\mu_0}{\gamma_0}\right)\exp(\alpha^2T)\ ;\ 0<T\le T_0\\ -\dfrac{u_0\mu_0}{\gamma_0}\exp(\alpha^2T)\ ;\ T>T_0\end{cases}\quad X=0,$$

$$\alpha^2=\frac{1}{R_d}\left(\frac{u_0^2}{4D_0}+\gamma_0\right)\tag{37}$$

Applying the Laplace transform on Eq. (37), we have

$$-D_0\frac{d\bar{K}}{dX}+\frac{u_0}{2}\bar{K}=\frac{u_0C_0}{(p-\alpha^2)}[1-\exp\{-(p-\alpha^2)T_0\}]$$

$$-\frac{u_0\mu_0}{\gamma_0(p-\alpha^2)}\ ;\ X=0\tag{38}$$

Now using the condition Eq. (38) in place of Eq. (30), for eliminating the arbitrary constants c_1 and c_2 in Eq. (32), the particular solution of Eq. (29) satisfying the conditions in Eqs. (31) and (38) may be obtained as

$$c_1=\frac{u_0C_0}{\sqrt{R_dD_0}}\frac{[1-\exp\{-(p-\alpha^2)T_0\}]}{(p-\alpha^2)(\sqrt{p}+\beta)}$$

$$-\frac{u_0}{\sqrt{R_dD_0}}\frac{\mu_0}{\gamma_0}\frac{1}{(p-\alpha^2)(\sqrt{p}+\beta)}$$

$$+\frac{u_0}{\sqrt{R_dD_0}}\left(C_i-\frac{\mu_0}{\gamma_0}\right)\frac{1}{(p-\beta^2)(\sqrt{p}+\beta)}\quad \text{at } X=0$$

and $c_2=0$ at $X\to\infty$

Thus, the particular solution in the Laplace domain may be written as

$$\bar{K}(X,p)=\frac{u_0C_0}{\sqrt{R_dD_0}}\frac{[1-\exp\{-(p-\alpha^2)T_0\}]}{(p-\alpha^2)(\sqrt{p}+\beta)}\exp\left\{-X\sqrt{pR_d/D_0}\right\}$$

$$-\frac{u_0}{\sqrt{R_dD_0}}\frac{\mu_0}{\gamma_0}\frac{1}{(p-\alpha^2)(\sqrt{p}+\beta)}\exp\left\{-X\sqrt{pR_d/D_0}\right\}$$

$$+\frac{u_0}{\sqrt{R_dD_0}}\left(C_i-\frac{\mu_0}{\gamma_0}\right)\frac{1}{(p-\beta^2)(\sqrt{p}+\beta)}\exp\left\{-X\sqrt{pR_d/D_0}\right\}$$

$$-\left(C_i-\frac{\mu_0}{\gamma_0}\right)\frac{\exp\left\{-\dfrac{u_0}{2D_0}X\right\}}{(p-\beta^2)}\tag{39}$$

where $\beta^2=\dfrac{u_0^2}{4R_dD_0}$, $\alpha^2=\dfrac{1}{R_d}\left(\dfrac{u_0^2}{4D_0}+\gamma_0\right)$.

Now, applying inverse Laplace transform, the analytical solution of advection-dispersion equation for varying pulse type input point source may be written in terms of $c(x,T)$ by using back transformations Eq. (24), Eq. (16) and Eq. (14) as

$$c(x,T)=\frac{\mu_0}{\gamma_0}+\left(C_0-\frac{\mu_0}{\gamma_0}\right)F_1(x,T)+\left(C_i-\frac{\mu_0}{\gamma_0}\right)F_2(x,T)$$

$$;\ 0<T\le T_0\tag{40a}$$

$$c(x,T)=\frac{\mu_0}{\gamma_0}+\left(C_0-\frac{\mu_0}{\gamma_0}\right)F_1(x,T)$$

$$-C_0F_1(x,T-T_0)+\left(C_i-\frac{\mu_0}{\gamma_0}\right)F_2(x,T)$$

$$;\ T>T_0\tag{40b}$$

where

$$F_1(x,T)=\frac{u_0}{\left\{u_0+(u_0^2+4\gamma_0D_0)^{1/2}\right\}}$$

$$\exp\left[\left(\frac{\{u_0-(u_0^2+4\gamma_0D_0)^{1/2}\}}{2D_0}\right)\frac{x}{f(mt)}\right]$$

$$\times \mathrm{e}rfc\left[\frac{\left\{R_dx/f(mt)-\left(u_0^2+4\gamma_0D_0\right)^{1/2}T\right\}}{2\sqrt{D_0R_dT}}\right]$$

$$+\frac{u_0}{\left\{u_0-(u_0^2+4\gamma_0D_0)^{1/2}\right\}}$$

$$\exp\left[\left(\frac{\{u_0+(u_0^2+4\gamma_0D_0)^{1/2}\}}{2D_0}\right)\frac{x}{f(mt)}\right]$$

$$\times \mathrm{e}rfc\left[\frac{\left\{R_dx/f(mt)+\left(u_0^2+4\gamma_0D_0\right)^{1/2}T\right\}}{2\sqrt{D_0R_dT}}\right]$$

$$+\frac{u_0^2}{2\gamma_0D_0}\exp\left(\frac{u_0}{D_0}\frac{x}{f(mt)}-\frac{\gamma_0}{R_d}T\right)$$

$$\mathrm{e}\,rfc\left[\frac{\{R_d x/f(mt)+u_0T\}}{2\sqrt{D_0R_dT}}\right],$$

$$F_2(x,T)=\exp\left\{-\frac{\gamma_0 T}{R_d}\right\}\left[1-\frac{1}{2}\mathrm{e}\,rfc\left\{\frac{R_d x/f(mt)-u_0T}{2\sqrt{D_0R_dT}}\right\}\right.$$

$$-\left(\frac{u_0^2}{\pi R_d D_0}T\right)^{1/2}\exp\left\{-\frac{\left(R_d x/f(mt)-u_0T\right)^2}{4D_0R_0T}\right\}$$

$$+\frac{1}{2}\left(1+\frac{u_0x/f(mt)}{D_0}+\frac{u_0^2}{D_0R_0}T\right)$$

$$\left.\exp\left\{\frac{u_0x/f(mt)}{D_0}\right\}\mathrm{e}\,rfc\left\{\frac{R_d x/f(mt)+u_0T}{2\sqrt{D_0R_dT}}\right\}\right]$$

and $T=\int_0^t \frac{dt}{f(mt)}$.

V. Illustrations and Discussions

The concentration values are evaluated from the solutions in Eq. (34a, b) and (40a, b) in finite domain $0\le x(\text{meter})\le 10$ of semi-infinite region for various parameters in the presence of the source ($t<t_0$) and in the absence of the source ($t>t_0$). The concentration values (c/C_0) are evaluated assuming reference concentration as $C_0=1.0$, in a finite domain along longitudinal direction. Initially the source concentration, $C_i=0.1$. Seepage velocity $u_0=1.05\,(\text{meter/day})$ and dispersion coefficient $D_0=1.29\,(\text{meter}^2/\text{day})$, $\gamma=0.04$, zero order production $\mu=0.0021$ are considered and unsteady flow parameter $m=0.1\ (\text{day})^{-1}$.

In both the cases analytical solutions (34a, b) and (40a, b) illustrated with numerical example for increasing and decreasing function for various time and retardation factors. Once the source of the pollution is removed, the input concentration becomes zero for uniform and starts decreasing for varying pulse type nature with time. Fig 1a is drawn for various time $t(\text{day})=1.5$, 2.5, 3.5 and 4.5 and fix retardation $R_d=1.25$ and unsteady parameter $m=0.1\,(\text{day})^{-1}$ when source pollutant entering in the domain. The concentration level at particular position is increases with time and decreasing with position. Fig 1b is drawn for various time $t(\text{day})=5.5$, 6.5, 7.5 and 8.5 and fix retardation $R_d=1.25$ and unsteady parameter $m=0.1\ (\text{day})^{-1}$ when source pollutant not entering in the domain. Time of eliminations is taken $t_0(\text{day})=5.0$ in both figures. In both the figures, solid curves show the concentration values at different time while dashed and dotted curves illustrates behavior of concentration at particular time $t(\text{day})=1.5$ and 5.5 with another retardation $R_d=1.45$ and unsteady parameter $m=0.5\,(\text{day})^{-1}$. It is observed that the solute concentration is higher for the higher unsteady parameter and slower for the lower retardation factor, which are shows with solid, dashed and dotted curves in Fig 1a, b.

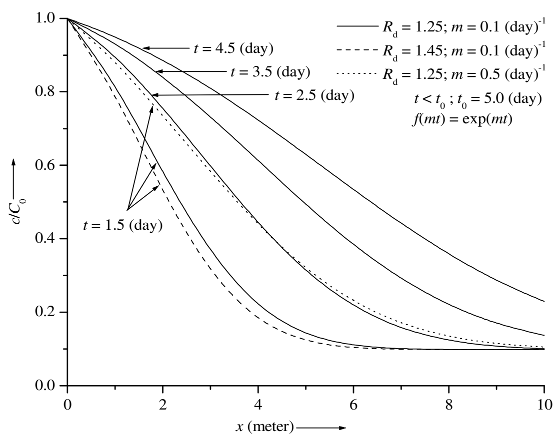

Figure 1a. Distribution of the solute concentration for solution (34a) in the presence of the source $(t<t_0)$, represented by four solid curves for $f(mt)=\exp(mt)$ and compare with the another retardation factor (dashed curve) and another unsteady parameter (dotted curve) at one time $t=1.5(\text{day})$.

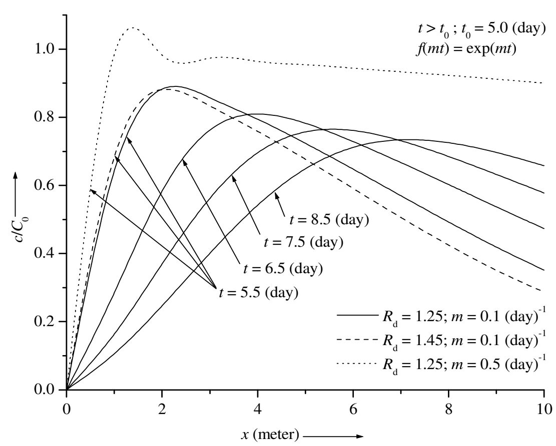

Figure 1b. Distribution of the solute concentration for solution (34b) in the absence of the source $(t>t_0)$, represented by four solid curves for $f(mt)=\exp(mt)$ and compare with the another retardation factor (dashed curve) and another unsteady parameter (dotted curve) at one time $t=5.5(\text{day})$.

Fig. 2a, b shows the comparison of concentration values evaluated from solution (34a, b) at ($t<t_0$) $t(\text{day})=2.5$ and 3.5, and ($t>t_0$) $t(\text{day})=6.5$ and 7.5 for various functions (increasing, decreasing and sinusoidal). It is observed that solute concentrations of increasing function attenuated faster than the decreasing and sinusoidal.

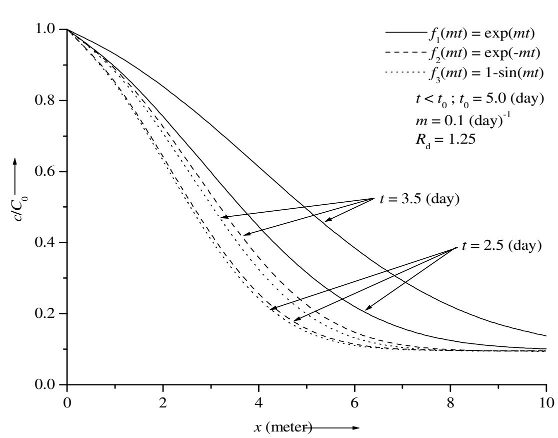

Figure 2a. Comparison of the solute concentration for solution (34a) in the presence of the source $(t < t_0)$, represented by solid curve for $f(mt) = \exp(mt)$, dashed curve for $f(mt) = \exp(-mt)$ and dotted curve for $f(mt) = 1 - \sin(mt)$ at one retardation factor and unsteady parameter at time $t(\text{day}) = 2.5$ and 3.5.

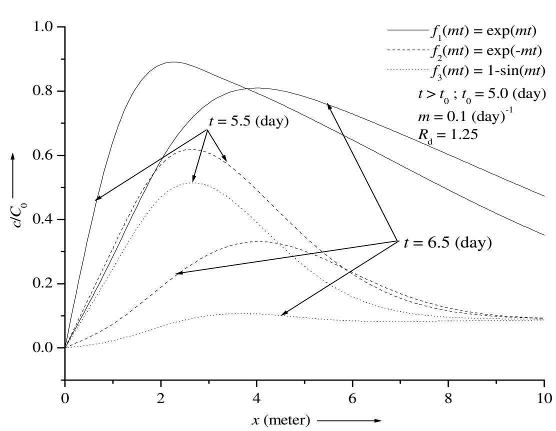

Figure 2a. Comparison of the solute concentration for solution (34b) in the absence of the source $(t > t_0)$, represented by solid curve for $f(mt) = \exp(mt)$, dashed curve for $f(mt) = \exp(-mt)$ and dotted curve for $f(mt) = 1 - \sin(mt)$ at one retardation factor and unsteady parameter at time $t(\text{day}) = 5.5$ and 6.5.

Fig. 3a, b show the comparison of solute concentration values evaluated from solution (34a, b) at a particular position $x = 5.0$ (meter) at time $(t < t_0)$ $t(\text{day}) = 0.0$ to 4.5, and $(t > t_0)$ $t(\text{day}) = 5.5$ to 8.5 for three expressions (i) $f(mt) = \exp(mt)$, (ii) $f(mt) = \exp(-mt)$, and (iii) $f(mt) = 1 - \sin(mt)$. It is observed that the solute concentration is higher for increasing function than decreasing and sinusoidal at particular position.

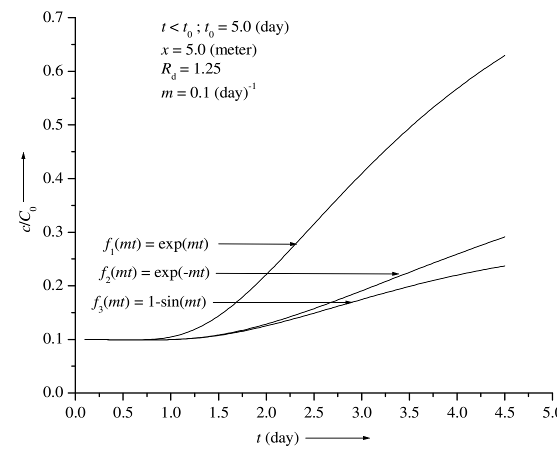

Figure 3a. Comparison of the solute concentration for solution (34a) in the presence of the source $(t < t_0)$, represented by solid curve for $f(mt) = \exp(mt)$, dashed curve for $f(mt) = \exp(-mt)$ and dotted curve for $f(mt) = 1 - \sin(mt)$ at particular position $x(\text{meter}) = 5.0$.

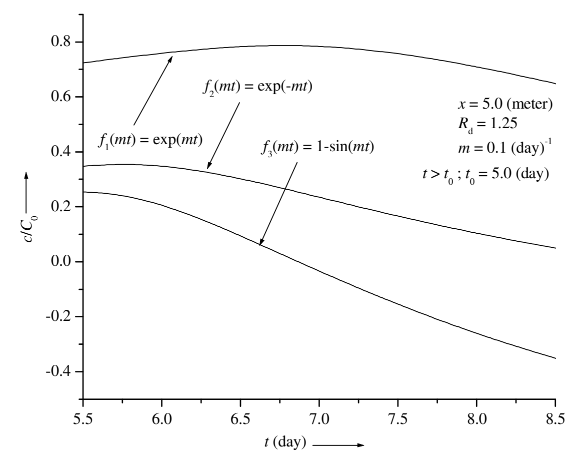

Figure 3b. Comparison of the solute concentration for solution (34b) in the presence of the source $(t > t_0)$, represented by solid curve for $f(mt) = \exp(mt)$, dashed curve for $f(mt) = \exp(-mt)$ and dotted curve for $f(mt) = 1 - \sin(mt)$ at particular position $x(\text{meter}) = 5.0$.

Fig 4a illustrates the solute transport from the point source along the longitudinal direction described by Eq. (40a), i.e., in the presence of the source concentration. The input concentration, (c/C_0) at the origin, $x = 0$ is increasing with time. Fig 4b illustrates the concentration behavior in absence of source concentration described by Eq. (40b). The input concentration decreases with increasing time. In both figures, solid curves shows the concentration values at various time $t(\text{day}) = 1.5$, 2.5, 3.5 and 4.5 for fix retardation $R_d = 1.25$ and unsteady parameter $m = 0.1\ (\text{day})^{-1}$, while dashed and dotted curve are drawn for another retardation $R_d = 1.45$ and unsteady parameter $m = 0.1\ (\text{day})^{-1}$ for fix time $t(\text{day}) = 1.5$ and 5.5 respectively. It is find that the solute concentration is higher for the lower retardation factor. At the origin concentration value is lower for higher unsteady parameter and these trends reverse away from the source.

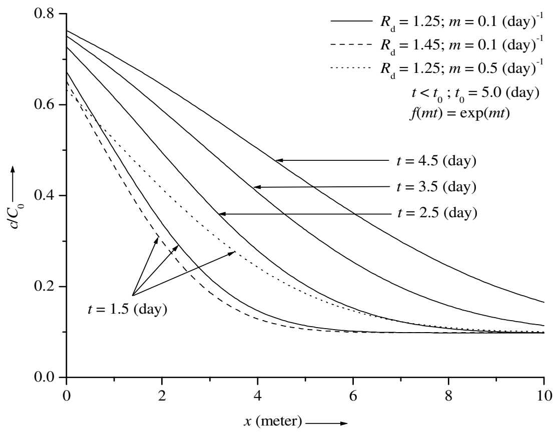

Figure 4a. Distribution of the solute concentration for solution (40a) in the presence of the source $(t < t_0)$, represented by four solid curves for $f(mt) = \exp(mt)$ and compare with the another retardation factor (dashed curve) and another unsteady parameter (dotted curve) at one time $t = 1.5(\text{day})$.

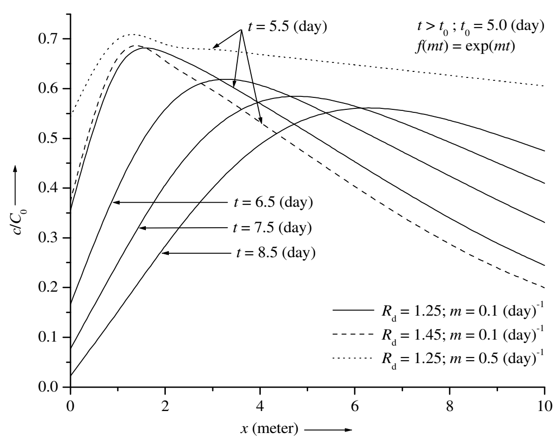

Figure 4b. Distribution of the solute concentration for solution (40b) in the absence of the source $(t > t_0)$, represented by four solid curves for $f(mt) = \exp(mt)$ and compare with the another retardation factor (dashed curve) and another unsteady parameter (dotted curve) at one time $t = 5.5(\text{day})$.

Fig. 5a, b shows the comparison of concentration levels evaluated from solution (40a, b) at ($t < t_0$) $t(\text{day}) = 2.5$ and 3.5, and ($t > t_0$) $t(\text{day}) = 6.5$ and 7.5 for various functions (increasing, decreasing and sinusoidal). It is also observed that solute concentrations of increasing functions attenuated higher than the decreasing and sinusoidal.

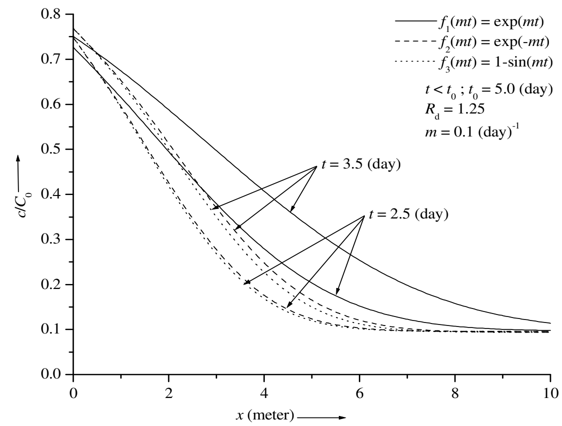

Figure 5a. Comparison of the solute concentration for solution (40a) in the presence of the source $(t < t_0)$, represented by solid curve for $f(mt) = \exp(mt)$, dashed curve for $f(mt) = \exp(-mt)$ and dotted curve for $f(mt) = 1 - \sin(mt)$ at one retardation factor and unsteady parameter at time $t(\text{day}) = 2.5$ and 3.5.

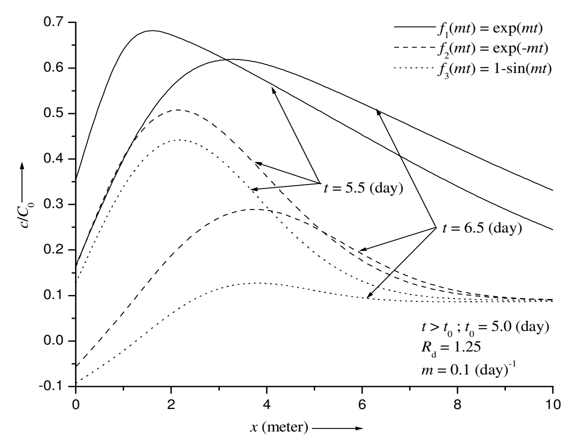

Figure 5b. Comparison of the solute concentration for solution (40b) in the absence of the source $(t > t_0)$, represented by solid curve for $f(mt) = \exp(mt)$, dashed curve for $f(mt) = \exp(-mt)$ and dotted curve for $f(mt) = 1 - \sin(mt)$ at one retardation factor and unsteady parameter at time $t(\text{day}) = 5.5$ and 6.5.

Fig. 6a, b show the comparison of concentration values evaluated from solution (40a, b) for a particular position $x = 5.0$ (meter) at time ($t < t_0$) $t(\text{day}) = 0.0$ to 4.5, and ($t > t_0$) t(day) = 5.5 to 8.5 for three expressions (i) $f(mt) = \exp(mt)$, (ii) $f(mt) = \exp(-mt)$, and (iii) $f(mt) = 1 - \sin(mt)$. It may be observed that the solute concentration is higher in exponentially increasing function than the exponentially decreasing function and sinusoidal nature at particular position.

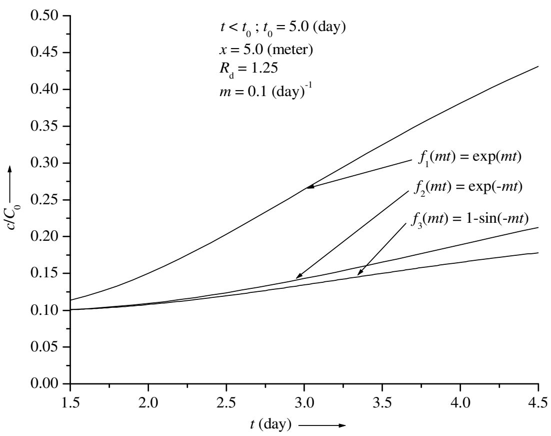

Figure 6a. Comparison of the solute concentration for solution (40a) in the presence of the source ($t < t_0$), represented by solid curve for $f(mt) = \exp(mt)$, dashed curve for $f(mt) = \exp(-mt)$ and dotted curve for $f(mt) = 1 - \sin(mt)$ at particular position $x(\text{meter}) = 5.0$.

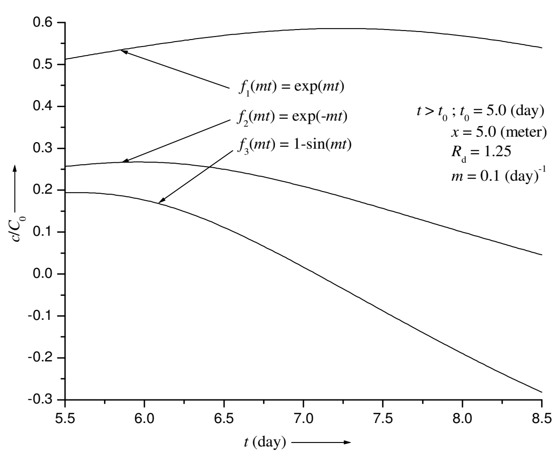

Figure 6b. Comparison of the solute concentration from solution (40b) in the absence of the source ($t > t_0$), represented by solid curve for $f(mt) = \exp(mt)$, dashed curve for $f(mt) = \exp(-mt)$ and dotted curve for $f(mt) = 1 - \sin(mt)$ at particular position $x(\text{meter}) = 5.0$.

VI. CONCLUSIONS

This study is presented semi analytical solutions for describing the solute transport of dissolved substances in porous media with temporally-dependent dispersion. In this paper, analytical solutions are describing temporally dependent solute concentration for a uniform and varying pulse type input source in a semi-infinite one-dimensional longitudinal domain. The proposed model assumes that the first order decay and zero order production are inversely proportional to dispersion coefficients. The Laplace transform technique is applied herein to solve the temporally-dependent advection-dispersion equation. The solutions in the both problems are obtained in terms of a general time variable function $f(mt)$ and may be used for a variety of time-dependent expressions (for an increasing and decreasing nature both). It is observed that the solute transport along the medium for an expression of an increasing nature is faster than decreasing and sinusoidal. The analytical solutions are obtained in the present work for pulse type input conditions may be useful in examining the degradation levels of the surface as well as below-surface hydro-environment, particularly in assessing the rehabilitation time period of a polluted flow domain once the sources of the pollution are eliminated.

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