Fuzzy Monoids in a Fuzzy Finite State Automaton with Unique Membership Transition on an Input Symbol

T. Rajaretnam and S.K. Ayyaswamy

***Abstract*—In this paper we consider the fuzzy finite state automaton with unique membership transition on an input symbol (uffsa) (Hopcroft and Ullman, 1979). It is proved and illustrated the existence of two different fuzzy monoids $F(M)$ and S_M from the fuzzy transition function of the given uffsa M. Also it is proved that $F(M)$ and S_M are anti-isomorphic as monoids.**

***Index Terms*—fuzzy finite state automaton, fuzzy languages , fuzzy monoids.**

***MSC 2010 Codes* – 03D05,03E72,68Q70**

I. Introduction

In a fuzzy finite state automaton(ffsa), there may be more than one fuzzy transition from a state on an input symbol with a given membership value [4, 7]. This development was followed by the postulation called deterministic fuzzy finite state automaton (dffsa) [4], that is, from a state for an input symbol there will be atmost one transition. However, it only acts as a deterministic fuzzy recognizer, that is, for any fuzzy recognizer M there is a deterministic fuzzy recognizer M_1 with the same behaviour in the sense , for any string x, the membership value of x in the language generated by M and is that of in M_1 need not be the same. In [6] an uffsa is intoduced by incorporating a condition that the membership function has a unique membership transition on an input symbol, that is, from a state for an input symbol with a given membership value there will be atmost one transition,here there may be another transition from the state for the same input symbol with different membership value , so uffsa is much simpler than ffsa.For any string x, the membership value of x in the language generated by ffsa and is that of in the corresponding uffsa will be the same.

In [4] two distinct semigroups have been generated from the fuzzy transition function of a fuzzy finite state machine (ffsm), we generate two monoids $F(M)$ and S_M from the fuzzy transition function of an uffsa M. These monoids (underlying semigroups) are not only different from the two semigroups mentioned above, but also that these could be generated only in uffsa's and are not possible in ffsm's. The existence of such monoids are proved and illustrated with examples.Also it is proved that $F(M)$ and S_M are anti-isomorphic as monoids.

Dr. T. Rajaretnam is with the P.G. and Research Dept. of Mathematics, St. Josephś College (Autonomous), Trichy-620002, India. E-mail: t_rajaretnam@yahoo.com

Mr. S.K. Ayyaswamy is with the P.G. and Research Dept. of Mathematics, St. Josephś College (Autonomous), Trichy-620002, India.

II. Preliminaries

We recall some basic definitions on fuzzy languages and fuzzy automata [1, 4–7].

Definition II.1. *Let X, Y and Z be non-empty sets and let μ be a fuzzy subset of $X \times Y$ and γ be a fuzzy subset of $Y \times Z$. Define the fuzzy subset $\mu \circ \gamma$ of $X \times Z$ by $(\mu \circ \gamma)(x, z) = \vee\{\mu(x, y) \wedge \gamma(y, z) \mid y \in Y\} \forall x \in X$ and $\forall z \in Z$. If μ is a fuzzy subset of $X \times X$, we define $\mu' = \mu$ and $\mu^{n+1} = \mu \circ \mu^n \, \forall n \in \mathcal{N}$.*

Definition II.2. *Let Σ be a finite alphabet and L the fuzzy subset of Σ^*, i.e., $L : \Sigma^* \to [0, 1]$ is called the **fuzzy language** over Σ. For $x \in \Sigma^*$, $L(x)$ is the membership value (degree) of x.*

Definition II.3. *A **fuzzy finite state machine (ffsm)** is a triple $M = (Q, \Sigma, \mu)$ where Q and Σ are finite non-empty sets and μ is a fuzzy subset of $Q \times \Sigma \times Q$ i.e., $\mu : Q \times \Sigma \times Q \to [0, 1]$.*

This structure includes only the fuzzy transition function and does not have the fuzzy initial states and fuzzy final states.

Definition II.4. *A **fuzzy finite state automaton (ffsa)** is a quintuple $M = (Q, \Sigma, \mu, i, f)$ where*

(i) Q is a finite non-empty set of states.

(ii) Σ is a finite non-empty set of input symbols.

(iii) the fuzzy subset $\mu : Q \times \Sigma \times Q \to [0, 1]$ is a function, called the fuzzy transition function,

(iv) i is a fuzzy subset of Q, i.e., $i : Q \to [0, 1]$ called the fuzzy subset of initial states, and

(v) f is a fuzzy subset of Q, i.e., $f : Q \to [0, 1]$ called the fuzzy subset of final states.

Definition II.5. *Let $M = (Q, \Sigma, \mu, i, f)$ be an* ffsa*, the extended fuzzy transition function for M is the fuzzy subset $\mu^* : Q \times \Sigma^* \times Q \to [0, 1]$ has been defined as follows: for all $p, q \in Q$, $a \in \Sigma$, $x \in \Sigma^*$,*

$$\mu^\star(p, \lambda, q) = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } p \neq q \end{cases}$$

$$\mu^\star(p, xa, q) = \vee\{\mu^\star(p, x, r) \wedge \mu(r, a, q) | r \in Q\}$$

Definition II.6. *Let $M = (Q, \Sigma, \mu, i, f)$ be an ffsa. Let $x \in \Sigma^*$. Then x is said to be **recognized** by M if*

$$\deg_M(x) = \vee\Big\{\{i(p) \wedge \mu^*(p, x, q) \wedge f(q) \mid q \in Q\} \mid p \in Q\Big\} > 0$$

Definition II.7. *The **fuzzy language accepted by an ffsa** $M = \big(Q, \Sigma, \mu, i, f\big)$ is a fuzzy subset of Σ^* and is denoted by L_M, $L_M : \Sigma^* \to [0, 1]$ is defined by*
$L_M(x) = \vee\Big\{\big\{i(p) \wedge \mu^*\big(p, x, q\big) \wedge f(q) \mid q \in Q\big\} \mid p \in Q\Big\}$

Definition II.8. *A **deterministic fuzzy finite state automaton** is an ordered five tuple $M = \big(Q, \Sigma, \mu, i, f\big)$ such that*

(i) there exists a unique $s_0 \in Q$ such that $i(s_0) > 0$; s_0 is called the initial state,

(ii) $\mu : Q \times \Sigma \times Q \to [0, 1]$ and if $\forall q \in Q$, $a \in \Sigma$, $\mu\big(q, a, p\big) > 0$ and $\mu\big(q, a, p'\big) > 0$ for some $p, p' \in Q$, then $p = p'$.

(iii) $\forall x \in \Sigma^*$, there exists a unique $q_x \in Q$ such that $\mu^*\big(s_0, x, q_x\big) > 0$.

Let $F = \big\{q \in Q \mid f(q) > 0\big\}$. F is called the set of final states of M.

Definition II.9. *[6] A fuzzy finite state automaton with unique membership transition on an input symbol is denoted by **uffsa** and is defined by*
$M = (Q, \Sigma, \mu, i, f)$, where

(i) Q is a finite non-empty set of states

(ii) Σ is a finite non-empty set of input symbols.

(iii) the fuzzy subset $\mu : Q \times \Sigma \times Q \to [0, 1]$ such that if for any $p \in Q, a \in \Sigma$, $\mu(p, a, q) = \mu(p, a, q')$ for some $q, q' \in Q$ then $q = q'$.

(iv) i is a fuzzy subset of Q, i.e., $i : Q \to [0, 1]$, called the fuzzy subset of initial states.

(v) f is a fuzzy subset of Q, i.e., $f : Q \to [0, 1]$ called the fuzzy subset of final states.

Theorem II.10. *Let $M = (Q, \Sigma, \mu, i, f)$ be an* uffsa.
Then $\mu^*\big(p, xy, q\big) = \vee\big\{\mu^*\big(p, x, r\big) \wedge \mu^*\big(r, y, q\big) \mid r \in Q\big\}\forall p, q \in Q$ and $\forall x, y \in \Sigma^*$.

III. Fuzzy Monoid

In this section we prove the existence of the monids $F(M)$ and $S_M$in an uffsa M.For some results are also obtained.

Definition III.1. *Let $M = (Q, \Sigma, \mu, i, f)$ be an* uffsa. *For $a \in \Sigma$, we define $f_a : Q \to Q$ by $f_a(p) = q$ if q is the state which has the maximum membership value from p on a and $\mu\ (p, a, q) > 0$. and $f_a(p) = p$ if there is no transition from p on a. For $x \in \Sigma^*$, we define $f_x : Q \to Q$ by (i) $f_\lambda(p) = p$, and inductively (ii) $f_{ax}(p) = f_x(q)$, where q is such that $f_a(p) = q$*

Theorem III.2. *Let $M = (Q, \Sigma, \mu, i, f)$ be an* uffsa *and let $F(M) = \{f_x \mid x \in \Sigma^*\}$, then $F(M)$ is a finite monoid (submonoid of Q^Q) under $\circ$, the composition of functions.*

Proof: Let $f_x, f_y \in F(M), x, y \in \Sigma^*$. For $p \in Q$,

$$\begin{aligned}(f_x \circ f_y)(p) &= f_x\big(f_y(p)\big)\\ &= f_x(q), \quad \text{where } q \text{ is such that } f_y(p) = q\\ &= s, \quad \text{where } s \text{ is such that } f_x(q) = s.\end{aligned}$$

Now $f_{yx}(p) = f_x(q) = s$. Therefore $(f_x \circ f_y)(p) = f_{yx}(p)$. Since p is arbitrary, $f_x \circ f_y = f_{yx} \in F(M)$, a unique function. Therefore $F(M)$ is closed under composition $\circ$. Let $f_x, f_y, f_z \in F(M)$. Now $f_x \circ (f_y \circ f_z) = f_x \circ (f_{zy}) = f_{zyx} = (f_{yx}) \circ f_z = (f_x \circ f_y) \circ f_z$. Thus $\circ$ is associative. For $p \in Q$, $(f_x \circ f_\lambda)(p) = f_x\big(f_\lambda(p)\big) = f_x(p) = f_\lambda\big(f_x(p)\big) = (f_\lambda \circ f_x)(p)$. Since p is arbitrary, $f_x \circ f_\lambda = f_x = f_\lambda \circ f_x$. Hence f_λ is the identity element which is in $F(M)$. Therefore $\big(F(M), \circ\big)$ is a finite monoid. Since Im(μ) is finite, $F(M)$ is finite. ■

Corollary III.3. *$\big(F(M), \circ\big)$ is a finite fuzzy monoid.*

Proof: Let $x \in \Sigma^*$, $p \in Q$, $x = a_1 a_2 \cdots a_n$, $a_i \in \Sigma$, $i = 1, 2, \ldots, n$. Let $f_x(p) = q$ and the sequence of membership values in the path be $\mu\big(p, a_1, p_1\big)$, $\mu\big(p_1, a_2, p_2\big)$, $\ldots$, $\mu\big(p_{n-1}, a_n, p_n\big)$, where $p_n = q$. Define $\mu_1 : F(M) \to [0, 1]$ by $\mu_1(f_{a_1 a_2 \cdots a_n}) = \vee\{\mu\big(p, a_1, p_1\big) \wedge \mu\big(p_1, a_2, p_2\big) \wedge \cdots \wedge \mu\big(p_{n-1}, a_n, p_n\big) \mid p \in Q\}$ Therefore $\mu_1(f_x \circ f_y) = \mu_1(f_{yx}) \geq \wedge\big(\mu_1(f_y), \mu_1(f_x)\big)$. Thus $\big(F(M), \circ\big)$ is a fuzzy monoid. ■

Theorem III.4. *$F(M)$ is an anti-homomorphic image of Σ^*.*

Proof: Let $\cdot$ be the concatenation operator. Then $\big(\Sigma^*, \cdot\big)$ is a semigroup with identity element λ.
Define $\phi : \big(\Sigma^*, \cdot\big) \to \big(F(M), \circ\big)$ by $\phi(x) = f_x\ \forall x \in \Sigma^*$.
Clearly ϕ is well-defined, onto, and $\phi(xy) = \phi(y) \circ \phi(x)$, $\forall\, x, y \in \Sigma^*$. ■

Theorem III.5. *Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. Define a relation $\equiv$ on Σ^* by $x \equiv y$ if and only if $f_x = f_y$, $\forall x, y \in \Sigma^*$. Then $\equiv$ is a congruence relation on Σ^*.*

Proof: Easily we can verify that $\equiv$ is an equivalence relation. We show that $\equiv$ is a congruence relation. Let $x, y \in \Sigma^*$ and $x \equiv y$. Then $f_x = f_y$, that is, $f_x(p) = f_y(p)\ \forall p \in Q$. Now for any $z \in \Sigma^*$, $f_{zx}(p) = f_x\big(f_z(p)\big) = f_y\big(f_z(p)\big) = f_{zy}(p)\forall p \in Q$. Therefore $f_{zx} = f_{zy}$, and so $zx \equiv zy$. Similarly $xz \equiv yz$. Therefore $\equiv$ is a congruence relation on Σ^*. ■

Example III.6. Consider the uffsa $M = (Q, \Sigma, \mu, i, f)$, where $Q = \{q_1, q_2, q_3\}$, $\Sigma = \{a, b\}$, $\mu : Q \times \Sigma \times Q \to [0, 1]$ is defined as follows:

$$\mu\big(q_1, a, q_1\big) = 1 \qquad \mu\big(q_1, b, q_2\big) = 0.6$$
$$\mu\big(q_1, a, q_2\big) = 0.8 \qquad \mu\big(q_2, a, q_3\big) = 1$$
$$\mu\big(q_1, b, q_1\big) = 1.$$

$i : Q \to [0, 1]$ such that $i(q_1) = 1$. $f : Q \to [0, 1]$ such that $f(q_3) = 1$.
The fuzzy transition diagram is shown below.
The fuzzy regular language accepted by M is
$L_M : \Sigma^* \to [0, 1]$ such that

$$L_M(x) = \begin{cases} 0.8, & \text{if } \ x \in \{a, b\}^* aa\\ 0.6, & \text{if } \ x \in \{a, b\}^* ba\\ 0, & \text{otherwise}\end{cases}$$

For example, $f_{\{a,b\}^*}(q_1) = q_1$ $f_{a\{a,b\}^*}(q_2) = q_3$ $f_{b\{a,b\}^*}(q_2) = q_2$.
Now, $aba \equiv aaba$. Since

$$f_{aba}(q_1) = q_1, \qquad f_{aba}(q_2) = q_3, \qquad f_{aba}(q_3) = q_3,$$
$$f_{aaba}(q_1) = q_1, \qquad f_{aaba}(q_2) = q_3 \qquad f_{aaba}(q_3) = q_3.$$

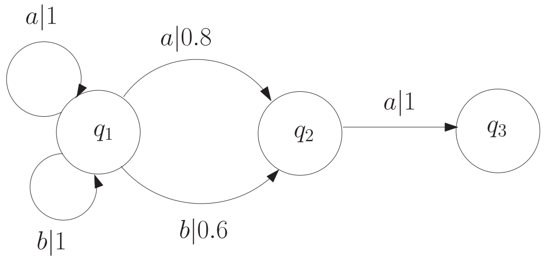

Fig. 1: Transition diagram of the uffsa

However, aba is not congruent to ba, since $f_{aba}(q_2) = q_3$ while $f_{ba}(q_2) = q_2$. Note that, any two strings beginning with a are equivalent and any two strings beginning with b are equivalent; and any other two strings are not equivalent. Therefore, there are only three equivalence classes namely, $[\lambda]$, $[a\{a,b\}^*]$, $[b\{a,b\}^*]$. Also, $\Sigma^* = [\lambda] \cup [a\{a,b\}^*] \cup [b\{a,b\}^*]$. Next we compute $F(M) = \{f_x \mid x \in \Sigma^*\}$. We have $f_{a\{a,b\}^*} = f_a$ and $f_{b\{a,b\}^*} = f_b$. Therefore $F(M) = \{f_\lambda, f_a, f_b\}$. The following is the required table for operations.

$\circ$	f_λ	f_a	f_b
f_λ	f_λ	f_a	f_b
f_a	f_a	f_{aa}	f_{ba}
f_b	f_b	f_{ab}	f_{bb}

But $f_{aa} = f_a$, $f_{bb} = f_b$, $f_{ba} = f_b$, $f_{ab} = f_a$. Therefore the operation table changes as follows:

$\circ$	f_λ	f_a	f_b
f_λ	f_λ	f_a	f_b
f_a	f_a	f_a	f_b
f_b	f_b	f_a	f_b

Thus $\big(F(M), \circ\big)$ is a finite monoid. The fuzzy function $\mu_1 : F(M) \to [0,1]$ is defined by $\mu_1(f_\lambda) = 1, \quad \mu_1(f_a) = 1, \quad \mu_1(f_b) = 1$.

Theorem III.7. *Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa, then $\Sigma^*/\equiv$ is a monoid.*

Proof: Let $x \in \Sigma^*$, $[x] = \{y \in \Sigma^* \mid x \equiv y\}$, $\Sigma^*/\equiv = \{[x] \mid x \in \Sigma^*\}$. Define a binary operation $*$ on $\Sigma^*/\equiv$ by $\forall [x], [y] \in \Sigma^*/\equiv, \quad [x] * [y] = [yx]$ Clearly $*$ is well defined and $[\lambda]$ is the identity element which is in $\Sigma^*/\equiv$. Hence $\Sigma^*/\equiv$ is a monoid. ∎

Theorem III.8. *Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa, then $\Sigma^*/\equiv$ is isomorphic to $F(M)$.*

Proof: Let $M = (Q, \Sigma, \mu, i, f)$ be the uffsa. $\Sigma^*/\equiv = \{[x] \mid x \in \Sigma^*\}$ and $F(M) = \{f_x \mid x \in \Sigma^*\}$ $\big(\Sigma^*/\equiv, *\big)$ and $\big(F(M), \circ\big)$ are monoids. Define $\phi : \big(\Sigma^*, \cdot\big) \to \big(F(M), \circ\big)$ by $\phi(x) = f_x \forall x \in \Sigma^*$. By Theorem III.4, ϕ is an anti-epimorphism. Define $g : \big(\Sigma^*/\equiv, *\big) \to \big(F(M), \circ\big)$ by $g[x] = \phi(x) \forall [x] \in \Sigma^*/\equiv$. We show that, g is well defined. Let $[x], [y] \in \Sigma^*/\equiv$. Now $[x] = [y]$, implies that $x \equiv y$. Therefore $f_x = f_y$. Since ϕ is onto, there exists $x, y \in \Sigma^*$ such that $\phi(x) = f_x$ and $\phi(y) = f_y$. Therefore $\phi(x) = \phi(y)$, implies that $g[x] = g[y]$. Thus g is well defined. Now we prove g is a homomorphism. Let $[x], [y] \in \Sigma^*/\equiv$. $g\big([x] * [y]\big) = g[yx] = \phi(yx) = f_x \circ f_y = g[x] \circ g[y]$ and $g\big([\lambda]\big) = \phi(\lambda) = f_\lambda$. Therefore g is a homomorphism of monoids. To prove g is one–one, take $[x], [y] \in \Sigma^*/\equiv$ and $g[x] = g[y]$. Now $\phi(x) = \phi(y)$, implies that $f_x = f_y$. Therefore $x \equiv y$, implies that $[x] \equiv [y]$. Therefore g is one–one. Finally we prove, g is onto. Let $f_x \in F(M)$. Since ϕ is onto there exists an $x \in \Sigma^*$ such that $\phi(x) = f_x$, which implies that $g[x] = f_x$. Hence g is an isomorphism of monoids. ∎

Definition III.9. *Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. For $p, q \in Q$, $x \in \Sigma^*$, $x = a_1 a_2 \ldots a_n$. Let $f_x(p) = q$ and the sequence of membership values in the path (unique) be $\mu\big(p, a_1, p_1\big), \mu\big(p_1, a_2, p_2\big), \ldots, \mu\big(p_{n-1}, a_n, q\big)$.*
Define $\mu^M\big(p, x, q\big) = \mu\big(p, a_1, p_1\big) \wedge \mu\big(p_1, a_2, p_2\big) \wedge \cdots \wedge \mu\big(p_{n-1}, a_n, q\big)$.
Therefore $\mu^M : Q \times \Sigma^* \times Q \to [0,1]$ is defined as follows:

$$\mu^M\big(p, a, q\big) = \begin{cases} \mu\big(p, a, q\big), & \textit{if} \quad \mu\big(p, a, q\big) = \vee\{\mu\big(p, a, r\big), \\ & \qquad\qquad\qquad\qquad \mid r \in Q\}, \\ 0, & \qquad \textit{otherwise}. \end{cases}$$

Inductively, $\mu^M\big(p, xa, q\big) = \mu^M\big(p, x, r\big) \wedge \mu^M\big(r, a, q\big)$, for some $r \in Q$. Since M is an uffsa, r will be unique.

Lemma III.10. *Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. Then for any $x, y \in \Sigma^*$ $\mu^M\big(p, xy, q\big) = \mu^M\big(p, x, r\big) \wedge \mu^M\big(r, y, q\big)$, some $\ r \in Q \ \forall p, q \in Q$.*

Proof: Let $p, q \in Q$ and $x, y \in \Sigma^*$. We prove the result by induction on $|y| = n$.Let $n = 0$, then $y = \lambda$, $xy = x\lambda = x$.

$$\mu^M\big(p, xy, q\big) = \mu^M\big(p, x\lambda, q\big) = \mu^M\big(p, x, q\big) \wedge \mu^M\big(q, \lambda, q\big), \quad \text{since} \quad \mu^M\big(q, \lambda, q\big) = 1$$

Therefore $\mu^M\big(p, xy, q\big) = \mu^M\big(p, x, r\big) \wedge \mu^M\big(r, y, q\big)$, such that $r = q \in Q$. Thus the result is true for $n = 0$. Suppose the result is true for all $y \in \Sigma^*$ such that $|y| \leq n-1$. Let $y = ua$ where $u \in \Sigma^*$ and $|u| = n-1$, $n > 0$. Now

$$\begin{aligned} \mu^M\big(p, xy, q\big) &= \mu^M\big(p, xua, q\big) \\ &= \mu^M\big(p, xu, s\big) \wedge \mu^M\big(s, a, q\big), \quad \text{some } s \in Q \\ &= \mu^M\big(p, x, r\big) \wedge \mu^M\big(r, u, s\big) \wedge \mu^M\big(s, a, q\big), \\ &\qquad\qquad \text{some } r, s \in Q \\ &= \mu^M\big(p, x, r\big) \wedge \mu^M\big(r, ua, q\big) \\ &= \mu^M\big(p, x, r\big) \wedge \mu^M\big(r, y, q\big), \quad r \in Q \end{aligned}$$

Thus the result is true for $|y| = n$. Hence the result. ∎

Definition III.11. *Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. For $x \in \Sigma^*$, define the fuzzy subset $x^M : Q \times Q \to [0,1]$ by $x^M(p,q) = \mu^M\big(p, x, q\big) \forall p, q \in Q$.*

Theorem III.12. *Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. Let $S_M = \{x^M \mid x \in \Sigma^*\}$. Then $(S_M, \circ)$ is a finite monoid.*

Proof: Let $x^M, y^M, z^M \in S_M$, $\forall p, q \in Q$. By Definition II.1,

$$\begin{aligned}(x^M \circ y^M)(p,q) &= \vee\{x^M(p,r) \wedge y^M(r,q) \mid r \in Q\} \\ &= \mu^M\big(p,\, x,\, r\big) \wedge \mu^M\big(r,\, y,\, q\big), \\ &\quad (\text{for some } r \in Q) \\ &= \mu^M\big(p,\, xy,\, q\big) \\ &= (xy)^M(p,q)\end{aligned}$$

Thus $\big(x^M \circ y^M\big) = (xy)^M$. Therefore S_M is closed under $\circ$. Associative law is also satisfied. In fact, $\big(x^M \circ y^M\big) \circ z^M = (xy)^M \circ z^M = (xyz)^M = x^M \circ (yz)^M = x^M \circ \big(y^M \circ z^M\big)$ $\lambda^M \in S_M$, and $\big(x^M \circ \lambda^M\big) = (x\lambda)^M = x^M = (\lambda x)^M = \big(\lambda^M \circ x^M\big)$. Therefore λ^M is the identity element which is in S_M. Im(μ) is finite, implies that S_M is finite. Thus $\big(S_M, \circ\big)$ is a finite monoid. ■

Theorem III.13. *Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. Then S_M and $F(M)$ are anti-isomorphic as monoids.*

Proof: Define $\phi : (S_M, \circ) \to (F(M), \circ)$ by $\phi(x^M) = f_x\ \forall x^M \in S_M$. We show that, ϕ is well-defined. Let $x^M, y^M \in S_M$. Now $x^M = y^M$, implies that $x^M(p,q) = y^M(p,q) \forall p,q \in Q$. That is $\mu^M\big(p,\, x,\, q\big) = \mu^M\big(p,\, y,\, q\big)$, and this implies that $f_x(p) = f_y(p) \forall p \in Q$. Therefore $f_x = f_y$, and so$\phi(x^M) = \phi(y^M)$. Hence ϕ is well-defined. To prove, ϕ is an anti-homomorphism of monoids, let $x^M, y^M \in S_M$. Now $\phi\big(x^M \circ y^M\big) = \phi\big((xy)^M\big) = f_{xy} = f_y \circ f_x = \phi\big(y^M\big) \circ \phi\big(x^M\big)$. Also $\phi\big(\lambda^M\big) = f_\lambda$, $\lambda^M \in S_M$. Therefore ϕ is an anti-homormophism of monoids. Next we prove ϕ is one–one. Let $x^M, y^M \in S_M$ and $\phi\big(x^M\big) = \phi\big(y^M\big)$. Therefore $f_x = f_y$, implies that $f_x(p) = f_y(p) = q \in Q$. Therefore $\mu^M\big(p,\, x,\, q\big) = \mu^M\big(p,\, y,\, q\big)$. In uffsa for $r \neq q$, $\mu^M\big(p,\, x,\, r\big) = \mu^M\big(p,\, y,\, r\big) = 0$. Therefore, $\mu^M\big(p,\, x,\, q\big) = \mu^M\big(p,\, y,\, q\big)\ \forall p,q \in Q$, implies that $x^M(p,q) = y^M(p,q)$. Hence $x^M = y^M$. Therefore ϕ is one–one. Finally we prove ϕ is onto. Let $f_x \in F(M)$, $x \in \Sigma^*$, $x^M \in S_M$. Therefore we have $\phi\big(x^M\big) = f_x$. Thus ϕ is onto. Hence ϕ is an anti-isomorphism. ■

REFERENCES

[1] Alexandru Mateescu, A., Salomaa, A., Salomaa, K. and Sheng Yi, “Lexical Analysis with a simple finite–fuzzy–automaton model, *Journal of Uni-Comp.Sci.* **1**, No.5 (1995), 292–311.

[2] S.Bozapalidis,O.L.Bozapalidou,On the recognizability of fuzzy languages I and II, *Fuzzy Sets and Systems* 157(2006)23942402;

[3] S.Bozapalidis,O.L.Bozapalidou,On the recognizability of fuzzy languages I and II,*Fuzzy Sets and Systems* 159(2008)107113

[4] D.Dubois,H.Prade,*Fuzzy Sets and Systems:Theory and Applications* ,Academic Press,NewYork,1980.

[5] S.Eilenberg, *Automata,Languages and Machines* ,Vol.B,Academic Press,NewYork,1976.

[6] Hopcroft, J.E. and Ullman, J.D., *Introduction to Automata theory, Languages and Computation* , Addison-Wesley, 1979.

[7] A.Kandel,C.Lee, *Fuzzy Switching and Automata Theory and Applications*,CraneRussak,London,1979.

[8] Lee, E.T. and Zadeh, L.A., Note on fuzzy languages, *Inform. Sci*, **1** (1969) 421–434.

[9] Malik, D.S. and Mordeson, J.N., *Fuzzy Automata and languages, theory and applications*, CRC, 2002.

[10] Malik, D.S. and Mordeson, J.N., On Fuzzy recognizers, *Kybernets*, **28** (1999) 47–60.

[11] Rajaretnam, T. and Ayyaswamy,A., Fuzzy finite State Automaton with Unique Membership transition on an input Symbol. *The journal of Combinatorial Mathematics and combinatorial computing*, **69** (2009) 151–164.

[12] Sen M.K. and Chowdhry, G. Local behaviour of fuzzy automata. *The journal of fuzzy Mathematics*, **9**, No.4, 2001.