On Inequalities involving First Three Moments

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***Abstract*—We derive inequalities involving first three moments of a discrete random variable defined over a finite interval. The refinements of some known inequalities are obtained. It is shown that the bounds are helpful in locating the extreme roots of a polynomial with all the roots real.**

***Index Terms*— Brunk inequalities, Central moments, Polynomials, roots.**

***MSC 2010 Codes* —60E15, 26C10.**

I. INTRODUCTION

Let a discrete random variable is defined over the finite interval $[a, b]$, $a < b$, with probabilities functions $p_i\,(i = 1, 2, \dots, n)$. The central moments and moments about the origin are respectively defined as

$$\mu_r = \sum_{i=1}^{n} p_i (x_i - \mu_1')^r \tag{1.1}$$

and

$$\mu_r' = \sum_{i=1}^{n} p_i {x_i}^r, \tag{1.2}$$

where $p_i \geq 0$ and

$$\sum_{i=1}^{n} p_i = 1. \tag{1.3}$$

Let $a \leq x_i \leq b$, then

$$\mu_1'^{\,2} \leq \mu_2' \leq (a+b)\mu_1' - ab \tag{1.4}$$

and

$$a\mu_2' + \frac{(\mu_2' - a\mu_1')^2}{\mu_1' - a} \leq \mu_3' \leq b\mu_2' - \frac{(b\mu_1' - \mu_2')^2}{b - \mu_1'} . \tag{1.5}$$

See [1]-[3], for more details. For the central moments we have

$$\mu_2 \leq (b - \mu_1')(\mu_1' - a) \tag{1.6}$$

and

$$\frac{\mu_2^2 - (\mu_1' - a)^2 \mu_2}{\mu_1' - a} \leq \mu_3 \leq \frac{(b - \mu_1')^2 \mu_2 - \mu_2^2}{b - \mu_1'} . \tag{1.7}$$

For the history and motivation of these inequalities, see [4]-[6].

If $x_i\,(i = 1, 2, \dots, n)$ are arranged in ascending order $x_1 \leq x_2 \leq \dots \leq x_n$ then we also have [7]

$$\mu_2' \geq (x_{j-1} + x_j)\mu_1' - x_{j-1}x_j , \tag{1.8}$$

for $j = 2, 3, \dots, n$. The inequality (1.8) gives a refinement of the first inequality (1.4) for that value of j for which $x_{j-1} \leq \mu_1' \leq x_j$.

We analyze inequalities (1.5) and (1.7) in detail and give refinements and alternative proofs of these inequalities. We obtain a lower bound for the largest root and upper bound for the smallest root of a polynomial equation when all its roots are real. Our bounds compare favorably than those obtained by the Brunk inequalities, see [8]-[9], (Example 1 and 2, below).

II. MAIN RESULTS

Theorem 2.1. Let x_i be arranged in ascending order and $a \leq x_i \leq b$, $i = 1, 2, \dots, n$. Then

$$\mu_3' \leq (x_{j-1} + x_j + b)\mu_2' - (x_{j-1}x_j + x_{j-1}b + x_j b)\mu_1' + x_{j-1}x_j b \tag{2.1}$$

and

$$\mu_3' \geq (a + x_{j-1} + x_j)\mu_2' - (a x_{j-1} + a x_j + x_{j-1}x_j)\mu_1' + a x_{j-1}x_j , \tag{2.2}$$

where $j = 2, 3, \dots, n$.

Proof. If $x_i \leq b$, then

$$(x_i - x_{j-1})(x_i - x_j)(x_i - b) \leq 0 \tag{2.3}$$

for $i = 1, 2, \dots, n$ and $j = 2, 3, \dots, n$. So

$$x_i^3 \leq (x_{j-1} + x_j + b)x_i^2 - (x_{j-1}x_j + x_{j-1}b + x_j b)x_i + x_{j-1}x_j b. \tag{2.4}$$

Multiply both sides of (2.4) by $p_i\ (i = 1, 2, \dots, n)$ and add n inequalities, we get

$$\sum_{i=1}^{n} p_i x_i^3 \leq (x_{j-1} + x_j + b)\sum_{i=1}^{n} p_i x_i^2 - (x_{j-1}x_j + x_{j-1}b + x_j b)\sum_{i=1}^{n} p_i x_i + x_{j-1}x_j b \sum_{i=1}^{n} p_i . \tag{2.5}$$

Use (1.2) and (1.3), the inequality (2.5) immediately gives the inequality (2.1).

The inequality (2.2) follows on using similar arguments. We note that if $x_i \geq a$, then

$$(x_i - a)(x_i - x_{j-1})(x_i - x_j) \geq 0 \tag{2.6}$$

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for $i = 1, 2, \dots, n$ and $j = 2, 3, \dots, n$. ■

The inequalities (2.1), (2.2) and (1.5) give the bounds for the third order moment in terms of the first and second order moments. In the following corollary and remark we show that the inequalities (2.1) and (2.2) provide the refinements of the inequalities in (1.5).

Corollary 2.1. The equivalent expressions for (2.1) and (2.2) are

$$\mu_3' \leq b\mu_2' - \frac{(b\mu_1'-\mu_2')^2}{b-\mu_1'} + \frac{\big(\mu_2' - (x_{j-1}+b)\mu_1' + x_{j-1}b\big)\big(\mu_2' - (x_j+b)\mu_1' + x_j b\big)}{b-\mu_1'} \tag{2.7}$$

and

$$\mu_3' \geq a\mu_2' + \frac{(\mu_2'-a\mu_1')^2}{\mu_1'-a} - \frac{\big(\mu_2' - (a+x_{j-1})\mu_1' + ax_{j-1}\big)\big(\mu_2' - (a+x_j)\mu_1' + ax_j\big)}{\mu_1'-a} \tag{2.8}$$

respectively, $j = 2, 3, \dots, n$.

Proof. It is evident from (2.1) that

$$\mu_3' \leq b\mu_2' - \frac{(b\mu_1'-\mu_2')^2}{b-\mu_1'} + \alpha, \tag{2.9}$$

where

$$\alpha = \big(x_{j-1}+x_j+b\big)\mu_2' - \big(x_{j-1}x_j + x_{j-1}b + x_j b\big)\mu_1' + x_{j-1}x_j b - b\mu_2' + \frac{(b\mu_1'-\mu_2')^2}{b-\mu_1'}. \tag{2.10}$$

From (2.10), we find that

$$\alpha = \frac{\mu_2'^2 + \alpha_1\mu_2' + \alpha_2}{b-\mu_1'}, \tag{2.11}$$

where

$$\alpha_1 = \big(x_{j-1}+x_j\big)(b-\mu_1') - 2b\mu_1'$$

and

$$\alpha_2 = \big(x_{j-1}x_j + x_{j-1}b + x_j b + b^2\big)\mu_1'^2 - \big(2x_{j-1}x_j + x_{j-1}b + x_j b\big)b\mu_1' + b^2 x_{j-1}x_j .$$

A little computation then shows that

$$\alpha = \frac{\big(\mu_2' - (x_{j-1}+b)\mu_1' + x_{j-1}b\big)\big(\mu_2' - (x_j+b)\mu_1' + x_j b\big)}{b-\mu_1'}. \tag{2.12}$$

Combining (2.9) and (2.12), we get (2.7).
Similarly, from the inequality (2.2), we have

$$\mu_3' \leq a\mu_2' + \frac{(\mu_2'-a\mu_1')^2}{\mu_1'-a} - \beta, \tag{2.13}$$

where

$$\beta = a\mu_2' + \frac{(\mu_2'-a\mu_1')^2}{\mu_1'-a} - \big(a+x_{j-1}+x_j\big)\mu_2' + \big(x_{j-1}a + x_j a + x_{j-1}x_j\big)\mu_1' - ax_{j-1}x_j . \tag{2.14}$$

From (2.14)

$$\beta = \frac{\mu_2'^2 + \beta_1\mu_2' + \beta_2}{\mu_1'-a}, \tag{2.15}$$

where

$$\beta_1 = \big(x_{j-1}+x_j\big)(a-\mu_1') - 2a\mu_1'$$

and

$$\beta_2 = \big(x_{j-1}a + x_j a + x_{j-1}x_j + a^2\big)\mu_1'^2 - \big(2x_{j-1}x_j + x_{j-1}a + x_j a\big)a\mu_1' + a^2 x_{j-1}x_j .$$

So,

$$\beta = \frac{\big(\mu_2' - (a+x_{j-1})\mu_1' + ax_{j-1}\big)\big(\mu_2' - (a+x_j)\mu_1' + ax_j\big)}{\mu_1'-a}. \tag{2.16}$$

Combining (2.13) and (2.16), we get (2.8). ■

Remark 2.1. The inequalities in Theorem 2.1 are valid for any $j = 2, 3, \dots, n$. We show that there is one such j ($j = 2, 3, \dots, n-1$) for which (2.1) provides a refinement of the second inequality (1.5). Similarly, (2.2) provides a refinement of the first inequality (1.5) for at least one j ($j = 3, 4, \dots, n$).

It suffices to show that there is a value of j ($j = 2, 3, \dots, n-1$) for which $\alpha \leq 0$. From (2.12) we find that $\alpha \leq 0$ if and only if

$$\big(x_j+b\big)\mu_1' - x_j b \leq \mu_2' \leq \big(x_{j-1}+b\big)\mu_1' - x_{j-1}b \tag{2.17}$$

for at least one $j = 2, 3, \dots, n-1$. It is clear from (1.8) that $\mu_2' \geq (x_{n-1}+b)\mu_1' - x_{n-1}b$, $j = n$. If $\mu_2' \leq (x_{n-2}+b)\mu_1' - x_{n-2}b$, then (2.17) is true for $j = n-1$. If $\mu_2' \geq (x_{n-2}+b)\mu_1' - x_{n-2}b$ and $\mu_2' \leq (x_{n-3}+b)\mu_1' - x_{n-3}b$, then (2.17) is true for $j = n-2$, and so on. This means that if μ_2' lies in the interval $[(x_{n-1}+b)\mu_1' - x_{n-1}b,\ (a+b)\mu_1' - ab]$, then it must also lie in one of the intervals (2.17) for at least one $j = 2, 3, \dots, n-1$. So $\alpha \leq 0$ for at least one j and hence (2.1) provides a refinement of the second inequality (1.5).

Likewise, it follows from (2.16) that $\beta \leq 0$ if and only if

$$\big(a+x_{j-1}\big)\mu_1' - ax_{j-1} \leq \mu_2' \leq \big(a+x_j\big)\mu_1' - ax_j \tag{2.18}$$

for at least one j ($j = 3, 4, \dots, n$). If μ_2' lies in the interval $[(a+x_2)\mu_1' - ax_2,\ (a+b)\mu_1' - ab]$, then it also lies in one of the intervals (2.18). So (2.2) gives a refinement of the first inequality (1.5).

Theorem 2.2. For $a \leq x_i \leq b$, we have

$$\mu_3' \leq \big(2x_j + b\big)\mu_2' - \big(x_j^2 + 2bx_j\big)\mu_1' + x_j^2 b \tag{2.19}$$

and

$$\mu_3' \geq (2x_j + a)\mu_2' - (x_j^2 + 2ax_j)\mu_1' + x_j^2 a, \quad (2.20)$$

where $i, j = 1, 2, \ldots, n$.

Proof. The proof of the Theorem 2.2 follows on using similar arguments used in the proof of the Theorem 2.1. The inequalities (2.19) and (2.20) follow respectively from the inequalities

$$(x_i - x_j)^2 (x_i - b) \leq 0 \quad (2.21)$$

and

$$(x_i - x_j)^2 (x_i - a) \geq 0, \quad (2.22)$$

for $i, j = 1, 2, \ldots, n$. ■

Corollary 2.2. For $x_i \leq b, (i = 1, 2, \ldots, n)$ we have

$$\mu_3' - (\mu_2')^{3/2} \leq \frac{4}{27}(x_{j-1} + x_j + b)^3 + x_{j-1}x_j b - (x_{j-1}x_j + x_{j-1}b + x_j b)\mu_1'$$

for $j = 2, 3, \ldots, n$. (2.23)

Proof. Subtracting $(\mu_2')^{3/2}$, on both sides of (2.1), we get

$$\mu_3' - (\mu_2')^{3/2} \leq (x_{j-1} + x_j + b)\mu_2' - (\mu_2')^{3/2} - (x_{j-1}x_j + x_{j-1}b + x_j b)\mu_1' + x_{j-1}x_j b. \quad (2.24)$$

Let

$$f(\mu_2') = (x_{j-1} + x_j + b)\mu_2' - (\mu_2')^{3/2} - (x_{j-1}x_j + x_{j-1}b + x_j b)\mu_1' + x_{j-1}x_j b. \quad (2.25)$$

The derivative

$$f'(\mu_2') = (x_{j-1} + x_j + b) - \tfrac{3}{2}(\mu_2')^{1/2} \quad (2.26)$$

vanishes at

$$\mu_2' = \tfrac{4}{9}(x_{j-1} + x_j + b)^2, \quad (2.27)$$

where $f(\mu_2')$ has maxima. On substituting value of μ_2' from (2.27) in (2.24), we get (2.23) on simplifications. ■

The inequalities obtained above involve the first three moments about origin of a discrete random variable. We now prove the inequalities involving the central moments.

Theorem 2.3. For $a \leq x_i \leq b, i = 1, 2, \ldots, n$, we have

$$\mu_3 \leq (x_{j-1} - \mu_1')(x_j - \mu_1')(b - \mu_1') + (x_{j-1} + x_j + b - 3\mu_1')\mu_2 \quad (2.28)$$

and

$$\mu_3 \geq (\mu_1' - a)(\mu_1' - x_{j-1})(x_j - \mu_1') + (a + x_{j-1} + x_j - 3\mu_1')\mu_2 \quad (2.29)$$

for $j = 2, 3, \ldots, n$.

Proof. From (2.1) and (2.2), we respectively have

$$\mu_3' - \mu_1'^3 \leq (x_{j-1} - \mu_1')(x_j - \mu_1')(b - \mu_1') + (x_{j-1} + x_j + b)\mu_2 \quad (2.30)$$

and

$$\mu_3' - \mu_1'^3 \geq (\mu_1' - a)(\mu_1' - x_{j-1})(x_j - \mu_1') + (a + x_{j-1} + x_j)\mu_2, \quad (2.31)$$

for $j = 2, 3, \ldots, n$.

We use the well known relations

$$\mu_2' = \mu_2 + \mu_1'^2 \quad (2.32)$$

and

$$\mu_3' = \mu_3 + 3\mu_1'\mu_2 + \mu_1'^3. \quad (2.33)$$

Then (2.28) and (2.29) follow respectively from (2.30) and (2.31). ■

Corollary 2.3. The equivalent expressions for (2.28) and (2.29) are

$$\mu_3 \leq \frac{(b - \mu_1')^2\mu_2 - \mu_2^2}{b - \mu_1'} + \frac{\big(\mu_2 - (b - \mu_1')(\mu_1' - x_{j-1})\big)\big(\mu_2 - (b - \mu_1')(\mu_1' - x_j)\big)}{b - \mu_1'} \quad (2.34)$$

and

$$\mu_3 \geq \frac{\mu_2^2 - (\mu_1' - a)^2\mu_2}{\mu_1' - a} - \frac{\big(\mu_2 - (\mu_1' - a)(x_j - \mu_1')\big)\big(\mu_2 - (\mu_1' - a)(x_{j-1} - \mu_1')\big)}{\mu_1' - a} \quad (2.35)$$

respectively, for $j = 2, 3, \ldots, n$.

Proof. The inequality (2.28) can be written as

$$\mu_3 \leq \frac{(b-\mu_1')^2 \mu_2 - \mu_2^2}{b - \mu_1'} + \gamma, \quad (2.36)$$

where

$$\gamma = (x_{j-1} - \mu_1')(x_j - \mu_1')(b - \mu_1') + (x_{j-1} + x_j + b - 3\mu_1')\mu_2 - \frac{(b - \mu_1')^2\mu_2 - \mu_2^2}{b - \mu_1'}. \quad (2.37)$$

On simplification, we get

$$\gamma = \frac{\big(\mu_2 - (b-\mu_1')(\mu_1' - x_{j-1})\big)\big(\mu_2 - (b-\mu_1')(\mu_1' - x_j)\big)}{b - \mu_1'}. \quad (2.38)$$

Combining (2.36) and (2.38), we get (2.34).

From the inequality (2.29), we have

$$\mu_3 \geq \frac{\mu_2^2 - (\mu_1' - a)^2 \mu_2}{\mu_1' - a} - \delta \quad (2.39)$$

where

$$\delta = \frac{\mu_2^2 - (\mu_1' - a)^2\mu_2}{\mu_1' - a} - (\mu_1' - a)(\mu_1' - x_{j-1})(x_j - \mu_1') - (a + x_{j-1} + x_j - 3\mu_1')\mu_2 . \tag{2.40}$$

On simplification we get

$$\delta = \frac{\left(\mu_2 - (\mu_1' - a)(x_j - \mu_1')\right)\left(\mu_2 - (\mu_1' - a)(x_{j-1} - \mu_1')\right)}{\mu_1' - a} . \tag{2.41}$$

Combining (2.39) and (2.41), we get (2.35). ■

Remark 2.2. The inequalities in Theorem 2.3 provide the refinements of the inequalities in (1.7).

III. An Application

As an immediate application we obtain bounds for the largest and smallest root of a polynomial equation when all the roots of the polynomial are real. We give examples to show that our bounds give better estimates than those given by the Brunk inequalities. Consider the n^{th} degree monic polynomial equation

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \tag{3.1}$$

with all the roots real. Let $x_1, x_2, \dots, x_n$ be the roots of (3.1). Then by Vieta's formula,

$$\sum_{i=1}^{n} x_i = -a_1, \tag{3.2}$$

and

$$\sum_{i<j}^{n} x_ix_j = a_2. \tag{3.3}$$

Let $\overline{x}$ be the mean and S^2 be the variance of the n roots $x_1, x_2, \dots, x_n$. It follows from (3.2) and (3.3) that

$$\overline{x} = \frac{-a_1}{n} , \tag{3.4}$$

$$S^2 = \frac{(n-1)a_1^2 - 2na_2}{n^2} \tag{3.5}$$

and

$$\mu_2' = S^2 + \overline{x}^2 = \frac{1}{n}(a_1^2 - 2a_2). \tag{3.6}$$

Laguerre proved that [8],

$$\frac{-a_1}{n} - S\sqrt{n-1} \le x_j \le \frac{-a_1}{n} + S\sqrt{n-1} \tag{3.7}$$

for $j = 1, 2, \dots, n$.
Let a be the smallest and b be the largest root of $f(x) = 0$. Then Brunk inequalities [9] give

$$b \ge \overline{x} + \frac{S}{\sqrt{n-1}} \tag{3.8}$$

and

$$a \le \overline{x} - \frac{S}{\sqrt{n-1}} . \tag{3.9}$$

Lemma 3.1. Let x_i $(i = 1,2,\dots,n)$ be the roots of the polynomial equation (3.1). Then

$$\sum_{i=1}^{n} x_i^3 = -a_1^3 + 3a_1a_2 - 3a_3 . \tag{3.10}$$

Proof. On using well known Newton's identity, we have

$$\sum_{i=1}^{n} x_i^3 = \left(\sum_{i=1}^{n} x_i\right)^3 - 3\sum_{i=1}^{n} x_i \sum_{i<j}^{n} x_i x_j + 3\sum_{i<j<k}^{n} x_i x_j x_k . \tag{3.11}$$

Use the relations between roots and coefficients of polynomial equation (3.1), we have

$$\sum_{i<j<k}^{n} x_i x_j x_k = -a_3 . \tag{3.12}$$

Combining (3.2), (3.3) and (3.12) with (3.11), we get (3.10). ■

Example 1. Let

$$f(x) = x^5 - 4x^4 - 66x^3 + 64x^2 + 1025x + 900. \tag{3.13}$$

Let $x_1 \le x_2 \le \cdots \le x_5$ be the roots of the polynomial equation (3.13). On using the relations between the roots and coefficients of a polynomial equation, we have

$$\mu_1' = \frac{1}{5}\sum_{i=1}^{5} x_i = \frac{4}{5} , \tag{3.14}$$

$$\mu_2' = \frac{1}{5}\sum_{i=1}^{5} x_i^2 = \frac{148}{5} \tag{3.15}$$

and

$$\mu_3' = \frac{1}{5}\sum_{i=1}^{5} x_i^3 = \frac{664}{5} . \tag{3.16}$$

Substituting the values of μ_1', μ_2' and μ_3' respectively from (3.14), (3.15) and (3.16) in first inequality (1.5), we find that

$$181x_1^2 - 682x_1 - 4812 \ge 0 . \tag{3.17}$$

Here x_1 is the smallest root of $f(x) = 0$. From (3.17) either $x_1 \ge 7.3735$ or $x_1 \le -3.6055$. Since $x_1 \ngtr 7.3735$, we conclude that $x_1 \le -3.6055$.
Likewise, it follows from the second inequality (1.5) that

$$181x_5^2 - 682x_5 - 4812 \ge 0 . \tag{3.18}$$

Here x_5 is the largest root of $f(x) = 0$. From (3.18) either $x_5 \ge 7.3735$ or $x_5 \le -3.6055$. Since $x_5 \nless -3.6055$, we must have $x_5 \ge 7.3735$.
From the Brunk inequalities (3.8) and (3.9), we respectively have $x_5 \ge 3.4904$ and $x_1 \le -1.8907$. The inequalities in (1.5) give better estimates for the largest and smallest roots of

$f(x) = 0$. The roots of $f(x) = 0$ are $-5, -4, -1, 5$ and 9. ■

Example 2. Let

$$g(x) = x^5 - 17x^3 + 12x^2 + 52x - 48 . \quad (3.19)$$

Let $x_1 \le x_2 \le \cdots \le x_5$ be the roots of the polynomial equation (3.19). On using the relations between the roots and coefficients of a polynomial equation, we have

$$\mu_1' = \frac{1}{5}\sum_{i=1}^{5} x_i = 0 , \quad (3.20)$$

$$\mu_2' = \frac{1}{5}\sum_{i=1}^{5} {x_i}^2 = \frac{34}{5} \quad (3.21)$$

and

$$\mu_3' = \frac{1}{5}\sum_{i=1}^{5} {x_i}^3 = -\frac{36}{5} . \quad (3.22)$$

Substituting the values of μ_1', μ_2' and μ_3' respectively from (3.20), (3.21) and (3.22) in first inequality (1.5), we find that

$$85x_1^2 + 90x_1 - 578 \ge 0 . \quad (3.23)$$

Here x_1 is the smallest root of $g(x) = 0$. From (3.23) either $x_1 \ge 2.1315$ or $x_1 \le -3.1903$. Since $x_1 \ngtr 2.1315$, we conclude that $x_1 \le -3.1903$.
Likewise, it follows from the second inequality (1.5) that

$$85x_5^2 + 90x_5 - 578 \ge 0 . \quad (3.24)$$

Here x_5 is the largest root of $g(x) = 0$. From (3.24) either $x_5 \ge 2.1315$ or $x_5 \le -3.1903$. Since $x_5 \nless -3.1903$, we must have $x_5 \ge 2.1315$.
From the Brunk inequalities (3.8) and (3.9), we respectively have $x_5 \ge 1.3038$ and $x_1 \le -1.3038$. The inequalities in (1.5) give better estimates for the largest and smallest roots of $g(x) = 0$. The roots of $g(x) = 0$ are $-4, -2, 1, 2$ and 3. ■

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