Estimation of Smallest Mortality Rate Deceleration Parameter and Life Expectancy

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***Abstract*—Areawise approximation of Logistic Frailty mortality curve to Gompertz mortality curve enables us to estimatate the smallest mortality rate deceleration parameter, and the onset of Mortality Deceleration age. Using the smallest mortality rate deceleration parameter also we estimate the life expectancy.**

***Index Terms*—Logistic Frailty model, Mortality rate deceleration parameter, Centroid, Life Expectancy, Mortality plateaus, Age at which onset of mortality deceleration.**

***MSC 2010 Codes* – 92Bxx, 92D10, 62F10.**

I. Introduction

MORTALITY curves are often described by the Gompertz equation:

$$\mu(t) = ae^{bt} \tag{1}$$

where a describes mortality rate at birth, and b is the rate of exponential increase in mortality with age. Recent experimental work involving large cohorts has documented a significant deceleration of mortality rates in advanced ages [1], [2], [3]. In such cases, the Gompertz equation is a poor description of mortality dynamics. And the logistic frailty model has been shown to provide a better description of mortality rates in such cases:

$$\overline{\mu}(t) = \frac{ae^{bt}}{1 + \frac{a\sigma^2}{b}(e^{bt} - 1)} \tag{2}$$

[4]. Early in life (x near zero), mortality increases exponentially at a rate determined by a and b. The parameter σ^2 determines the extent to which mortality rates decelerate late in life. Higher values of σ^2 indicate greater decleration. Note that when $\sigma^2 = 0$, Equation (2) reduces to Equation (1) [5]. The corresponding survival function is

$$\overline{S}(t) = (1 + \frac{a\sigma^2}{b}(e^{bt} - 1))^{-\frac{1}{\sigma^2}}. \tag{3}$$

Most commonly this mortality deceleration is measured by the life-table aging rate, introduced by Horiuchi and colleagues [6], but also other methods were used previously. Mortality would decelerate along various trajectories rather than merely plateau [7], and it is better to consider departure from the Gompertz law rather than just convergence of mortality to a plateau level [8].

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Manuscript received November 10, 2011; revised January 27, 2012.

In [9], by equating the center of mass of the area under the curve of Logistic Frailty mortality function and the area under the curve of Gompertz mortality function, we obtained the following equations for σ^2 and t^*

$$\int_0^{t^*} ln(1+\frac{a\sigma^2}{b}(e^{bt}-1))dt = \frac{e^{bt^*} - bt^* - 1}{b(e^{bt^*} - 1)} ln(1+\frac{a\sigma^2}{b}(e^{bt^*}-1)), \tag{4}$$

$$\frac{a\sigma^2}{b} e^{bt^*} = 2[1 - \frac{x}{(1+x)ln(1+x)}], \tag{5}$$

where $x = \frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}}$, $t^* > 0$.

Using (4) and (5) we also proved

Theorem 1: For a given a, b there exists $\sigma^2 > 0$ satisfying the estimation

$$\frac{a\sigma^2}{b} \leq 0.0064 \tag{6}$$

provided $ln(1 - \frac{a\sigma^2}{b})$ is significantly equal to $\frac{a\sigma^2}{b}$.

II. Estimation of smallest possible value of σ^2

Rewriting equation (5), we get

$$\frac{x(1 - \frac{a\sigma^2}{b})}{2} = 1 - \frac{x}{(1+x)ln(1+x)}$$

or, equivalently

$$\frac{x(1 - \frac{a\sigma^2}{b})}{2} + \frac{x}{(1+x)ln(1+x)} = 1. \tag{7}$$

Since $\frac{a\sigma^2}{b} = 0.0064$ is uninteresting we consider the general case $\frac{a\sigma^2}{b} < 0.0064$. Then from (7) we obtain

$$\frac{0.9936x}{2} + \frac{x}{(1+x)ln(1+x)} < 1. \tag{8}$$

Using Mathematica 5.1, let us solve the inequality (8) for x. The solution x can be found from the following Table.

c	$\lim_{x\to c} \frac{0.9936x}{2} + \frac{x}{(1+x)ln(1+x)}$
0.00009	1
0.0001	1
0.00011	1
0.00012	1
0.00013	1
0.00014	1
0.00015	1
0.000159	1
0.00016	0.999999
0.00017	0.999999
0.0002	0.999999
0.001	0.999997

Clearly, the Table values reveal that

$$x > 0.000159. \tag{9}$$

Now substitution of $x = 0.000159$ into (5) yields

$$\frac{a\sigma^2}{b} = 0.000132. \tag{10}$$

Theorem 2: The smallest possible value of $\frac{a\sigma^2}{b}$ is 0.000132, i.e., $\sigma^2 = \frac{0.000132b}{a}$.

Proof

On the contrary, let us suppose that $\frac{a\sigma^2}{b} < 0.000132$. Notice that equation (7) can also be written as

$$\frac{a\sigma^2}{b} = 1 - \frac{1 - \frac{x}{(1+x)ln(1+x)}}{\frac{x}{2}}. \tag{11}$$

Hence (11) becomes

$$1 - \frac{1 - \frac{x}{(1+x)ln(1+x)}}{\frac{x}{2}} < 0.000132. \tag{12}$$

Solving (12) for x by using Mathematica 5.1 (see 1), we get

$$x < 0.00015842,$$

which is a contradiction to (9). Hence we can conclude that

$$\frac{a\sigma^2}{b} > 0.000132. \tag{13}$$

Here $y = 1 - \frac{1-\frac{x}{(1+x)ln(1+x)}}{\frac{x}{2}}$ and the straight line is $y = 0.000132$.

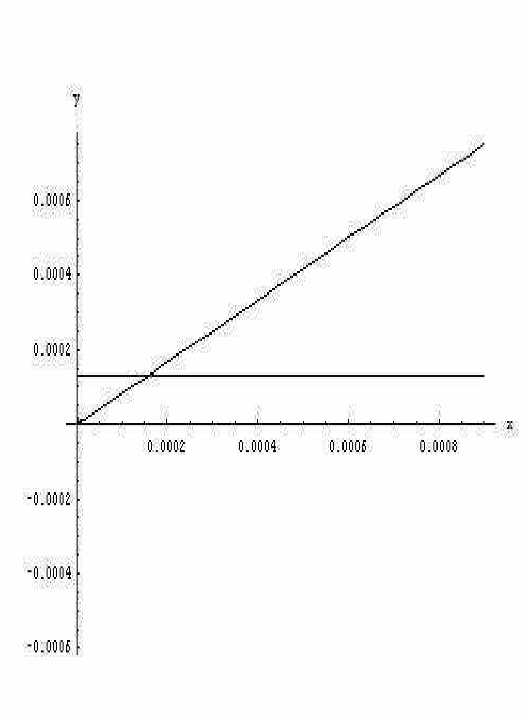

Fig. 1: Solution of Eq. (12) for x

III. Estimation of e^{bt^*}

Recalling $x = \frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}}$, it follows that

$$x > \frac{\frac{a\sigma^2}{b}}{1 - \frac{a\sigma^2}{b}},$$

or, equivalently

$$x > \frac{1}{1 - \frac{a\sigma^2}{b}} - 1. \tag{14}$$

From (14), we obtain

$$\frac{1}{(1+x)ln(1+x)} < \frac{1 - \frac{a\sigma^2}{b}}{-ln(1 - \frac{a\sigma^2}{b})},$$

which gives

$$\frac{x}{(1+x)ln(1+x)} < \frac{\frac{a\sigma^2}{b}e^{bt^*}}{-ln(1 - \frac{a\sigma^2}{b})}. \tag{15}$$

Using (15) in (5), finally we get

$$\frac{a\sigma^2}{b}e^{bt^*} > 2\left(1 - \frac{\frac{a\sigma^2}{b}e^{bt^*}}{-ln(1 - \frac{a\sigma^2}{b})}\right),$$

or, equivalently

$$e^{-bt^*} < \frac{a\sigma^2}{2b} + \frac{\frac{a\sigma^2}{b}}{-ln(1 - \frac{a\sigma^2}{b})}. \tag{16}$$

Also observe that the right hand side expression of (16) has only one real fixed point at $\frac{a\sigma^2}{b} = 0.864663$. Hence

$$e^{-bt^*} < 0.864665 \Leftrightarrow e^{bt^*} > 1.15652.$$

A simple comparison with experimental data (see Table 1) shows that the above estimation is sharp.

IV. Estimation of Life Expectancy

Let us estimate life expectancy using the smallest possible value of σ^2. In view of (13), from (2) we get

$$\overline{\mu}(t) < \frac{ae^{bt}}{1 + 0.000132(e^{bt} - 1)},$$

which gives

$$\overline{S}(t) > (1 + 0.000132(e^{bt} - 1))^{\frac{-a}{0.000132b}},$$

since $\overline{\mu}(t) = -\frac{1}{\overline{S}(t)}\frac{d\overline{S}(t)}{dt}$ and $\overline{S}(0) = 1$. Hence

$$\begin{aligned}\text{Life expectancy} &= \int_0^\infty \overline{S}(t)dt \\ &> \int_0^\infty (1 + 0.000132(e^{bt} - 1))^{\frac{-a}{0.000132b}} dt.\end{aligned}$$

The numerical values of last integral for a given a and b listed in Table 1. Thus we see that the estimated life expectancy given in the last column is very near to actual life expectancy.

Table 1 (reprinted from [10], [11], [12])

Species	a	b	σ^2	$\frac{a\sigma^2}{b} > 0.000132$	$e^{bt^*} > 1.15652$	$LifeExpectancy$	$EstimatedLifeExpectancy$
A	0.00053	0.290	1.42	0.002595	1.19998	22.75	19.92
A	0.000041	1.700	9.40	0.000227	1.21072	10.37	8.32
B	0.000301	0.28	1.78	0.00191	1.2018	26.35	22.60
B	0.0002	0.318	2.39	0.001514	1.20241	26.27	21.72
B	0.000593	0.309	1.89	0.003628	1.20032	22.27	18.53
B	0.000448	0.243	1.55	0.00286	1.19983	27.48	23.73
B	0.000394	0.285	1.90	0.002624	1.19965	25.31	21.13
B	0.000034	0.128	0.86	0.000226	1.19796	63.73	61.93
B	0.000163	0.101	0.52	0.000841	1.20029	60.82	58.46
B	0.000251	0.091	0.43	0.001186	1.20159	61.13	58.87
B	0.001548	0.099021	0.13	0.002074	1.20008	37.57	36.94

Here A represents Parage Aegeria and B represents Drosophila Melanogaster.

V. Conclusion

Notice that considering the areawise approximation, we are able to get estimation for the smallest possible value of σ^2. It is worth mentioning that we obtained the estimation for e^{bt^*}.

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