On a Theorem of Ankeny and Rivlin

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(Dedicated to Late Professor G.M. Qazi)

***Abstract*—If $p(z)$ is a polynomial of degree n and does not vanish in $|z|<1$, then it was shown by Dewan, Hans and Kaur (2010) that**

$$\{M(p,R)\}^s \le \left(\frac{R^{ns}+1}{2}\right)\{M(p,1)\}^s$$

In this paper,we have improved the above inequality by involving some or all the coefficients of $p(z)$.

***Index Terms*—Polynomial, Inequality, Derivatives, Zeros.**

***MSC 2010 Codes* – 30A10, 30C10, 30C15**

I. Introduction and Statement of results

FOR an arbitrary entire function $f(z)$, let

$$M(f,r)=\max_{|z|=r}|f(z)|$$

and

$$m(f,k)=\min_{|z|=k}|f(z)|$$

Then for a polynomial $p(z)$ of a degree n, it is a simple consequence of a Maximum Modulus Principle (see [1, Vol. 1, p. 137, Prob. III, 269]) that

$$M(p,r)\le R^n M(p,1) \quad \text{for } R\ge 1\,.$$

The result is best possible and equality holds for

$$p(z)=\alpha z^n$$

where $|\alpha|=1$.

While concerning the estimate of $|p'(z)|$ in terms of $|p(z)|$ on $|z|=1$ for the class of polynomials having no zeros in $|z|<1$, it was conjectured by P. Erdös and later by Lax [2] that if $p(z)\neq 0$ in $|z|<1$, then

$$M(p',1)\le \frac{n}{2}M(p,1)\,. \tag{1.1}$$

The result is best possible and equality holds for

$$p(z)=\alpha+\beta z^n$$

where $|\alpha|=|\beta|$.

For the class of polynomials $p(z)$ of degree n not vanishing in $|z|<k$, $k\ge 1$, Malik [3] proved

$$\max_{|z|=1}|p'(z)|\le \frac{n}{1+k}\max_{|z|=1}|p(z)|\,. \tag{1.2}$$

Chan and Malik [4] generalized (1.3) in a different direction and proved that if

$$p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu, \quad (1\le\mu\le n))$$

is a polynomial of degree n, which does not vanish in $|z|<k$, $k\ge 1$, then

$$\max_{|z|=1}|p'(z)|\le \frac{n}{1+k^\mu}\max_{|z|=1}|p(z)|\,. \tag{1.3}$$

Inequality (1.4) was independently proved by Qazi [5, Lemma 1], who also under the same hypothesis proved that

$$\max_{|z|=1}|p'(z)|\le n\Lambda \tag{1.4}$$

where

$$\Lambda=\left\{\frac{1+\left(\frac{\mu}{n}\right)\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\left(\frac{\mu}{n}\right)\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}\max_{|z|=1}|p(z)|\right\}$$

The following result which is due to Gardner, Govil and Weems [6] is of independent interest, because it provides generalizations and refinements of inequalities (1.2), (1.3), (1.4) and (1.5).

Theorem I.1. *If $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$ is a polynomial of degree n, having no zeros in the disk $|z|<k$, $k\ge 1$, then for $1\le\mu\le n$*

$$\begin{aligned}&\max_{|z|=1}|p'(z)|\\ &\le n\left\{\frac{1+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}{1+k^{\mu+1}+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}+k^{2\mu})}\right\}\\ &\quad\times\left(\max_{|z|=1}|p(z)|-m\right).\end{aligned} \tag{1.5}$$

Clearly $m=\min_{|z|=k}|p(z)|$.

It was shown by Ankeny and Rivlin [7] that if $p(z)\neq 0$ in $|z|<1$, then inequality (1.1) can be replaced by a sharper inequality.

Theorem I.2. *If $p(z)$ is a polynomial of degree n, which does not vanish in $|z|<1$, then*

$$M(p,R)\le\left(\frac{R^n+1}{2}\right)M(p,1), \quad R\ge 1\,. \tag{1.6}$$

Recently Dewan et al. [8] proved the following generalization as well as an improvement of Theorem B.

Theorem I.3. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<1$, then for every positive integer s*

$$\{M(p,R)\}^s\le\left(\frac{R^n+1}{2}\right)\{M(p,1)\}^s, \quad R\ge 1\,. \tag{1.7}$$

In this paper we improve Theorem C, by involving some or all the coefficients of $p(z)$. More precisely, we prove

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Theorem I.4. *If*

$$p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu a^\nu, \quad (1 \le \mu \le n)$$

is a polynomial of degree $|z| < k$, $k \ge 1$, having no zeros in s then every positive integer s, we have

$$\{M(p,R)\}^s \le \left[1 + R^{ns}\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}\right\}\right] \times \{M(p,1)\}^s \tag{1.8}$$

Theorem I.5. *If*

$$p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu a^\nu, \quad (1 \le \mu \le n)$$

is a polynomial of degree n having no zeros in $|z| < k$, $k \ge 1$, then every positive integer s, we have

$$\{M(p,R)\}^s \le \left\{1 + R^{ns}\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}\right\}\right\}\{M(p,1)\}^s - R^{ns}\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}\right\}\Bigg\}m(p,k), \quad R \ge 1 \tag{1.9}$$

II. Proofs of the Theorems

Proof of Theorem 1: Let

$$M(p,1) = \max_{|z|=1} |p(z)|$$

Since $p(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, therefore, by inequality (1.5), we have

$$|p'(z)| \le n\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}M(p,1), \quad \text{for } |z| = 1\,.$$

Now $p'(z)$ is a polynomial of degree $n-1$, therefore, it follows by (1.1) that for all $r \ge 1$, and $0 \le \theta < 2\pi$,

$$|p'(re^{i\theta})| \le n\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}r^{n-1}M(p,1). \tag{2.1}$$

Also for each θ, $0 \le \theta < 2\pi$ and $R \ge 1$, we have

$$\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s = \int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt = \int_1^R s\{p(te^{i\theta})\}^{s-1}p'(te^{i\theta})e^{i\theta}dt\,.$$

This implies

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \le s\int_1^R |\{p(te^{i\theta})\}^{s-1}|\,|p'(te^{i\theta})e^{i\theta}|dt\,.$$

Which on combining with inequality (1.1) and (2.1), we get

$$\begin{aligned}|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| &\le ns\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})} \times \int_1^R t^{ns-1}\{M(p,1)\}^s dt \\ &= R^{ns}\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}\right\}\{M(p,1)\}^s\,.\end{aligned}$$

Which implies

$$\begin{aligned}|\{p(Re^{i\theta})\}|^s &\le |\{p(e^{i\theta})\}|^s + R^{ns}\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}\right\} \times \{M(p,1)\}^s \\ &\le \left[1 + R^{ns}\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}\right\}\right]\{M(p,1)\}^s\,.\end{aligned} \tag{2.2}$$

Hence from (2.2) we conclude that

$$\{M(p,R)\}^s \le \left[1 + R^{ns}\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}\right\}\right]\{M(p,1)\}^s\,.$$

This completes the proof of Theorem 1. ■

Proof of Theorem 2: The proof of Theorem 2 follows on the same lines as that of Theorem 1 by using inequality (1.6) instead of (1.5). But for the sake of completeness we give a brief outline of the proof. Since $p(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, therefore, by inequality (1.6), we have

$$\max_{|z|=1}|p'(z)| \le n\left\{\frac{1+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}{1+k^{\mu+1}+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}+k^{2\mu})}\right\} \times (\max_{|z|=1}|p(z)| - m) \quad \text{for } |z| = 1\,.$$

Now $p'(z)$ is a polynomial of degree $n-1$, therefore, it follows by (1.1) that for all $r \ge 1$, and $0 \le \theta < 2\pi$,

$$|p'(re^{i\theta})| \le n\left\{\frac{1+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}{1+k^{\mu+1}+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}+k^{2\mu})}\right\}r^{n-1} \times (\max_{|z|=1}|p(z)| - m). \tag{2.3}$$

Also for each θ, $0 \le \theta < 2\pi$ and $R \ge 1$, we have

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \le s\int_1^R |p(te^{i\theta})^{s-1}|\,|p'(t^{i\theta})|dt\,.$$

Which on combining with inequality (1.1) and (2.3), we get

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \le R^{ns}\left\{\frac{1+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}{1+k^{\mu+1}+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}+k^{2\mu})}\right\} \times \{\{M(p,1)\}^s - m(p,k)\}.$$

Which implies

$$|p(Re^{i\theta})|^s$$
$$\leq (\{M(p,1)\})^s + R^{ns}\left\{\frac{1+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}{1+k^{\mu+1}+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}+k^{2\mu})}\right\}$$
$$\times \{\{M(p,1)\}^s - m(p,k)\}.$$

From which the proof of Theorem 2 follows. ■

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